Annales Univ. Sci. Budapest., Sect. Comp. 50 (2020) 283–298

LOCAL LIMIT THEOREMS FOR POISSON'S BINOMIAL IN THE CASE OF INFINITE LIMITING EXPECTATION

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Communicated by Oleg Klesov

(Received January 29, 2020; accepted June 10, 2020)

Abstract. Let $V_n = X_{1,n}+X_{2,n}+\dots+X_{n,n}$ where the $X_{i,n}$'s are Bernoulli random variables which take the value 1 with probability $b(i;n)$. Let $\lambda_n = \sum_{i=1}^{n} b(i;n)$, $\lambda = \lim_{n\to\infty} \lambda_n$, and $m_n = \max_{1\le i\le n} b(i;n)$. We derive asymptotic results for $P(V_n = k)$ that hold without assuming that $\lambda < +\infty$ or $m_n \to 0$. Also, we do not assume k to be fixed, but instead, our results hold uniformly for all k which satisfy particular growth conditions with respect to n. These results extend known Poisson local limit theorems to the case when $\lambda = +\infty$. While our results apply to triangular arrays, without the assumption that $m_n \to 0$ they continue to hold for sums of Bernoulli random variables. In this setting, our growth conditions cover a range of values for k not centered at λ_n, thus complementing known local limit theorems based on approximation by the normal distribution. In addition, we show that our local limit theorems apply to a scheme of dependent random variables introduced in the work of B.A. Sevast'yanov.

Key words and phrases: Local limit teorem, Poisson distribution, infinite expectation, weak convergence.
2010 Mathematics Subject Classification: 60F05, 62E20, 62E15.
https://doi.org/10.71352/ac.50.283

1. Introduction

1.1. Preliminaries

In this paper we consider triangular arrays of random variables $\{X_{i,n}, 1 \leq \leq i \leq n; n \in \mathbb{N}\}$ such that, for every n, the $X_{i,n}$, $1 \leq i \leq n$, are Bernoulli random variables. If for every n, the variables $X_{i,n}$, $1 \leq i \leq n$, are independent Bernoulli random variables, we say that the array is row-wise independent.

Given a triangular array $\{X_{i,n}, 1 \leq i \leq n; n \in \mathbb{N}\}$, we let $P(X_{i,n} = 1) = = b(i;n)$,

$$V_n = X_{1,n} + X_{2,n} + \cdots + X_{n,n} \tag{1.1}$$

and

$$b(i_1, i_2, ..., i_k; n) = P(X_{i_1,n} = X_{i_2,n} = \cdots = X_{i_k,n} = 1).$$

We denote

$$\lambda_n = \sum_{i=1}^{n} b(i;n), \qquad \lambda = \lim_{n\to\infty} \lambda_n,$$

$$\alpha_n = \sum_{i=1}^{n} \ln\left(1 - b(i;n)\right), \qquad \beta_n = \sum_{i=1}^{n} \frac{b(i;n)}{1 - b(i;n)}, \qquad m_n = \max_{1\leq i\leq n} b(i;n).$$

In this paper we derive Poisson-type local limit theorems for $P(V_n = k)$ that hold when the $X_{i,n}$'s are assumed to be independent Bernoulli random variables and $\lambda = +\infty$. In particular we give sufficient conditions under which

$$P(V_n = k) \sim P(V_n = 0)\,\frac{\lambda_n^k}{k!}$$

or

$$P(V_n = k) \sim P(V_n = 0)\,\frac{\beta_n^k}{k!}.$$

Under additional assumptions we also prove exact Poisson asymptotic behavior

$$P\left(V_n = k\right) \sim e^{-\lambda_n}\,\frac{\lambda_n^k}{k!}.$$

While our results apply to triangular arrays, when no assumption on $m_n \to 0$ is made, they continue to hold for sums of Bernoulli random variables. Also, we do not assume k to be fixed, but instead, our results hold uniformly for all k which satisfy particular growth conditions with respect to n. These growth conditions are valid on intervals that are not contained in those for which (1.2) or (1.3) hold, thus complementing known local limit theorems based on approximation by the normal distribution and providing new results in the setting of sums of Bernoulli random variables.

In addition, we show that our local limit theorems apply to a scheme of dependent random variables introduced in the work of B.A. Sevast'yanov [18], in which local limit theorems were derived under assumptions $\lambda < +\infty$ and $m_n \to 0$.

In 1.2 we provide the relevant background and motivation for this work. Section 2 contains the theorems and proofs.

