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REMARKS ON THE NOTION OF HOMO-DERIVATIONS

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Dedicated to the 70th birthday of Professor Antal Járai

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Abstract. The purpose of this paper is to study the (different) notions of homo-derivations. These are additive mappings f of a ring R that also fulfill the identity

$$f(xy) = f(x)y + xf(y) + f(x)f(y) \qquad (x, y \in R)\,,$$

or (in case of the other notion) the system of equations

$$\begin{array}{rcll} f(xy) & = & f(x)f(y) & \\ f(xy) & = & f(x)y + xf(y) & \end{array} \qquad (x, y \in R)\,.$$

Our primary aim is to investigate the above equations without additivity as well as the following Pexiderized equation

$$f(xy) = h(x)h(y) + xk(y) + k(x)y.$$

The obtained results show that under rather mild assumptions homo-derivations can be fully characterized, even without the additivity assumption.

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1. Introduction and preliminaries

The main aim of this paper is to present some characterization theorems concerning homomorphisms, derivations and also homo-derivations. Thus, at first, we list some notions and preliminary results that will be used in the sequel. All of these statements and definitions can be found in Kuczma [7] and in Zariski–Samuel [13] and also in Kharchenko [6].

Homomorphisms and derivations

Definition 1.1. Let P and Q be (not necessarily unital) rings. A function $f\colon P \to Q$ is called a *homomorphism* (of P into Q) if it is additive, i.e.

$$f(x+y) = f(x) + f(y) \quad (x, y \in P)$$

and also

$$f(xy) = f(x)f(y) \qquad (x, y \in P)$$

holds.

If moreover, f is one-to-one, then f is called a *monomorphism*. If f is onto, then f is called an *epimorphism*. A homomorphism which is a monomorphism and an epimorphism is called an *isomorphism*. In case $P = Q$, the function f is termed to be an *endomorphism*.

Definition 1.2. Let Q be a (not necessarily unital) ring and let P be a subring of Q. A function $f\colon P \to Q$ is called a *derivation* if it is additive and also satisfies the so-called *Leibniz rule*, i.e. equation

$$f(xy) = f(x)y + xf(y) \quad (x, y \in P)\,.$$

Remark 1.1. Let Q be a ring and let P be a subring of Q. Functions $f\colon P \to Q$ fulfilling the Leibniz rule only, will be termed *Leibniz functions*.

Among derivations one can single out so-called inner derivations, similarly as in the case of automorphisms.

Definition 1.3. Let R be a ring and $b \in R$, then the mapping $\mathrm{ad}_b\colon R \to R$ defined by

$$\mathrm{ad}_b(x) = [x, b] = xb - bx \qquad (x \in R)$$

is a derivation. A derivation $f\colon R \to R$ is termed to be an *inner derivation* if there is a $b \in R$ so that $f = \mathrm{ad}_b$. We say that a derivation is an *outer derivation* if it is not inner.

An another fundamental example for derivations is the following.

Remark 1.2. Let $\mathbb{F}$ be a field, and let in the above definition $P = Q = \mathbb{F}[x]$ be the ring of polynomials with coefficients from $\mathbb{F}$. For a polynomial $p \in \mathbb{F}[x]$, $p(x) = \sum_{k=0}^{n} a_k x^k$, define the function $f\colon \mathbb{F}[x] \to \mathbb{F}[x]$ as

$$f(p) = p',$$

where $p'(x) = \sum_{k=1}^{n} k a_k x^{k-1}$ is the derivative of the polynomial p. Then the function f clearly fulfills

$$\begin{aligned} f(p+q) &= f(p) + f(q) \\ f(pq) &= pf(q) + qf(p) \end{aligned}$$

for all $p, q \in \mathbb{F}[x]$. Hence f is a derivation.

Clearly, commutative rings admit only *trivial* inner derivations. At the same time, it not so evident whether commutative rings (or fields) *do* or *do not* admit nontrivial outer derivations. To answer this problem partially, here we recall Theorem 14.2.1 from Kuczma [7].

Theorem 1.1. *Let $\mathbb{K}$ be a field of characteristic zero, let $\mathbb{F}$ be a subfield of $\mathbb{K}$, let S be an algebraic base of $\mathbb{K}$ over $\mathbb{F}$, if it exists, and let $S = \emptyset$ otherwise. Let $f\colon \mathbb{F} \to \mathbb{K}$ be a derivation. Then, for every function $u\colon S \to \mathbb{K}$, there exists a unique derivation $g\colon \mathbb{K} \to \mathbb{K}$ such that $g|_{\mathbb{F}} = f$ and $g|_S = u$.*

In [4], El Sofy introduced the notion of homo-derivations. After that several results appeared where the authors proved commutativity results for the domain of these mappings, see e. g. [1, 2, 8, 11]. It is objectionable however that there have not been made attempt to characterize or to compare these notions. One of the main purpose of this work is to clarify these problems.

Definition 1.4 (El Sofy [4])**.** Let Q be a ring and let P be a subring of Q. A function $f\colon P \to Q$ is called a *homo-derivation* if it is additive and also satisfies the equation

$$f(xy) = f(x)y + xf(y) + f(x)f(y) \quad (x, y \in P)\,.$$

We remark that there can be found some other ways of introducing homo-derivations. Here we present a further definition as it appears in [10].

Definition 1.5 (Mehdi Ebrahimi–Pajoohesh [10])**.** Let Q be a ring and let P be a subring of Q. A function $f\colon P \to Q$ is called a *homo-derivation* if it is a homomorphism and also satisfies the Leibniz rule.

Polynomials and the Levi-Cività equation

As we will see in the second section, the notion of exponential polynomials and the so-called Levi-Cività functional equation will play a distinguished role while proving our results.

