REDUCTION MODULO SIM AND UNIVERSALITY

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To the memory of Professor Dr. János Galambos and Professor Dr. Gisbert Stoyan with great respect

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Abstract. The purpose of this note is to make higher mathematics more accessible to learners who are not necessarily majors in mathematics. Mathematics is in one aspect a method of classification, which is done by equivalence relation and in another, uniqueness, which is also accomplished by the same. Starting from congruence between two triangles and coming into the underlying principle of reduction modulo an equivalence relation, advanced high school students could master this essential aspect of mathematics. Uniqueness or neutral element is that which works as neutral in the new world of a quotient space, which is also interpreted as annihilating those entities which are hard to treat, an example being reduction modulo measure zero set.

1. Universal algebraic structure

In mathematics, constructing a new object from existing ones rests on two methods, one is forming a Cartesian product and the other a quotient space. In [1] we fully used them to simplify many homomorpshism type theorems appearing in algebra.

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Definition 1.1. Given a certain system $\mathcal{R}$, it is possible to construct a universal system $\mathcal{S}$ with a certain mapping $\sigma : \mathcal{R} \to \mathcal{S}$ which in a certain sense preserves the mathematical structure. *Universality* means that given another similar system $\mathcal{T}$ and a mapping $\tau : \mathcal{R} \to \mathcal{T}$ which preserves the relevant mathematical structure, then there exists a unique mapping $t : \mathcal{S} \to \mathcal{T}$ such that

$$t \circ \sigma = \tau. \tag{1.1}$$

We call such a system $\mathcal{S}$ and σ *universal.* t is uniquely determined by (1.1) by its behavior on $\sigma(\mathcal{R})$.

There is a similar notion in the case of a Cartesian (direct) product and direct sum and is regarded as duality in a categorical setting (when the product exists). In [1] we did not make much difference in constructions. However, look at a universal mapping property given on [3, p. 24]: Let K be a field with valuation v and K_v be its completion with a natural injection $\iota_v : K \to K_v$. Suppose L is a field which is complete with respect to its valuation η and that there exists an isometric field injection $\lambda : K \to L$. Then there exists a unique field injection $\rho : K_v \to L$ such that

$$\rho \circ \iota_v = \lambda. \tag{1.2}$$

Looking at the proof, one sees that the quotient space is used not only in order to simplify the proof but to assure the uniqueness of the metric induced by the valuation. If this aspect is separated, then the above universality is the one for a direct sum (i.e. Cauchy sequences) and the natural injection.

In this note we will develop the idea that reduction modulo "sim", a suitable equivalence relation, will give rise to a universal system, which simplifies quite a lot of constructions in whole spectrum of mathematics. We call this process *reduction modulo sim yielding universality.*

For the quotient space $\mathcal{S} = \mathcal{R}/\sim$ we let $\pi = \pi_\sim : \mathcal{R} \to \mathcal{S} = \mathcal{R}/\sim$ be the *canonical projection.* In $\mathcal{S}$ those elements (their totality being $\mathcal{U}$) of $\mathcal{R}$ satisfying the relation $\sim$ are a sigleton, symbolically $\mathcal{U} = \mathfrak{o}$, say which means that we may treat the elements of $\mathcal{S}$ as if they are in $\mathcal{R}$ with the neutral element $\mathfrak{o}$. Symbolically, $(a+\mathfrak{o}) * (b+\mathfrak{o}) = a*b+\mathfrak{o}$, a reminiscent of Bachman–Landau o-notation. Universality (1.1) accompanies with this. If $\tau : \mathcal{R} \to \mathcal{T}$ preserves the relevant mathematical structure and $\tau(\mathcal{U}) = 0$, then there exists a unique mapping $t : \mathcal{S} \to \mathcal{T}$ such that

$$t \circ \pi = \tau. \tag{1.3}$$

This describes the behavior of t on $\pi(\mathcal{R})$ by τ and so if it exists, it is unique.

In a certain algebraic system $\mathcal{R}$, if a certain relation $\sim$ determines a certain sub-algebraic system, then the quotient system $\mathcal{R}$ modulo $\mathcal{U}$ and the canonical projection $\pi = \pi_\sim : \mathcal{R} \to \mathcal{S} = \mathcal{R}/\sim$ is an example. In $\mathcal{S}$, $\mathcal{U}$ works as a neutral element $\mathfrak{o}$.