1.2. The setting and background

The problem of approximating the limiting local behavior of the distribution of V_n, known as Poisson-binomial or Poisson's Binomial distribution, has a long history in probability. This problem has been studied in two main directions:

$$(a) \quad \lambda = +\infty \qquad \text{with a Gaussian limit}$$

$$(b) \quad 0 < \lambda \leq +\infty \qquad \text{with a Poisson limit.}$$

The first local limit result for case (a) is the well known De Moivre–Laplace Theorem (1795), see P. Gorrochurn [9] for a historical account. This theorem states that if $X_{i,n} = X_i$, and $X_1, X_2, ..., X_n$ are independent and identically distributed Bernoulli random variables, with $0 < b(i;n) = p < 1$, then as $n \to +\infty$,

$$\sup_{k:|k-np|\leq\varphi(n)} \left| \frac{P\left(V_n = k\right)}{\frac{1}{\sqrt{2\,\pi np(1-p)}} e^{-(k-np)^2/(2np(1-p))}} - 1 \right| \longrightarrow 0 \tag{1.2}$$

for $\varphi(n) = o(np(1-p))^{2/3}$. For a proof see A.N. Shiryaev [19], for example.

Several extensions of (1.2) without the assumption that the X_i's are identically distributed have been proposed. For example, let $b(i;n) = p_i, 1 - p_i = q_i$, and $B_n^2 = Var(V_n)$. Under the assumption that $0 < p_i < 1$, M.M. Mamatov [15] proved the existence of an absolute constant C such that

$$\left| B_n P\left(V_n = k\right) - (\sqrt{2\pi})^{-1}\, e^{-\frac{(k-\lambda_n)^2}{2B_n^2}} \right| < C\, \frac{\sum_{i=1}^n p_i q_i \left(p_i^2 + q_i^2\right)}{B_n^3}. \tag{1.3}$$

Further extensions which cover arbitrary lattice distributions can be found in B.V. Gnedenko [8], R. Giuliano-Antonini, M. Weber [7] and V.V. Petrov [16], for example. In such results, however, as in (1.3), the differences of the probabilities are considered instead of the quotients.

It is important to note that the asymptotic ratio of probabilities cannot be recovered from their asymptotic differences when each term in the difference goes to zero. In fact, if one is interested in ratios, and the random variables are i.i.d. Bernoulli, (1.3) is not an improvement over (1.2). By using the notation in (1.2), expression (1.3) reduces to

$$(1.4) \qquad e^{-\frac{(k-np)^2}{2np(1-p)}} \left| \frac{P\left(V_n = k\right)}{\frac{1}{\sqrt{2\pi np(1-p)}} e^{-(k-np)^2/(2np(1-p))}} - 1 \right| < \frac{C'}{\sqrt{np(1-p)}},$$

where $C' \geq C/4$. It follows from (1.4) that if

$$|k - np| > \sqrt{np(1-p)}\,\sqrt{\ln(np(1-p))}$$

the factor in front of the absolute value in (1.4) is eventually less than the upper bound, thus giving no useful information on the size of the ratio inside the absolute value. Hence (1.3) does not imply (1.2).

Note that in general the estimate in (1.4) cannot be improved (see V.V. Petrov [16], p. 197).

Case (b), concerning Poisson limits, was first investigated under the assumption that the $X_{i,n}$'s are independent and $\lambda < +\infty$. In this setting it is well known that V_n converges pointwise to a Poisson random variable with parameter λ if and only if

$$(i^a) \lim_{n\to+\infty} m_n = 0 \quad \text{or} \quad (i^b) \lim_{n\to+\infty} \sum_{i=1}^{n} b^2(i;n) = 0,$$

see [23].

In general, condition (i^b) implies condition (i^a); in the case that $\lambda < +\infty$, conditions (i^a) and (i^b) are actually equivalent.

Pointwise convergence of V_n to a Poisson random variable T as well as local limit theorems can be obtained by showing that the variation distance goes to 0 as n goes to ∞. The variation distance, D, is given by

$$D = \sup_{k\geq 0} \left|P(V_n \leq k) - P(T \leq k)\right|.$$

In this setting, one of the first results for pointwise convergence can be found in R. Von Mises [22]. Subsequently, various methods were employed to improve rates of convergence, or equivalently, to find tighter upper bounds for D, see for example J.V. Prohorov [17], L. Le Cam [14], W. Vervatt [12], [21], G. Simons, N.L. Johnson [20], and Y.H. Wang [23].

Poisson approximation to V_n in the case when the $X_{i,n}$'s are dependent has also been the object of much investigation and several approaches have been proposed; see for example J. L. Hodges, Jr., L. Le Cam [10], B. Freedman [5], L.H.Y. Chen [3], A. D. Barbour, L. Holst, S. Janson [2], and R. Arratia, L. Goldstein, L. Gordon (1990) [1].