In view of Theorem 10.1 of [12], if $(G,\cdot)$ is an Abelian group, then any function $f\colon G\to\mathbb{C}$ satisfying the so-called *Levi-Cività functional equation*, that is,

$$f(x\cdot y)=\sum_{i=1}^{n}g_i(x)h_i(y)\qquad(x,y\in G) \tag{1.1}$$

for some positive integer n and functions $g_i,h_i\colon G\to\mathbb{C}$ $(i=1,\dots,n)$, is an exponential polynomial of order at most n.

At the same time, in equation (1.1) not only the function f, but also the mappings $g_i,h_i\colon G\to\mathbb{C}$ $(i=1,\dots,n)$ are assumed to be unknown.

If either the functions $g_1,\dots,g_n$ or the functions $h_1,\dots,h_n$ are *linearly dependent*, then the number n appearing in equation (1.1) can be reduced and in this case the general solution of the equation can contain arbitrary functions, we shall call this case *degenerate*.

Alternatively, if the functions $h_1,\dots,h_n$ are *linearly independent*, then $g_1,\dots,g_n$ are linear combinations of the translates of f, hence they are exponential polynomials of order at most n, too. Moreover, they are built up from the same additive and exponential functions as the function f. Roughly speaking this is Theorem 10.4 of [12] which is the following.

Theorem 1.2. *Let G be an Abelian group, n be a positive integer and $f,g_i,h_i:\colon G\to\mathbb{C}$ $(i=1,\dots,n)$ be functions so that both the sets $\{g_1,\dots,g_n\}$ and $\{h_1,\dots,h_n\}$ are linearly independent. The functions $f,g_i,h_i\colon G\to\mathbb{C}$ $(i=1,\dots,n)$ form a* non-degenerate *solution of equation (1.1) if and only if*

(a) *there exist positive integers $k,n_1,\dots,n_k$ with $n_1+\dots+n_k=n$;*

(b) *there exist different nonzero complex exponentials $m_1,\dots,m_k$;*

(c) *for all $j=1,\dots,k$ there exists linearly independent sets of complex additive functions*

$$\left\{a_{j,1},\dots,a_{j,n_j-1}\right\};$$

(d) *there exist polynomials $P_j,Q_{i,j},R_{i,j}\colon\mathbb{C}^{n_j-1}\to\mathbb{C}$ for all $i=1,\dots,n$; $j=1,\dots,k$ in n_j-1 complex variables and of degree at most n_j-1;*

so that we have

$$f(x) = \sum_{j=1}^{k} P_j\left(a_{j,1}(x), \ldots, a_{j,n_j-1}(x)\right) m_j(x),$$

$$g_i(x) = \sum_{j=1}^{k} Q_{i,j}\left(a_{j,1}(x), \ldots, a_{j,n_j-1}(x)\right) m_j(x)$$

and

$$h_i(x) = \sum_{j=1}^{k} R_{i,j}\left(a_{j,1}(x), \ldots, a_{j,n_j-1}(x)\right) m_j(x)$$

for all $x \in G$ *and* $i = 1, \ldots, n$.

Let G be a groupoid and $\mathbb{F}$ be field. Given $(A_{i,j})_{i,j} \in \mathscr{M}_{n\times n}(\mathbb{F})$ and $(\Gamma_{i,j}^{(k)})_{i,j} \in \mathscr{M}_{n\times n}(\mathbb{F})$, in [9] McKiernan studied the following problems.

(1) Find all functions $h, f_i, g_i \colon G \to \mathbb{F}$ $(i = 1, \ldots, n)$ satisfying the equation

$$h(xy) = \sum_{i,j=1}^{n} A_{i,j} f_i(x) g_j(y) \qquad (x, y \in G)\,. \tag{1.2}$$

(2) Find all functions $g_i \colon G \to \mathbb{F}$ $(i = 1, \ldots, n)$ satisfying the system of equations

$$g_k(xy) = \sum_{i,j=1}^{n} \Gamma_{i,j}^{(k)} g_i(x) g_j(y) \qquad (x, y \in G, k = 1, \ldots, n) \tag{1.3}$$

The solutions are obtained by showing that the two problems are essentially equivalent, then transforming to a matrix problem and applying one of his earlier results concerning a multiplicative matrix equation. All in all, the main result of [9] is that if

(a) G is a (not necessarily commutative) semigroup,

(b) $\mathbb{F}$ is an algebraically closed field with $\operatorname{char}(\mathbb{F}) \geq (n-1)!$,

(c) $\det(A_{i,j}) \neq 0$,

(d) $f_1, \ldots, f_n$ are linearly independent,

(e) $g_1, \ldots, g_n$ are linearly independent,

then the solutions of equation (1.2) as well as (1.3) are exponential polynomials.

In what follows [12, Lemma 4.2], that is, the statement below will be utilized several times.

Theorem 1.3. *Let G be an Abelian group, $\mathbb{K}$ a field, X a $\mathbb{K}$-linear space and $\mathscr{V}$ a translation invariant linear space of X-valued functions on G. Let k_i be nonnegative integers, $n \geq 1$, $m_i \colon G \to \mathbb{K}$ different nonzero exponentials, $A_i \colon G^{n_i} \to X$ symmetric, k_i-additive functions and $q_i \colon G \to X$ polynomials of degree at most $k_i - 1$ $(i = 1, \ldots, n)$. If the function*

$$\sum_{i=1}^{n} (\operatorname{diag}(A_i) + q_i)\, m_i$$

belongs to $\mathscr{V}$, then there exist polynomials $r_i \colon G \to X$ of degree at most $k_i - 1$ such that $(\operatorname{diag}(A_i) + r_i)\, m_i$ belongs to $\mathscr{V}$ for $i = 1, \ldots, n$.