In particular, if $\mathcal{U} = \mathfrak{o}$ is a maximal sub-system satisfying $\sim$, then in $\mathcal{S}$, there are only two sub-systems $\mathcal{R}$ and $\mathfrak{o}$.

Example 1.1. One of the guiding principles of [1] is that the residue classes are for making a surjective homomorphism $f : G \to \operatorname{Im} f \subset G'$ injective. In other words, $f(a) = f(b)$ should imply $a = b$. For this a natural construction is reduction modulo $\sim$: $f(a) = f(b)$ yielding $G/\sim$. It turns out that this amounts to reduction modulo the kernel of f in the case of algebraic homomorphisms, thus universality arises: Let $N \triangleleft G$ be a normal subgroup of a group $G = \mathcal{R}$. Then we let $\mathcal{S} = G/N$ and $\pi_N = \sigma : G \to G/N$. For a group homomorphism $\varphi = \tau : G \to \mathcal{T}$ such that $\operatorname{Ker} \varphi \supset N$, there exists a unique $\bar{\varphi}_N = t : \mathcal{S} \to \mathcal{T}$ such that (1.1) is satisfied. Since $\bar{\varphi} : G/\operatorname{Ker}\varphi \xrightarrow{\sim} \operatorname{Im}\varphi$ is the homomorphism theorem, it follows that $N = \operatorname{Ker}\varphi$ is the threshold normal subgroup for which $\bar{\varphi}_N : G/N \xrightarrow{\sim} \operatorname{Im}\varphi$ for Λ-groups and that the homomorphism theorem is a special case of universality.

This example gives almost all proof of the following

Theorem 1.2. *Suppose G, G' are two Λ-groups and $f : G \to G'$ is a Λ-homomorphism. Then $N = \operatorname{Ker} f$ is a normal Λ-subgroup of G and $\operatorname{Im} f$ is a Λ-subgroup of G'.*

(i) *A Λ-subgroup $H \supset N$ of G and a Λ-subgroup H' of $\operatorname{Im} f$ is in one-to-one correspondence under the image and the inverse image of each other:*

$$H' = f(H), \quad H = f^{-1}(H') \tag{1.4}$$

and

$$H/N \cong f(H) \tag{1.5}$$

holds true up to $H = G$.

(ii) *$H \triangleleft G$ and $H' \triangleleft \operatorname{Im} f$ are equivalent and we have the third homomorphism theorem*

$$G/H \cong \operatorname{Im} f/H' = f(G)/f(H) \cong (G/N)/(H/N). \tag{1.6}$$

Proof. If H' is a subgroup of $\operatorname{Im} f$, then $H := f^{-1}(H') \supset f^{-1}(e') = N$, so that the inclusion-order is preserved.

Let $f|_H$ be the restriction of f to H. Since $\operatorname{Ker} f|_H \subset \operatorname{Ker} f = N$ always holds, it follows that $\operatorname{Ker} f|_H = \operatorname{Ker} f = N$ in view of $H \supset N$. Hence Example 1.1 implies (1.5). Cf. Table 1. (ii) Suppose $H' \triangleleft \operatorname{Im} f$. Choose $\mathcal{T} = f(G)/H'$

in Example 1.1. Then the kernel of the epimorphism $\pi_{H'} \circ f : G \to f(G)/H'$ is H and we have

$$G/H \cong f(G)/H'. \tag{1.7}$$

This proves the first isomorphism in (1.6). Substituting (1.5) with G resp. H in (1.7) proves the last isomorphism of (1.6). ■

proposition	group	homom	N
Example 1.1	G	φ	N
Theorem 1.2	H	$f\|_H$	$\operatorname{Ker} f\|_H = \operatorname{Ker} f$

Table 1. Correspondence of ingredients

Example 1.3. A distance function $\rho : X \to \mathbb{R}$ is called a pseudo-distance function if it satisfies the axioms save for the uniqueness: $d(a, b) = 0$ implies $a = b$. In the same vein as in Example 1.1 one considers the equivalence classes $X/\sim$ with $\sim$: $d(a, b) = 0$. Then in the equivalence class, uniqueness holds. More precisely it is stated as the following theorem.