The case of approximating $P(V_n = k)$ by a Poisson random variable T_n with parameter λ_n, when $\lim_{n\to+\infty} \lambda_n = +\infty$, has been considered by a smaller number of researchers; it has generated very interesting results.

P. Deheuvels, D. Pfeifer [4], show that under condition (i^a),

$$D \sim \frac{1}{\sqrt{2\pi e}} \frac{\sum_{i=1}^{n} b^2(i;n)}{\lambda_n}. \tag{1.5}$$

Much broader work has been done by H-K Hwang, Kowalski and Nikeghbali. H-K Hwang derived a unified scheme for Poisson approximation to discrete random variables. These results are based on the assumption that the generating function of the random variables satisfies certain asymptotic regularity conditions, see H-K Wang [11]. However these results are approximation theorems, not limit theorems.

Kowalski and Nikeghbali opened a new direction in the theory of convergence of random variables by defining a new type of convergence, called mod Poisson convergence [13]. Limited to our setting however, the general asymptotic estimates their theory provides are comparable to (1.5), which again do not translate into meaningful bounds for the ratio of probabilities for the range of values we consider in this work.

2. Statement of theorems and proofs

2.1. Independent case

We begin by considering the case of independent random variables and obtain a local limit theorem that holds for $\lambda \le +\infty$.

Theorem 2.1. *Let the array* $\{X_{i,n}, 1 \le i \le n; n \in \mathbb{N}\}$ *be row-wise independent, and suppose* $0 < \lambda \le +\infty$. *If*

$$\lim_{n\to+\infty} m_n = 0, \tag{A1}$$

then for any function $\phi(n)$ *such that* $\phi(n)\, m_n \to 0$ *as* $n \to +\infty$,

$$\sup_{\{k:\, k^2 \le \phi(n)\}} \left| \frac{P(V_n = k)}{P(V_n = 0)\lambda_n^k / k!} - 1 \right| \longrightarrow 0 \qquad \text{as } n \to +\infty.$$

Proof. Suppose $k \geq 1$, and let $\phi(n)$ be such that $\phi(n)\, m_n \to 0$ as $n \to +\infty$. Then

$$P\left(V_n = k\right) = \sum_{\substack{B \subset \{1,2,\cdots,n\} \\ |B|=k}} \Big(\prod_{i \in B} b(i;n) \prod_{j \in B^c} (1 - b(j;n)) \Big) \geq \tag{2.1}$$

$$\geq P\left(V_n = 0\right) \sum_{\substack{B \subset \{1,2,\cdots,n\} \\ |B|=k}} \Big(\prod_{i \in B} b(i;n) \Big).$$

We rewrite the sum in the above lower bound as follows,

$$\begin{aligned} \sum_{\substack{B \subset \{1,2,\cdots,n\} \\ |B|=k}} \Big(\prod_{i \in B} b(i;n) \Big) &= \sum_{\substack{A \subset \{1,2,\cdots,n\} \\ |A|=k-1}} \Big(\prod_{i \in A} b(i;n) \Big) \frac{\sum_{j \in A^c} b(j;n)}{k} = \\ &= \sum_{\substack{A \subset \{1,2,\cdots,n\} \\ |A|=k-1}} \Big(\prod_{i \in A} b(i;n) \Big) \frac{\lambda_n - \sum_{j \in A} b(j;n)}{k} \geq \qquad (2.2) \\ &\geq \frac{(\lambda_n - (k-1) m_n)}{k} \sum_{\substack{A \subset \{1,2,\cdots,n\} \\ |A|=k-1}} \Big(\prod_{i \in A} b(i;n) \Big). \end{aligned}$$

By repeating similar calculations, one obtains

$$\sum_{\substack{B \subset \{1,2,\cdots,n\} \\ |B|=k}} \Big(\prod_{i \in B} b(i;n) \Big) \geq \frac{\prod_{j=0}^{k-1} (\lambda_n - j m_n)}{k!},$$

and

$$\begin{aligned} P\left(V_n = k\right) &\geq P\left(V_n = 0\right) \frac{\prod_{j=0}^{k-1} (\lambda_n - j m_n)}{k!} \geq P\left(V_n = 0\right) \frac{(\lambda_n - k m_n)^k}{k!} = \\ &= P\left(V_n = 0\right) \frac{\lambda_n^k}{k!} \left(1 - \frac{k m_n}{\lambda_n}\right)^k = P\left(V_n = 0\right) \frac{\lambda_n^k}{k!} \left(1 - \frac{\frac{k^2 m_n}{\lambda_n}}{k}\right)^k \geq \\ &\geq P\left(V_n = 0\right) \frac{\lambda_n^k}{k!} \left(1 - \frac{k^2 m_n}{\lambda_n}\right) \qquad (2.3) \end{aligned}$$

for sufficiently large n, since $\left(1 - \dfrac{x}{k}\right)^k$ is increasing in k if $0 < x < 1$.