From this, with the choice $\mathscr{V} = \{0\}$ and $X = \mathbb{K}$ we get the following.

Proposition 1.4. *Let G be an Abelian group, $\mathbb{K}$ be a field and n be a positive integer. Suppose that for each $x \in G$*

$$p_1(x)m_1(x) + \cdots + p_n(x)m_n(x) = 0$$

holds, where $m_1, \ldots, m_n \colon G \to \mathbb{K}$ are different exponentials and $p_1, \ldots, p_n : : G \to \mathbb{K}$ are (generalized) polynomials. Then for all $i = 1, \ldots, n$ the polynomial p_i is identically zero.

2. Results

The functional equation of homo-derivations

As the theorem below shows, the notion of homo-derivations (in the sense of El Sofy [4]) can be characterized even *without* additivity supposition.

Theorem 2.1. *Let P and Q be rings such that P is a subring of Q and assume that ε is an arbitrary nonzero element of the center of Q. Function $h \colon P \to Q$ fulfills the functional equation*

$$h(xy) = h(x)y + xh(y) + \varepsilon h(x)h(y) \tag{2.1}$$

for all $x, y \in P$ if and only if there exists a multiplicative function $m \colon P \to Q$ such that

$$\varepsilon h(x) = m(x) - x \qquad (x \in P)\,.$$

Proof. Multiplying equation (2.1) with ε leads to

$$\varepsilon h(xy) = \varepsilon h(x)y + x\varepsilon h(y) + \varepsilon h(x)\varepsilon h(y) \qquad (x, y \in P)\,.$$

If we add xy to both sides of this equation, then

$$\varepsilon h(xy) + xy = \varepsilon h(x)y + x\varepsilon h(y) + \varepsilon h(x)\varepsilon h(y) + xy \qquad (x, y \in P)\,,$$

follows. Observe however that

$$\varepsilon h(x)y + x\varepsilon h(y) + \varepsilon h(x)\varepsilon h(y) + xy = (\varepsilon h(x) + x) \cdot (\varepsilon h(y) + y) \qquad (x, y \in P)\,,$$

yielding exactly that the mapping $m\colon P \to Q$ defined through

$$m(x) = \varepsilon h(x) + x \qquad (x \in P)$$

is multiplicative. ■

Remark 2.1. In the case the ring Q is unital, $\varepsilon = 1$ is a nonzero central element of the ring Q. Hence any solution of the above equation can be represented as

$$h(x) = m(x) - x \qquad (x \in P)\,,$$

with an appropriate multiplicative function m.

As an immediate corollary we get the following.

Corollary 2.1. *Let P be a subring of the ring Q and assume that ε is an arbitrary nonzero element of the center of Q. The additive mapping $a\colon P \to Q$ fulfills functional equation*

$$a(xy) = a(x)y + xa(y) + \varepsilon a(x)a(y)$$

for all $x, y \in P$ if and only if there exists a homomorphism $\varphi\colon P \to Q$ such that

$$\varepsilon a(x) = \varphi(x) - x \qquad (x \in P)\,.$$

In view of the above result, homo-derivations in the sense of El Sofy [4] on commutative rings can be characterized.

Corollary 2.2. *Let P be a subring of the commutative ring Q and $a\colon P \to Q$ be a homo-derivation in the sense of El Sofy* [4]. *Then and only then there exists a homomorphism $\varphi\colon P \to Q$ such that*

$$a(x) = \varphi(x) - x \qquad (x \in P)\,.$$

The proposition below considers the other notion of homo-derivations.

Proposition 2.2. *Let P be a subring of the ring Q, the function $f\colon P \to Q$ fulfills the system of equations*

$$\begin{array}{rcll} f(xy) & = & f(x)f(y) & \\ f(xy) & = & f(x)y + xf(y) & (x \in \mathbb{F}) \end{array}$$

if and only if there exists a non-zero constant $\alpha \in Q$ such that $\alpha \cdot f$ and $f \cdot \alpha$ are identically zero.

Proof. Assume that the function $f\colon P \to Q$ satisfies the above system. Then clearly, functional equation

$$f(x)f(y) = f(x)y + xf(y) \qquad (x, y \in P)$$

also holds. After some rearrangement we arrive at

$$f(x)\left[f(y) - y\right] = xf(y) \qquad (x \in P)\,.$$

Since f must be a Leibniz function,

$$f(y) - y = 0$$

cannot hold for all $y \in P$. Thus there exists $y^* \in P$ such that $f(y^*) - y^* \neq 0$, from which

$$f(x)\left[f(y^*) - y^*\right] = xf(y^*) \qquad (x \in P)\,.$$

In other words, there are constants $\alpha, \beta \in Q$ with $\alpha \neq 0$ such that

$$f(x)\alpha = x\beta \qquad (x \in P)\,.$$

Writing this back to the Leibniz equation $\beta = 0$ follows, yielding that $f \cdot \alpha$ is identically zero. Changing the role of x and y in the above argument, $\alpha \cdot f \equiv 0$ also follows. ■

Corollary 2.3. *Let P be a subring of the ring Q, a function $a\colon P \to Q$ is a homo-derivation in the sense of Mehdi Ebrahimi–Pajoohesh* [10] *if and only if there exists a nonzero constant $\alpha \in Q$ such that $\alpha \cdot a \equiv a \cdot \alpha \equiv 0$. Especially, if Q has no zero-divisors, then a has to be identically zero.*

On the functional equation $f(xy) = h(x)h(y) + xk(y) + k(x)y$

In this subsection we determine the solutions $f, h, k\colon \mathbb{F} \to \mathbb{K}$ of the functional equation

$$f(xy) = h(x)h(y) + xk(y) + k(x)y \qquad (x, y \in \mathbb{F})\,. \tag{2.2}$$

From this, the additive solutions of the same equation will follow immediately. Here we suppose that $\mathbb{K}$ *is an algebraically closed field with* $\operatorname{char}(\mathbb{K}) \neq 2$ *and* $\mathbb{F}$ *is a subfield of* $\mathbb{K}$.