Theorem 1.4. ([5, Theorem 4.15, p. 123]) *Let (X, d) be a pseudo-metric space and for each $x \in X$ let $[x]^-$ denote the set of all points $y \in X$ such that $d(x, y) = 0$. Let $\mathfrak{D}$ be the set of all elements $[x]^-$ and for two elements $A, B \in \mathfrak{D}$ let*

$$\rho(A, B) = d(A, B).$$

Then $(\mathfrak{D}, \rho)$ is a metric space whose topology is the quotient topology $X/\sim$ for $\mathfrak{D}$ and the projection of X onto $\mathfrak{D}$ is an isometry. Here $\sim$ is the relation $x \sim y \iff d(x, y) = 0$.

As is also stated on [3, p. 22], the universality for Cartesian product shows that there is a unique distance function which keeps uniqueness.

1.1. Ring theory

The following result is ubiquitously used throughout mathematical disciplines without much care about its origin.

Corollary 1.1. *Suppose R contains the identity 1. Then for any ideal $\mathfrak{a}_0 \subsetneq R$, there exists a maximal ideal $\mathfrak{a} \subset R$ containing $\mathfrak{a}_0$.*

Proof follows from Proposition below with exceptional set $E = \{1\}$.

Definition 1.2. A ring with unity is called a *simple ring* if the only ideals are 0 and R. A commutative ring with unity is called a *local ring* [resp, semi-local ring] if it contains a unique [resp. finitely many] maximal ideal.

In [6, p. 11, p. 13] the phrase "by localizing" is used to mean that the localization at a prime ideal $\mathfrak{p}$ is a local ring. In the same vein, we propose to say "by maximizing" to mean that for an ideal in the ring there exists a maximal ideal (by Corollary 1.1) by which we form a field.

Proposition 1.5. *Let E be a non-empty set $\subsetneq R$ and suppose there is an ideal $\mathfrak{a}_0$ of R such that $\mathfrak{a}_0 \cap E = \phi$. Then there exists a maximal ideal $\mathfrak{a}$ such that $\mathfrak{a} \supset \mathfrak{a}_0$ and $\mathfrak{a} \cap E = \emptyset$.*

For any totally ordered subset $\{\mathfrak{a}_\nu\}$ of the family of all ideals of R containing $\mathfrak{a}_0$ not meeting E, the union is an upper bound and **Zorn's lemma** applies.

Example 1.6. Suppose R is a ring with unity and $\mathfrak{m}$ is a two-sided maximal ideal. Then $R/\mathfrak{m}$ is a skew field. Especially, if further R is commutative, then it is a field. For if $x \notin \mathfrak{m}$, then $Rx = R$ or $xR = R$ according as it is a left or right ideal, so that x has the inverse element.

Example 1.7. Let R be a commutative ring with unity 1. Let $S \subset R$ be a multiplicatively closed subset, i.e. a multiplicative semi-group of R containing 1 but not 0. To construct a quotient ring, in the direct product $R \times S$ the equivalence relation is introduced

$$(a_1, s_1) \sim (a_2, s_2) \longleftrightarrow (a_1 s_2 - a_2 s_1)s = 0 \quad \text{for some} \quad s \in S. \tag{1.8}$$

The equivalence class containing (a, s) is denoted a/s. Then the quotient space $R/S = R/\sim = \{a/s | a \in R, s \in S\}$, often written $S^{-1}R$ is such that those elements which belong to S are 1/1–the identity in $S^{-1}R$ and that addition and multiplication are defined as usual. (1.8) assures well-defined-ness of these operations and $S^{-1}R$ becomes a ring. We denote the canonical projection π by S^{-1}:

$$S^{-1} : R \to S^{-1}R; \quad S^{-1}a = a/s \tag{1.9}$$

If we restrict S to be the set of all non zero-divisors of R, then the condition (1.8) reduces to $a_1 s_2 - a_2 s_1 = 0$ and $S^{-1}R$ is called the quotient ring denoted $q(R)$.