In order to derive an upper bound for $P(V_n = k)$,

$$\begin{aligned}
P\left(V_n = k\right) &= \sum_{\substack{B \subset \{1,2,\cdots,n\} \\ |B|=k}} \Big(\prod_{i \in B} b(i;n) \prod_{j \in B^c} (1 - b(j;n)) \Big) = \\
&= \sum_{\substack{B \subset \{1,2,\cdots,n\} \\ |B|=k}} \Big(\prod_{i \in B} \frac{b(i;n)}{1 - b(i;n)} \prod_{j=1}^{n} (1 - b(j;n)) \Big) \leq \\
&\leq P(V_n = 0) \sum_{\substack{B \subset \{1,2,\cdots,n\} \\ |B|=k}} \Big(\prod_{i \in B} \frac{b(i;n)}{1 - m_n} \Big) = \\
&= P(V_n = 0) \frac{1}{(1 - m_n)^k} \sum_{\substack{B \subset \{1,2,\cdots,n\} \\ |B|=k}} \Big(\prod_{i \in B} b(i;n) \Big).
\end{aligned}$$

From (2.2),

$$\sum_{\substack{B \subset \{1,2,\cdots,n\} \\ |B|=k}} \Big(\prod_{i \in B} b(i;n) \Big) \leq \frac{\lambda_n}{k} \sum_{\substack{A \subset \{1,2,\cdots,n\} \\ |A|=k-1}} \Big(\prod_{i \in A} b(i;n) \Big),$$

and by repeating similar calculations,

$$\sum_{\substack{A \subset \{1,2,\cdots,n\} \\ |A|=k}} \Big(\prod_{i \in A} b(i;n) \Big) \leq \frac{\lambda_n^k}{k!}.$$

Since for $0 \leq x < 1$

$$\frac{1}{(1 - \frac{x}{k})^k} \leq e^x \leq 1 + \frac{x}{1 - x},$$

the above calculation gives, for $x = k\, m_n$,

$$P\left(V_n = k\right) \leq P(V_n = 0) \frac{\lambda_n^k}{k!} \left(1 + \frac{k m_n}{1 - k m_n} \right)$$

if n is sufficiently large. We now have

$$-\epsilon_1(k, m_n) \leq \frac{P\left(V_n = k\right)}{P(V_n = 0)\lambda_n^k / k!} - 1 \leq \epsilon_2(k, m_n), \tag{2.4}$$

where $\epsilon_1(k, m_n) = \dfrac{k^2 m_n}{\lambda_n}$ and $\epsilon_2(k, m_n) = \dfrac{k m_n}{1 - k m_n}$. Thus, for $i = 1, 2$, as $n \to +\infty$,

$$\sup_{\{k\colon k^2 \leq \phi(n)\}} \epsilon_i(k, m_n) \to 0.$$

This, and the validity of the result in the case $k = 0$ completes the proof. ■

Remark 2.1. Theorem 2.1 holds without any assumption on the finiteness of λ. Estimates for m_n and simple additional information about λ_n, e.g. $\lambda_n \geq c > 0$ for all n, are sufficient to obtain accurate asymptotics for V_n via (2.4).

If $\lambda < +\infty$, then

$$\lim_{n\to+\infty} P(V_n = 0) = \lim_{n\to+\infty} e^{\alpha_n} = e^{-\lambda},$$

and thus, Theorem 2.1 implies a well known convergence result which we state as a corollary.

Corollary 2.1. *Let the array $\{X_{i,n}, 1 \leq i \leq n;\, n \in \mathbb{N}\}$ be row-wise independent. If the following hold,*

(A1) $\lim_{n\to+\infty} m_n = 0,$

(A2) $\lim_{n\to+\infty} \lambda_n = \lambda < +\infty,$

then for any function $\phi(n)$ such that $\phi(n) m_n \to 0$ as $n \to +\infty$,

$$\sup_{\{k:\, k^2 \leq \phi(n)\}} \left| \frac{P\left(V_n = k\right)}{e^{-\lambda}\lambda^k/k!} - 1 \right| \longrightarrow 0 \qquad \text{as } n \to +\infty.$$

By assuming $\lambda = +\infty$, we obtain a local limit theorem replacing the condition $m_n \to 0$ by the weaker assumption that m_n is bounded away from 1.