Observe that equation (2.2) is a special Levi-Cività equation. Therefore according to the value $\dim \operatorname{lin}(\mathrm{id}, h, k)$, we have to distinguish several cases. Clearly,

$$\dim \operatorname{lin}(\mathrm{id}, h, k) = 3$$

means that the mappings involved in the right hand side of (2.2) are linearly independent. Thus in the degenerate cases we have $\dim \operatorname{lin}(\mathrm{id}, h, k) < 3$.

Degenerate cases

Firstly, let us assume that $\dim \operatorname{lin}(\mathrm{id}, h, k) = 1$. In this situation there exist constants $\lambda_1, \lambda_2 \in \mathbb{K}$ such that

$$h(x) = \lambda_1 x \quad \text{and} \quad k(x) = \lambda_2 x \qquad (x \in \mathbb{F}).$$

Proposition 2.3. *Let* $\lambda_1, \lambda_2 \in \mathbb{K}$ *be arbitrarily fixed. Function* $f\colon \mathbb{F} \to \mathbb{K}$ *fulfills functional equation* (2.2) *if and only if*

$$f(x) = \left(\lambda_1^2 + 2\lambda_2\right) x \qquad (x \in \mathbb{F}).$$

Proof. In case

$$h(x) = \lambda_1 x \quad \text{and} \quad k(x) = \lambda_2 x \qquad (x \in \mathbb{F}),$$

our equation reduces to

$$f(xy) = \left(\lambda_1^2 + 2\lambda_2\right) xy \qquad (x, y \in \mathbb{F}),$$

from which the results follows immediately. ■

Secondly, assume that $\dim \operatorname{lin}(\mathrm{id}, h, k) = 2$, which can happen in different ways. If $\{\mathrm{id}, h\}$ are linearly dependent, that is

$$h(x) = \lambda x \qquad (x \in \mathbb{F})$$

holds with a certain $\lambda \in \mathbb{K}$, then we have the following.

Proposition 2.4. *Let* $\lambda \in \mathbb{K}$ *be an arbitrary constant. Functions* $f, k\colon \mathbb{F} \to \mathbb{K}$ *fulfill the functional equation*

$$f(xy) = \lambda^2 xy + xk(y) + k(x)y \qquad (x, y \in \mathbb{F})$$

if and only if there exists a Leibniz function $\delta\colon \mathbb{F} \to \mathbb{K}$ *such that*

$$\begin{array}{rcl} k(x) & = & k(1)x + \delta(x) \\ f(x) & = & \left(\lambda^2 + 2k(1)\right) x + \delta(x) \end{array} \qquad (x \in \mathbb{F}).$$

Proof. Under the above assumptions, equation (2.2) with $y = 1$ yields that

$$f(x) = (\lambda^2 + k(1))x + k(x) \qquad (x \in \mathbb{F}).$$

Writing this back into our equation, we get that

$$k(xy) + (\lambda^2 + k(1))xy = xk(y) + k(x)y + \lambda^2 xy \qquad (x, y \in \mathbb{F}).$$

This means that the function $k\colon \mathbb{F} \to \mathbb{K}$ fulfills

$$k(xy) + k(1)xy = xk(y) + k(x)y \qquad (x, y \in \mathbb{F}).$$

Thus, there exists a Leibniz function $\delta\colon \mathbb{F} \to \mathbb{K}$ such that

$$\begin{array}{rcl} k(x) & = & k(1)x + \delta(x) \\ f(x) & = & \left(\lambda^2 + 2k(1)\right)x + \delta(x) \end{array} \qquad (x \in \mathbb{F}). \quad \blacksquare$$

Corollary 2.4. *Let $\lambda \in \mathbb{K}$ be an arbitrary constant. The additive functions $f, k\colon \mathbb{F} \to \mathbb{K}$ fulfill the functional equation*

$$f(xy) = \lambda^2 xy + xk(y) + k(x)y \qquad (x, y \in \mathbb{F})$$

if and only if there exists a derivation $d\colon \mathbb{F} \to \mathbb{K}$ such that

$$\begin{array}{rcl} k(x) & = & k(1)x + d(x) \\ f(x) & = & \left(\lambda^2 + 2k(1)\right)x + d(x) \end{array} \qquad (x \in \mathbb{F}).$$

Our second case is when $\{\mathrm{id}, k\}$ are linearly dependent, that is if

$$k(x) = \lambda x \qquad (x \in \mathbb{F})$$

with a certain $\lambda \in \mathbb{K}$.