Let $\mathfrak{a} \subset R$ be a prime ideal. Then $S = R - \mathfrak{a}$ is a multiplicatively closed subset and we may form a quotient ring $q(R)$ denoted $R_\mathfrak{a}$ and called the *localization at* $\mathfrak{a}$.

Theorem 1.8. *If $\tau : R \to R'$ is a ring homomorphism such that $\tau(S) \subset U(R')$, where $U(R')$ is the unit group. Then there exists a unique $t : S^{-1}R \to R'$ such that $t \circ S^{-1} = \tau$.*

Definition 1.3. A commutative ring with 1 is called a *local ring* if there is only one maximal ideal and is called a *semi-local ring* if there are only finitely many maximal ideals.

Theorem 1.9. *Let R be a commutative ring with 1. R is a local ring if and only if the set $\mathfrak{m}$ of all non-unis of R forms an ideal.*

Proof. Suppose R is a local ring with $\mathfrak{n}$ its unique maximal ideal. ■

2. Modules over a ring

Here we shall elucidate the construction of a tensor product, alternating algebra etc.

2.1. Tensor product

Let R be a ring with unity. For a right R-module M and a left R-module N, a mapping $f : M \times N \to T$, T being an Abelian group, is called a *semi-linear map* if it is additive with respect to both variables and $f(xa, y) = f(x, ay)$.

Definition 2.1. For a right R-module M and a left R-module N, their *tensor product* $M \otimes N = M \otimes_R N = (M \otimes N, \otimes)$ is an Abelian group satisfying the following conditions. The map

$$\otimes : M \times N \to M \otimes N, (x, y) \to x \otimes y, \quad x \in M, y \in N \tag{2.1}$$

satisfies the semi-bilinearity conditions

$$\begin{aligned}(x_1 + x_2, y) &= (x_1, y) + (x_2, y)\\ (x, y_1 + y_2) &= (x, y_1) + (x, y_2)\\ (xa, y) &= (x, ay)\end{aligned} \tag{2.2}$$

and also universality condition (U) is satisfied:

(U) For any semi-bilinear map $\tau : M \times N \to T$ to an Abelian group, there exists a unique $t : M \otimes N \to T$ such that $\tau = t \circ \otimes$.

Theorem 2.1. *The tensor product $M \otimes_R N$ exists and is unique up to a isomorphism. Its elements are expressed as a finite linear combination*

$$\sum x_i \otimes y_i, (x_i \in M, y_i \in N). \tag{2.3}$$

Proof. Existence. Let $\mathcal{F} = \mathcal{F}_{M\times N}$ be a free Abelian group i.e. a free $\mathbb{Z}$-module with basis $M \times N$. The relation $\sim$ is to be set so as to make the reduction of a system satisfy the conditions for a tensor product, i. e. semi-bilinearity of the natural projection $\pi_\sim$. I.e. for $x, x_i \in M, y, y_j \in N, a \in R$, the reduction $\sim$ is to make (2.2) hold true in $\mathcal{F}/\sim$. I.e. two elements are equivalent if they differ by elements in (2.4), or simply their difference belongs to the subgroup $\mathcal{H} \subset \mathcal{F}$ generated by elements of the form

$$(2.4) \qquad \begin{aligned} &(x_1 + x_2, y) - (x_1, y) - (x_2, y), \\ &(x, y_1 + y_2) - (x, y_1) - (x, y_2), \\ &(xa, y) - (x, ay). \end{aligned}$$

Then $\mathcal{S} = \mathcal{F}/\sim = \mathcal{F}/\mathcal{H}$ as a factor group. The canonical projection is $\pi = \pi_\sim : : \mathcal{F} \to \mathcal{S}$. The restriction $\pi_{M \times N}$ is a semi-bilinear map and works as the map $\otimes$ in the definition. Any semi-bilinear map $\tau : M \times N \to T$ can be extended to a homomorphism $\mathcal{F} \to T$ since it maps $\mathcal{H}$ to 0. Hence we may define the map $t : \mathcal{F}/\mathcal{H} \to T$ by $t \circ \pi_\sim = \tau$. Hence $\mathcal{F}/\mathcal{H}$ satisfies the defining conditions in Definition 2.1 and passes as the tensor product $M \otimes N$.