Theorem 2.2. *Let the array $\{X_{i,n}, 1 \leq i \leq n;\, n \in \mathbb{N}\}$ be row-wise independent, and let $m_n < \beta < 1$ for all n. If*

(A3) $$\lim_{n\to+\infty} \lambda_n = +\infty,$$

then for any function $\phi(n)$ such that $\frac{\phi(n)}{\lambda_n} \to 0$ as $n \to +\infty$,

$$\sup_{\{k:\, k^2 \leq \phi(n)\}} \left| \frac{P\left(V_n = k\right)}{P(V_n = 0)\beta_n^k/k!} - 1 \right| \longrightarrow 0 \qquad \text{as } n \to +\infty.$$

Proof.

$$P\left(V_n = k\right) = \sum_{\substack{B\subset\{1,2,\cdots,n\}\\ |B|=k}} \left(\prod_{i\in B} b(i;n) \prod_{j\in B^c} (1 - b(j;n)) \right) =$$

$$= \sum_{\substack{B\subset\{1,2,\cdots,n\}\\ |B|=k}} \left(\prod_{i\in B} \frac{b(i;n)}{1 - b(i;n)} \prod_{j=1}^{n} (1 - b(j;n)) \right) =$$

$$= P(V_n = 0) \sum_{\substack{B \subset \{1,2,\cdots,n\} \\ |B|=k}} \Big(\prod_{i \in B} \frac{b(i;n)}{1 - b(i;n)} \Big) =$$

$$= P(V_n = 0) \sum_{\substack{A \subset \{1,2,\cdots,n\} \\ |A|=k-1}} \Big(\prod_{i \in A} \frac{b(i;n)}{1 - b(i;n)} \Big) \frac{\sum_{j \in A^c} \frac{b(j;n)}{1-b(j;n)}}{k} =$$

$$= P(V_n = 0) \sum_{\substack{A \subset \{1,2,\cdots,n\} \\ |A|=k-1}} \Big(\prod_{i \in A} \frac{b(i;n)}{1 - b(i;n)} \Big) \frac{\beta_n - \sum_{j \in A} \frac{b(j;n)}{1-b(j;n)}}{k}.$$

To obtain a lower bound, we bound the above sums as follows,

$$\sum_{\substack{A \subset \{1,2,\cdots,n\} \\ |A|=k-1}} \Big(\prod_{i \in A} \frac{b(i;n)}{1 - b(i;n)} \Big) \frac{\beta_n - \sum_{j \in A} \frac{b(j;n)}{1-b(j;n)}}{k} \geq$$

$$\geq \frac{1}{k} \left(\beta_n - \frac{(k-1)\beta}{(1-\beta)} \right) \sum_{\substack{A \subset \{1,2,\cdots,n\} \\ |A|=k-1}} \Big(\prod_{i \in A} \frac{b(i;n)}{1 - b(i;n)} \Big) =$$

$$= \frac{\beta_n}{k} \left(1 - \frac{(k-1)\beta}{\beta_n(1-\beta)} \right) \sum_{\substack{A \subset \{1,2,\cdots,n\} \\ |A|=k-1}} \Big(\prod_{i \in A} \frac{b(i;n)}{1 - b(i;n)} \Big) \geq$$

$$\geq \frac{\beta_n}{k} \left(1 - \frac{(k-1)\beta}{\lambda_n(1-\beta)} \right) \sum_{\substack{A \subset \{1,2,\cdots,n\} \\ |A|=k-1}} \Big(\prod_{i \in A} \frac{b(i;n)}{1 - b(i;n)} \Big).$$

Following similar calculations as those leading up to (2.3), we obtain

$$\sum_{\substack{B \subset \{1,2,\cdots,n\} \\ |B|=k}} \Big(\prod_{i \in B} \frac{b(i;n)}{1 - b(i;n)} \Big) \geq \frac{\beta_n^k}{k!} \left(1 - \frac{k\,\beta}{\lambda_n\,(1-\beta)} \right)^k.$$

Hence, by exploiting again the increasing property of $(1 - \frac{x}{k})^k$ for $0 < x < 1$ as a function of k,

$$P(V_n = k) \geq P(V_n = 0) \frac{\beta_n^k}{k!} \left(1 - \frac{\frac{k^2\beta}{\lambda_n(1-\beta)}}{k} \right)^k \geq$$

$$\geq P(V_n = 0) \frac{\beta_n^k}{k!} \left(1 - \frac{k^2\beta}{\lambda_n(1-\beta)} \right).$$