Proposition 2.5. *Let $\lambda \in \mathbb{F}$ be arbitrarily fixed. Functions $f, h\colon \mathbb{F} \to \mathbb{K}$ fulfill the functional equation*

$$f(xy) = h(x)h(y) + 2\lambda xy \qquad (x, y \in \mathbb{F})$$

if and only if there exists a multiplicative function $m\colon \mathbb{F} \to \mathbb{K}$ such that

$$\begin{array}{rcl} h(x) & = & h(1)m(x) \\ f(x) & = & h(1)^2 m(x) + 2\lambda x \end{array} \qquad (x \in \mathbb{F}).$$

Proof. Define the function $\widetilde{f}\colon \mathbb{F} \to \mathbb{K}$ through

$$\widetilde{f}(x) = f(x) - 2\lambda x \qquad (x \in \mathbb{F})$$

to deduce that

$$\widetilde{f}(xy) = h(x)h(y) \qquad (x, y \in \mathbb{F}).$$

This identity with $y = 1$ implies that

$$\widetilde{f}(x) = h(1)h(x) \qquad (x \in \mathbb{F}).$$

Therefore,

A) either $h(1) = 0$ from which

$$f(x) = 2\lambda x \qquad (x \in \mathbb{F})$$

and $h \equiv 0$ follows;

B) or $h(1) \neq 0$ from which we get that

$$\frac{h(xy)}{h(1)} = \frac{h(x)}{h(1)} \cdot \frac{h(y)}{h(1)} \qquad (x, y \in \mathbb{F}).$$

All in all, there exists a multiplicative function $m\colon \mathbb{F} \to \mathbb{K}$ such that

$$h(x) = h(1)m(x) \qquad (x \in \mathbb{F}).$$

From this we also obtain that

$$f(x) = h(1)^2 m(x) + 2\lambda x \qquad (x \in \mathbb{F}). \qquad \blacksquare$$

Remark 2.2. Similarly, as previously, the additive solutions of the above equation are of the form

$$\begin{array}{rcl} h(x) & = & h(1)\varphi(x) \\ f(x) & = & h(1)^2\varphi(x) + 2\lambda x \end{array} \qquad (x \in \mathbb{F}),$$

with a certain homomorphism $\varphi\colon \mathbb{F} \to \mathbb{K}$.

Finally, the last possibility is that $\{h, k\}$ is linearly dependent. In this case there are constants $\lambda_1, \lambda_2 \in \mathbb{K}$ not vanishing simultaneously such that

$$\lambda_1 k(x) + \lambda_2 h(x) = 0$$

for all $x \in \mathbb{F}$. Again we have the following alternative.

A) Either $\lambda_2 = 0$ and then $k \equiv 0$. In this case (2.2) is

$$f(xy) = h(x)h(y) \qquad (x, y \in \mathbb{F}).$$

Using Proposition 2.5 we finally get that there exists a multiplicative function $m\colon \mathbb{F} \to \mathbb{K}$ such that

$$\begin{array}{rcl} h(x) & = & h(1)m(x) \\ f(x) & = & h(1)^2 m(x) \end{array} \qquad (x \in \mathbb{F}).$$

B) Or $\lambda_2 \neq 0$ and then there exists a constant $\lambda \in \mathbb{K}$ such that

$$h(x) = \lambda k(x) \qquad (x \in \mathbb{F}).$$

In this case equation (2.2) is of the form

$$f(xy) = k(x)y + xk(y) + \lambda^2 k(x)k(y) \qquad (x, y \in \mathbb{F}).$$

Proposition 2.6. *Let $\lambda \in \mathbb{K}$. Functions $f, k\colon \mathbb{F} \to \mathbb{K}$ fulfill the functional equation*

$$f(xy) = xk(y) + k(x)y + \lambda^2 k(x)k(y) \qquad (x, y \in \mathbb{F})$$

if and only if

A) *in case $\lambda = 0$ and there exists a Leibniz function $\delta\colon \mathbb{F} \to \mathbb{K}$ such that*

$$\begin{array}{rcl} k(x) & = & k(1)x + \delta(x) \\ f(x) & = & 2k(1)x + \delta(x) \end{array} \qquad (x \in \mathbb{F}),$$

B) *in case $\lambda \neq 0$*

(a) *either there exists a constant $\gamma \in \mathbb{K}$ such that*

$$\begin{array}{rcl} k(x) & = & \gamma x \\ f(x) & = & \left(\lambda^2\gamma^2 + 2\gamma\right) x \end{array} \qquad (x \in \mathbb{F}).$$

(b) *or there exists a multiplicative function $m\colon \mathbb{F} \to \mathbb{K}$ and a constant $\gamma \in \mathbb{K}$ such that*

$$\begin{array}{rcl} k(x) & = & -\dfrac{1}{\lambda^2}x + \gamma m(x) \\ f(x) & = & -\dfrac{1}{\lambda^2}x + \gamma^2\lambda^2 m(x) \end{array} \qquad (x \in \mathbb{F}).$$

Proof. Observe, that our equation with $y = 1$ immediately yields that

$$f(x) = (1 + \lambda^2 k(1))k(x) + k(1)x \qquad (x \in \mathbb{F}).$$

If $\lambda = 0$, then from this we get that the mapping $\tilde{k}$ defined on $\mathbb{F}$ by

$$\tilde{k}(x) = k(x) - k(1)x \qquad (x \in \mathbb{F}),$$

is a Leibniz function.

If $\lambda \neq 0$, then define the function $h\colon \mathbb{F} \to \mathbb{K}$ by

$$h(x) = x + \frac{\lambda^2}{2}k(x) \qquad (x \in \mathbb{F})$$

to derive

$$f(xy) = k(x)h(y) + h(x)k(y) \qquad (x, y \in \mathbb{F}),$$

that is the same sine-type equation as in the proof of Proposition 2.4. Similarly as there, a careful adaptation shows that alternative (a) corresponds to the case when k and h are linearly dependent, while alternative (b) corresponds to the case when k and h are linearly independent. ■

Remark 2.3. If we would like to describe the additive solutions of the above functional equation, then the mapping δ should be replaced by a derivation $d\colon \mathbb{F} \to \mathbb{K}$, and the mapping m should be replaced by a homomorphism $\varphi\colon \mathbb{F} \to \mathbb{K}$.

The non-degenerate case

In this subsection we investigate the so-called non-degenerate case. More precisely, the problem to be studied is the following. Let $\mathbb{K}$ *be an algebraically closed field with* $\operatorname{char}(\mathbb{K}) \neq 2$ *zero and* $\mathbb{F}$ *be a subfield of* $\mathbb{K}$, and $f, h, k\colon \mathbb{F} \to \mathbb{K}$ be functions so that the system $\{\mathrm{id}, h, k\}$ is *linearly independent*. In what follows we determine the solutions of the functional equation

$$f(xy) = h(x)h(y) + xk(y) + k(x)y \qquad (x, y \in \mathbb{F}).$$

Using the results of Székelyhidi [12] and McKiernan [9] delineated in the first section, we derive immediately that the solutions $f, h, k\colon \mathbb{F} \to \mathbb{K}$ of the above equation are exponential polynomials of degree at most two. Depending on this degree we have three different possibilities, see pages 89–92 of [12] where the description of the functional equation

$$f(xy) = g_1(x)h_1(y) + g_2(x)h_2(y) + g_3(x)h_3(y)$$

can be found. This obviously covers our equation with the choice

$$\begin{array}{rcll} g_1(x) = h_1(x) & = & h(x) & \\ g_2(x) & = & x & \\ h_2(x) & = & k(x) & \qquad (x \in \mathbb{F}). \\ g_3(x) & = & k(x) & \\ h_3(x) & = & x & \end{array}$$

The first possibility is that there exist different nonzero multiplicative functions $m_1, m_2, m_3\colon \mathbb{F} \to \mathbb{K}$ and constants $\alpha_i, \beta_j^{(i)}, \gamma_i^{(j)} \in \mathbb{K}$, $i, j = 1, 2, 3$ such that

$$\begin{array}{rcll} f(x) & = & \alpha_1 m_1(x) + \alpha_2 m_2(x) + \alpha_3 m_3(x) & \\ g_i(x) & = & \beta_1^{(i)} m_1(x) + \beta_2^{(i)} m_2(x) + \beta_3^{(i)} m_3(x) & \qquad (x \in \mathbb{F}). \\ h_i(x) & = & \gamma_1^{(i)} m_1(x) + \gamma_2^{(i)} m_2(x) + \gamma_3^{(i)} m_3(x) & \end{array}$$

Condition $g_1 = h_1$ implies that

$$\beta_j^{(1)} = \gamma_j^{(1)} \qquad (j = 1, 2, 3)\,.$$

Similarly, from $h_2 = g_3$ we obtain that

$$\beta_j^{(3)} = \gamma_j^{(2)} \qquad (j = 1, 2, 3)\,.$$

Finally, from

$$g_2(x) = h_3(x) = x \qquad (x \in \mathbb{F})$$

we derive that one of the multiplicative functions m_1, m_2, m_3 is the identity, say m_1 and

$$\beta_2^{(2)} = \beta_3^{(2)} = 0 \qquad \beta_1^{(2)} = 1$$
$$\gamma_2^{(3)} = \gamma_3^{(3)} = 0 \qquad \gamma_1^{(3)} = 1.$$

Using this and our functional equation, we get that for the above constants

$$\begin{pmatrix} \beta_1^{(1)} & 1 & \beta_1^{(3)} \\ \beta_2^{(1)} & 0 & \beta_2^{(3)} \\ \beta_3^{(1)} & 0 & \beta_3^{(3)} \end{pmatrix} \cdot \begin{pmatrix} \beta_1^{(1)} & \beta_2^{(1)} & \beta_3^{(1)} \\ \beta_1^{(3)} & \beta_2^{(3)} & \beta_3^{(3)} \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}$$

has to hold. Especially, $\beta_2^{(1)} \cdot \beta_3^{(1)} = 0$, which yields that the coefficient of m_2 or that of m_3 is zero. All in all, there exists a multiplicative function $m\colon \mathbb{F} \to \mathbb{K}$ and constants such that

$$\begin{array}{rcll} f(x) & = & (\beta_1^2 + 2\gamma_1)x + \beta_2^2 m(x) & \\ h(x) & = & \beta_1 x + \beta_2 m(x) & (x \in \mathbb{F})\,. \\ k(x) & = & \gamma_1 x + \gamma_2 m(x) & \end{array}$$

In this case however the functions id, h and k span a linear space of dimension at most two. Notice that we are interested in the case $\dim \operatorname{lin}(\mathrm{id}, h, k) = 3$. Therefore the above type of solutions does not appear.

The second possibility is that there exist different multiplicative functions m_1, m_2 and a logarithmic function $l\colon \mathbb{F}^\times \to \mathbb{K}$ and constants such that

$$\begin{array}{rcll} f(x) & = & (\alpha_1 l(x) + \alpha_2)\, m_1(x) + \alpha_3 m_2(x) & \\ g_i(x) & = & \left(\beta_1^{(i)} l(x) + \beta_2^{(i)}\right) m_1(x) + \beta_3 m_2(x) & (x \in \mathbb{F}, i = 1, 2, 3)\,. \\ h_i(x) & = & \left(\gamma_1^{(i)} l(x) + \gamma_2^{(i)}\right) m_1(x) + \gamma_3 m_2(x) & \end{array}$$

In our case

$$\begin{array}{rcll} g_1(x) = h_1(x) & = & h(x) & \\ g_2(x) & = & x & \\ h_2(x) & = & k(x) & (x \in \mathbb{F})\,, \\ g_3(x) & = & k(x) & \\ h_3(x) & = & x & \end{array}$$

thus either m_1 or m_2 is the identity. If we would have

$$m_2(x) = x \qquad (x \in \mathbb{F}),$$

then after comparing the coefficients

$$\begin{array}{rcll} f(x) & = & \alpha_1 x + \alpha_2 m(x) & \\ h(x) & = & \beta_1 x + \beta_2 m(x) & (x \in \mathbb{F}). \\ k(x) & = & \gamma_1 x + \gamma_2 m(x) & \end{array}$$

would follow with certain constants. Similarly as above, in this case we would have $\dim \operatorname{lin}(\mathrm{id}, h, k) \leq 2$, contrary to our assumptions.