To obtain an upper bound,

$$P(V_n = 0) \sum_{\substack{A\subset\{1,2,\cdots,n\} \\ |A|=k-1}} \Big(\prod_{i\in A} \frac{b(i;n)}{1-b(i;n)}\Big) \frac{\beta_n - \sum_{j\in A} \frac{b(j;n)}{1-b(j;n)}}{k} \leq$$

$$\leq P(V_n = 0)\, \frac{\beta_n}{k} \sum_{\substack{A\subset\{1,2,\cdots,n\} \\ |A|=k-1}} \Big(\prod_{i\in A} \frac{b(i;n)}{1-b(i;n)}\Big).$$

So,

$$P(V_n = k) \leq P(V_n = 0)\, \frac{\beta_n^k}{k!}.$$

We now have

$$-\epsilon(k,\lambda_n) \leq \frac{P\,(V_n = k)}{P(V_n = 0)\beta_n^k/k!} - 1 \leq 0,$$

where

$$\epsilon(k,\lambda_n) = \frac{k^2\beta}{\lambda_n(1-\beta)}.$$

For any given $\phi(n)$ such that $\phi(n)/\lambda_n \longrightarrow 0$ as $n \to +\infty$,

$$\sup_{\{k:\, k^2\leq\phi(n)\}} \epsilon(k,\lambda_n) \to 0 \qquad \text{as } n \to +\infty.$$

This and the validity of our result in the case $k = 0$ completes the proof. ■

Remark 2.2. Since the only assumption on the behavior of m_n is to be bounded away from 1, Theorem 2.2 gives a new local limit theorem for arbitrary sums of independent Bernoulli random variables with $\lambda_n \to \infty$. Note that the range of values of k for which Theorem 2.2 holds differs from the range of values in (1.2) and (1.3). Hence our result provides new limiting results in the case of sums of Bernoulli random variables.

We single out an interesting special case of Theorems 2.1 and 2.2, where exact Poisson asymptotic behavior is obtained.

Theorem 2.3. *Let the array* $\{X_{i,n}, 1 \leq i \leq n;\, n \in \mathbb{N}\}$ *be row-wise independent. If*

(A3) $\lim_{n\to+\infty} \lambda_n = +\infty,$

(A4) $\lim_{n\to\infty} \sum_{i=1}^{n} b^2(i;n) = 0,$

then $m_n \to 0$, *and for every function* $\phi(n)$ *such that* $\phi(n)m_n \to 0$ *as* $n \to +\infty$,

$$\sup_{\{k:\, k^2 \le \phi(n)\}} \left| \frac{P\left(V_n = k\right)}{e^{-\lambda_n}\lambda_n^k/k!} - 1 \right| \longrightarrow 0 \qquad \text{as } n \to +\infty.$$

Proof. $(A4)$ and the inequality

$$\sum_{i=1}^{n} b^2(i;n) \ge m_n^2$$

give that $\lim_{n\to\infty} m_n = 0$, and therefore in our proof we can assume that for all n, and for i such that $1 \le i \le n$, $0 \le b(i;n) < 1/2$.

We first consider the case $k = 0$. By using the Taylor series expansion of $\ln(1-x)$, one obtains the following bounds for $0 < x < 1/2$,

$$-x - x^2 \le \ln(1-x) \le -x. \tag{2.5}$$

Using (2.5) with $x = b(i;n)$, one gets

$$\exp(-\sum_{i=1}^{n} b^2(i;n)) \le \frac{P\left(V_n = 0\right)}{e^{-\lambda_n}} \le \frac{\exp(-\sum_{i=1}^{n} b(i;n))}{e^{-\lambda_n}} = 1.$$

So we have that

$$\exp(-\sum_{i=1}^{n} b^2(i;n)) \le \frac{P(V_n = 0)}{e^{-\lambda_n}} \le 1. \tag{2.6}$$

It follows from $(A4)$ that both bounds go to 1 as n goes to infinity, and thus, the convergence holds for the case $k = 0$.

Since $m_n \to 0$, for $k \ge 1$, the proof of Theorem 2.1 applies, and (2.4) and (2.6) give

$$1 - \epsilon_1(k, m_n) \le \frac{P(V_n = k)}{e^{-\lambda_n}\frac{\lambda_n^k}{k!}} \le \exp(\sum_{i=1}^{n} b^2(i;n))\left(1 + \epsilon_2(k, m_n)\right).$$

The result follows from the above inequalities. ∎

2.2. Dependent case

In this section we extend our previous results to arrays $\{\tilde{X}_{i,n}, 1 \le i \le$ $\le n;\, n \in \mathbb{N}\}$ where for every n, $\tilde{X}_{i,n}$, $1 \le i \le n$, are dependent Bernoulli random variables which satisfy a scheme described by B.A. Sevast'yanov [18].

We use the notation introduced in Section 1.1 if the random variables are independent and modify this notation by adding ' ˜ ' when the the random variables are dependent. Let $S_{0,n} = \tilde{S}_{0,n} = 1$, and for $1 \le k \le n$ let

$$S_{k,n} = \sum_{i_1,\cdots,i_k} b(i_1;n)b(i_2;n)\cdots b(i_k;n)$$

and

$$\tilde{S}_{k,n} = \sum_{i_1,\cdots,i_k} \tilde{b}(i_1,i_2,...,i_k;n),$$

where each of the above sums is over all collections of mutually distinct indices $1 \le i_1 < i_2 < \cdots < i_k \le n$.

We consider *rare sets*, $I_k(n)$, to be particular collections of k mutually distinct indices. To simplify the notation, in what follows by $\sum_{I_k(n)}$ and $\sum_{I_k(n)^c}$ we denote summations over all collections of k indices in $I_k(n)$ and $I_k(n)^c$, respectively.

Theorem 2.4. *Let $\{\tilde{X}_{i,n}, 1 \le i \le n; n \in \mathbb{N}\}$ be a triangular array and $\{X_{i,n}, 1 \le i \le n; n \in \mathbb{N}\}$ a row-wise independent triangular array. If the following assumptions are satisfied:*

$$(B1) \qquad \lim_{n\to+\infty} \frac{\tilde{b}(i_1,i_2,...,i_k;n)}{b(i_1;n)b(i_2;n)\cdots b(i_k;n)} = 1 \quad \text{for all } (i_1,i_2,...,i_k) \in I_k^c(n)$$

$$(B2) \qquad \lim_{n\to+\infty} \frac{\tilde{S}_{k,n}}{\sum\limits_{I_k(n)^c} \tilde{b}(i_1,i_2,...,i_k;n)} = 1$$

$$(B3) \qquad \lim_{n\to+\infty} \frac{S_{k,n}}{\sum\limits_{I_k(n)^c} b(i_1;n)b(i_2;n)\cdots b(i_k;n)} = 1,$$

where the limits $(B1)$, $(B2)$, and $(B3)$ are uniform in k and in $(i_1,i_2,\cdots,i_k) \in$ $\in I_k^c(n)$, then

$$(2.7) \qquad \lim_{n\to+\infty} \frac{P(\tilde{V}_n = k)}{P(V_n = k)} = 1$$

uniformly in k. Moreover Theorems 2.1, 2.2, and 2.3 apply with V_n replaced by $\tilde{V}_n$.

Remark 2.3. There are some differences between the assumptions of Theorem 2.4 and those of Theorem 1 in B.A. Sevast'yanov [18]. In our theorem we require

$(B1)-(B3)$ to hold uniformly in k, which is not required in Theorem 1 of [18]. Since the proof in [18] is based on the method of moments, finiteness is a crucial assumption. By requiring uniformity in k, we are able to extend Sevast'yanov's result to $\lambda=+\infty$. Moreover, since we do not assume that $m_n\to 0$, Theorem 2.4 also applies to sums of dependent Bernoulli random variables and not only to triangular arrays. In the case $0<\lambda<+\infty$, conditions $(B1)-(B3)$ of Theorem 2.4 are equivalent to conditions (2) and (3) of Theorem 1 in [18]. In the setting where λ is not assumed to be finite, our formulation is preferable since when $\lambda=+\infty$, conditions $(B1)-(B3)$ can be satisfied even without assuming

$$\sum_{I_k(n)} b(i_1;n)b(i_2;n)\cdots b(i_k;n)\longrightarrow 0$$

or

$$\sum_{I_k(n)} \tilde{b}(i_1,i_2,...,i_k;n)\longrightarrow 0$$

as $n\to+\infty$.