The fact that

$$m_1(x) = x \qquad (x \in \mathbb{F}),$$

means that there exists a multiplicative function $m\colon \mathbb{F} \to \mathbb{K}$ and a logarithmic function $l\colon \mathbb{F}^\times \to \mathbb{K}$ and constants such that

$$\begin{array}{rcll} f(x) & = & (\alpha_1 l(x) + \alpha_2)\, x + \alpha_3 m(x) & \\ h(x) & = & (\beta_1 l(x) + \beta_2)\, x + \beta_3 m(x) & (x \in \mathbb{F}). \\ k(x) & = & (\gamma_1 l(x) + \gamma_2)\, x + \gamma_3 m(x) & \end{array}$$

Inserting this back into our equation the system of equations

$$\begin{array}{rcl} \beta_1 & = & 0 \\ \alpha_1 & = & \gamma_1 \\ \alpha_2 & = & \beta_2^2 + 2\gamma_2 \\ \alpha_3 & = & \beta_3^2 \end{array}$$

follow, that is,

$$\begin{array}{rcll} f(x) & = & \big(\gamma_1 l(x) + \beta_2^2 + 2\gamma_2\big)\, x + \beta_3^2 m(x) & \\ h(x) & = & \beta_2 x + \beta_3 m(x) & , \\ k(x) & = & (\gamma_1 l(x) + \gamma_2)\, x + \gamma_3 m(x) & \end{array}$$

where additionally $\gamma_3 + \beta_2\beta_3 = 0$ also has to hold. To guarantee the system $\{\mathrm{id}, h, k\}$ to be linearly independent, $\beta_3 \neq 0$ and $\gamma_1 \neq 0$ also has to be supposed.

The last possibility is that all the involved functions are exponential polynomials of degree two. Since

$$g_2(x) = h_3(x) = x \qquad (x \in \mathbb{F}),$$

the corresponding multiplicative function is the identity, that is, we have

$$\begin{aligned} f(x) &= \sum_{p,q=1}^{2} \alpha_{p,q} l_p(x) l_q(x) x + \sum_{p=1}^{2} \alpha_p l_p(x) x + \alpha x \\ h(x) &= \sum_{p,q=1}^{2} \beta_{p,q} l_p(x) l_q(x) x + \sum_{p=1}^{2} \beta_p l_p(x) x + \beta x \qquad (x \in \mathbb{F}) \\ k(x) &= \sum_{p,q=1}^{2} \gamma_{p,q} l_p(x) l_q(x) x + \sum_{p=1}^{2} \gamma_p l_p(x) x + \gamma x \end{aligned}$$

with certain constants and with linearly independent logarithmic functions $l_1, l_2 \colon \mathbb{F}^\times \to \mathbb{K}$. Substituting these representations into our equation, firstly

$$\alpha_{p,q} = \beta_{p,q} = \gamma_{p,q} = 0 \qquad (p, q \in \{1, 2\})$$

can be concluded, that is, in fact we have that

$$\begin{aligned} f(x) &= \sum_{p=1}^{2} \alpha_p l_p(x) x + \alpha x \\ h(x) &= \sum_{p=1}^{2} \beta_p l_p(x) x + \beta x \qquad (x \in \mathbb{F})\,. \\ k(x) &= \sum_{p=1}^{2} \gamma_p l_p(x) x + \gamma x \end{aligned}$$

Again, the comparison of the coefficients leads to

$$\beta_1 = 0 \qquad \beta_2 = 0 \qquad \alpha_1 = \gamma_1 \qquad \alpha_2 = \gamma_2 \qquad \alpha = \beta^2 + 2\gamma,$$

that is, there exist linearly independent logarithmic functions $l_1, l_2 \colon \mathbb{F}^\times \to \mathbb{K}$ and constants $\gamma_1, \gamma_2, \beta, \gamma \in \mathbb{K}$ such that

$$\begin{aligned} f(x) &= \gamma_1 l_1(x) x + \gamma_2 l_2(x) x + (\beta^2 + 2\gamma) x \\ h(x) &= \beta x \qquad\qquad (x \in \mathbb{F})\,. \\ k(x) &= \gamma_1 l_1(x) x + \gamma_2 l_2(x) x + \gamma x \end{aligned}$$

Observe that in this case id and h are linearly dependent, yielding that this possibility cannot occur in our situation. Summing up, the following statement holds true.

Theorem 2.7. *Let $f, h, k \colon \mathbb{F} \to \mathbb{K}$ be functions such that the system $\{\mathrm{id}, h, k\}$ is* linearly independent. *The functional equation*

$$f(xy) = h(x)h(y) + xk(y) + k(x)y \tag{2.3}$$

is fulfilled for any $x, y \in \mathbb{F}$ *if and only if there exists a multiplicative function* $m\colon \mathbb{F} \to \mathbb{K}$ *and a logarithmic function* $l\colon \mathbb{F}^\times \to \mathbb{K}$ *and constants* $\beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3 \in \mathbb{K}$ *such that*

$$\begin{aligned} f(x) &= \left(\gamma_1 l(x) + \beta_2^2 + 2\gamma_2\right) x + \beta_3^2 m(x) \\ h(x) &= \beta_2 x + \beta_3 m(x) \qquad\qquad (x \in \mathbb{F}), \\ k(x) &= (\gamma_1 l(x) + \gamma_2)\, x + \gamma_3 m(x) \end{aligned}$$

where additionally $\gamma_3 + \beta_2\beta_3 = 0$, $\beta_3 \neq 0$ *and* $\gamma_1 \neq 0$ *also have to hold.*

3. Further interpretations and open questions

The primary aim of this paper was to study (different) notions of homo-derivations on fields (with and without additivity supposition) as well as the Pexiderized version of this definition. At the same time, the results obtained here can be restated with the aid of the notion of alien functional equations. This concept was developed by J. Dhombres in the paper [3]. However, the interested reader should also consult the survey paper Ger–Sablik [5].