Proof. We use inclusion-exclusion, see J. Galambos, I. Simonelli [6], to express

$$P(\tilde{V}_n=k)=\sum_{l=0}^{n}(-1)^l\binom{k+l}{k}\tilde{S}_{k+l,n}. \tag{2.8}$$

We will compare this to

$$P(V_n=k)=\sum_{l=0}^{n}(-1)^l\binom{k+l}{k}S_{k+l,n}. \tag{2.9}$$

We begin by rewriting

$$\begin{aligned}P(\tilde{V}_n=k)&=\sum_{l=0}^{n}(-1)^l\binom{k+l}{k}\tilde{S}_{k+l,n}=\\&=\sum_{l=0}^{n}(-1)^l\binom{k+l}{k}\left(\sum_{I^c_{k+l}(n)}\tilde{b}(i_1,i_2,...,i_{k+l};n)+\sum_{I_{k+l}(n)}\tilde{b}(i_1,i_2,...,i_{k+l};n)\right)=\\&=\sum_{l=0}^{n}(-1)^l\binom{k+l}{k}\sum_{I^c_{k+l}(n)}\tilde{b}(i_1,i_2,...,i_{k+l};n)\left(1+\frac{\sum_{I_{k+l}(n)}\tilde{b}(i_1,i_2,...,i_{k+l};n)}{\sum_{I^c_{k+l}(n)}\tilde{b}(i_1,i_2,...,i_{k+l};n)}\right).\end{aligned}$$

Since we can write

$$\sum_{I^c_{k+l}(n)} \tilde{b}(i_1, i_2, ..., i_{k+l}; n) =$$

$$= \sum_{I^c_{k+l}(n)} \left(\frac{\tilde{b}(i_1, i_2, ..., i_{k+l}; n)}{b(i_1; n)b(i_2; n)\cdots b(i_{k+l}; n)} \cdot b(i_1; n)b(i_2; n)\cdots b(i_{k+l}; n) \right),$$

it follows from $(B1)$ that for $\epsilon > 0$ there exists n_0 such that for $n \geq n_0$ and all $l + k$,

$$(1 - \frac{\epsilon}{3}) \sum_{I^c_{k+l}(n)} b(i_1; n)b(i_2; n)\cdots b(i_{k+l}; n) \leq$$

$$\leq \sum_{I^c_{k+l}(n)} \tilde{b}(i_1, i_2, ..., i_{k+l}; n) \leq$$

$$\leq (1 + \frac{\epsilon}{3}) \sum_{I^c_{k+l}(n)} b(i_1; n)b(i_2; n)\cdots b(i_{k+l}; n).$$

From $(B3)$, it follows that there exists an n_1 such that for $n \geq n_1$ and all $l + k$,

$$(1 - \frac{\epsilon}{3}) \leq \frac{\sum_{I^c_{k+l}(n)} b(i_1; n)b(i_2; n)\cdots b(i_{k+l}; n)}{S_{k+l,n}} \leq (1 + \frac{\epsilon}{3}).$$

Lastly, since

$$\frac{\tilde{S}_{k,n}}{\sum_{I^c_{k+l}(n)} \tilde{b}(i_1, i_2, ..., i_{k+l}; n)} = 1 + \frac{\sum_{I_{k+l}(n)} \tilde{b}(i_1, i_2, ..., i_{k+l}; n)}{\sum_{I^c_{k+l}(n)} \tilde{b}(i_1, i_2, ..., i_{k+l}; n)},$$

it follows from $(B2)$ that there exists n_2 such that for $n \geq n_2$ and all $l + k$,

$$1 \leq 1 + \frac{\sum_{I_{k+l}(n)} \tilde{b}(i_1, i_2, ..., i_{k+l}; n)}{\sum_{I^c_{k+l}(n)} \tilde{b}(i_1, i_2, ..., i_{k+l}; n)} \leq (1 + \frac{\epsilon}{3}).$$

Let $N = \max\{n_0, n_1, n_2\}$. Then for $n \geq N$,

$$(1 - \frac{\epsilon}{3})^2 \sum_{l=0}^{n} (-1)^l \binom{k+l}{k} S_{k+l,n} \leq P(\tilde{V}_n = k) \leq$$

$$\leq (1 + \frac{\epsilon}{3})^3 \sum_{l=0}^{n} (-1)^l \binom{k+l}{k} S_{k+l,n}$$

which can be expressed as

$$(1-\frac{\epsilon}{3})^2 P(V_n=k) \leq P(\tilde{V}_n=k) \leq (1+\frac{\epsilon}{3})^3 P(V_n=k).$$

Thus, for all k,

$$(1-\frac{\epsilon}{3})^2 \leq \frac{P(\tilde{V}_n=k)}{P(V_n=k)} \leq (1+\frac{\epsilon}{3})^3.$$

Since ϵ is arbitrary, we have shown that the results in Theorems 2.1, 2.2, and 2.3 hold under the respective assumptions for this case of dependence. ■

Acknowledgements. The topic of this article was suggested to the first author by Professor János Galambos in personal communication. Even after his retirement from Temple University, Professor Galambos continued to inspire and contribute to the work of many around the world. He will be remembered not only as a distinguished mathematician, but also as an inspiring teacher and mentor.

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