Let X and Y be nonempty sets and $E_1(f) = 0$ and $E_2(f) = 0$ be two functional equations for the function $f\colon X \to Y$. We say that equations E_1 and E_2 are *alien*, if any solution $f\colon X \to Y$ of the functional equation

$$E_1(f) + E_2(f) = 0$$

also solves the system

$$\begin{aligned} E_1(f) &= 0 \\ E_2(f) &= 0. \end{aligned}$$

Furthermore, equations E_1 and E_2 are *strongly alien*, if any pair $f, g\colon X \to Y$ of functions that solves

$$E_1(f) + E_1(g) = 0$$

also yields a solution for

$$\begin{aligned} E_1(f) &= 0 \\ E_2(g) &= 0. \end{aligned}$$

The following statement shows that the multiplicative Cauchy equation and the Leibniz rule equation are *not strongly alien*, this is an easy consequence of Theorem 2.7.

Proposition 3.1. *Let $h, k\colon \mathbb{F} \to \mathbb{K}$ be functions such that the system* $\{\mathrm{id}, h, k\}$ *is* linearly independent. *The functional equation*

$$h(xy) + k(xy) = h(x)h(y) + xk(y) + k(x)y \tag{3.1}$$

is fulfilled for any $x, y \in \mathbb{F}$ if and only if there exists a multiplicative function $m\colon \mathbb{F} \to \mathbb{K}$ and a logarithmic function $l\colon \mathbb{F}^\times \to \mathbb{K}$ and constants $\beta, \gamma \in \mathbb{K}$, $\beta \neq 1$, $\gamma \neq 0$ such that

$$\begin{aligned} h(x) &= \beta x + (1-\beta)m(x) \\ k(x) &= \left(\gamma l(x) + \beta - \beta^2\right) x + (\beta^2 - \beta)m(x) \end{aligned} \qquad (x \in \mathbb{F})\,.$$

As the proposition below shows, the multiplicative Cauchy equation and the Leibniz rule equation are alien, cf. Proposition 2.2 and take $\lambda = \mu$ in the corollary below.

Corollary 3.1. *Let $\lambda, \mu \in \mathbb{K}$ be arbitrarily fixed constants not vanishing simultaneously. Function $f\colon \mathbb{F} \to \mathbb{K}$ fulfills the functional equation*

$$\lambda\left[f(xy) - f(x)y - xf(y)\right] + \mu\left[f(xy) - f(x)f(y)\right] = 0 \qquad (x, y \in \mathbb{F}) \tag{3.2}$$

if and only if

(A) *either $\lambda = 0$ and f is multiplicative;*

(B) *or $\mu = 0$ and f is a Leibniz function;*

(C) *or none of them is zero and*

  (a) *f is identically zero*

  (b) *or*

$$f(x) = \frac{\mu - \lambda}{\mu} \cdot x \qquad (x \in \mathbb{F})\,.$$

Remark 3.1. Under the assumptions of the previous corollary, the additive solutions of equation (3.2) are the following.

(A) either $\lambda = 0$ and f is a homomorphism;

(B) or $\mu = 0$ and f is a derivation;

(C) or none of them is zero and

  (a) f is identically zero

  (b) or

$$f(x) = \frac{\mu - \lambda}{\mu} \cdot x \qquad (x \in \mathbb{F})\,.$$

Open problems. In this paper equation (2.1) was considered under rather mild assumptions on the domain and also on the range. At the same time, while investigating functional equation (2.2) we always assumed that the range of the involved mappings is a commutative, algebraically closed field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) \neq 2$. The reason for this is that our main tools were Theorem 1.2 and the related results of McKiernan [9]. Clearly, the above equations can be studied without these assumptions. We remark that in case of the so-called *degenerate cases* a careful adaptation of the proofs shows that the same results hold true for commutative rings (at some places we have to assume that the range does not have any zero-divisors). Therefore we can formulate the following open questions.

(A) The algebraic closedness of the field $\mathbb{K}$ essential in our results. Nevertheless, if the field $\mathbb{K}$ is not algebraically closed, then we may take $\mathrm{algcl}(\mathbb{K})$ as the extended range. Using our method, $\mathrm{algcl}(\mathbb{K})$-valued solutions can be described and every $\mathbb{K}$-valued solution belongs to the above larger solution space. How can these solutions be recognized in the larger solution space?

(B) To apply the results of Székelyhidi [12] and McKiernan [9], the assumption $\mathbb{F} \subset \mathbb{K}$ be a field is sufficient, but may not be necessary. Observe that we only need $\mathbb{F}$ to be a (commutative) subring of $\mathbb{K}$. In case the range Q is only a (commutative) ring, then what else should be supposed about Q, to get the same results?
It is worth to note that if Q has no zero divisors, then up to isomorphism there exists a unique field of fractions that we may denote by $\mathbb{K}$. In this case $\mathbb{K}$-valued functions can be considered and if it is needed we should take the algebraic closure of this field (c.f. part (A)).

(C) In case $P \subset Q$ are (commutative) rings and we consider mappings from P to Q, then are there different solutions of (2.2) then presented here?

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