Uniqueness. Suppose in addition to $(M \otimes N, \otimes)$, the pair (T, τ) also satisfies the universality condition (U) is satisfied, i.e. there exists a $t_1 : T \to M \otimes N$ such that

$$(t_1 \circ t)(x \otimes y) = (t_1 \circ \tau)(x, y) = x \otimes y.$$

Applying the uniqueness in (U) to $T = M \otimes N$ and $\tau = \otimes$, we see that $t_1 \circ t = I_{M \otimes N}$. Changing the role, we have $t \circ t_1 = I_T$. Hence $t : M \otimes N \simeq T$. Identifying $t : M \otimes N$ and T, the maps $\otimes$ and T are identified and the uniqueness holds including the semi-bilinear map (2.1).

Linear combination expression. Let T be the set of all elements of the form (2.3) forms an Abelian group. The map $\otimes$ in (2.1) induces a semi-bilinear map $M \times N \to T$. Universality implies the existence of a homomorphism $t : M \otimes N \to T(\subset M \otimes N)$ such that $t(x \otimes y) = x \otimes y$. By the above argument, this t must be an isomorphism and $T = M \otimes N$. ■

2.2. Graded algebras

Definition 2.2. Let J be a commutative semi-group with unity 0, we shall confine to $J = \{0, 1, 2, \cdots\}$. The direct sum $M = \sum_{j \in J} M_j$ of R-modules M_j with index set J is called a graded R-module of type J. Every element in M_j is called a *homogeneous element* of degree j. Any submodule $N \subset M$ of the form $\sum_{j \in J} N_j$, $N_j \subset M_j$ is called a graded R-submodules.

If in a graded module $A = \sum A_k$ over a commutative ring R the product is defined so that $A_k A_l \subset A_{k+l}$ and A forms an algebra with respect to this product, then A is called a *graded algebra*.

Definition 2.3. Let R be a commutative ring with unity and let M be an R-module. Let $T^k(M)$ be the k-ple tensor product

$$T^k(M) = M \otimes \cdots \otimes M. \tag{2.5}$$

There is a natural isomorphism $T^k(M) \otimes T^l(M) \simeq T^{k+l}(M)$. With $T^0(M) = R$ form the graded module

$$T(M) = \sum_k T^k(M) = R \otimes (M) \otimes (M \otimes M) \otimes (M \times M \times M) \otimes \cdots . \tag{2.6}$$

The product of $x_1 \otimes \cdots \otimes x_k \in T^k(M)$ and $y_1 \otimes \cdots \otimes y_l \in T^l(M)$ is defined by

$$(x_1 \otimes \cdots \otimes x_k) \otimes (y_1 \otimes \cdots \otimes y_1) = x_1 \otimes \cdots \otimes x_k \otimes y_1 \otimes \cdots \otimes y_1 \in T^{k+l}(M) \tag{2.7}$$

according to the above natural isomorphism. (2.6) becomes a graded algebra, called the *tensor algebra* or a tensor product algebra.

Definition 2.4. A k-ple linear map $f : M^k \to N$ is called *alternating* if it satisfies one of the following equivalent conditions

(i) If $x_i \neq x_j$ for $i \neq j$ then $f(x_1, \cdots, x_k) = 0$.

(ii) If x_i, x_j with $i \neq j$ are changed, it changes sign:
$f(x_1, \cdots, x_j, \cdots, x_i, \cdots x_k) = -f(x_1, \cdots, x_i, \cdots, x_j, \cdots x_k)$.

In $T^k(M)$ we introduce a relation: $x \sim y$ if they differ by elements $x_1 \otimes \cdots \otimes \otimes x_k$ in which $x_i = x_j$ for some $i \neq j$. Or we let $\mathfrak{a}_k$ be the submodule consisting of those elements $x_1 \otimes \cdots \otimes x_k$ in which at least two components coincide and we define the R-module

$$\Lambda^k(M) = T^k(M)/\sim = T^k(M)/\mathfrak{a}_k; \quad \pi_\sim : T^k(M) \to \Lambda^k \tag{2.8}$$

and write

$$\pi_\sim(x_1 \otimes \cdots \otimes x_k) = x_1 \wedge \cdots \wedge x_k. \tag{2.9}$$

Hence $x_1 \wedge \cdots \wedge x_k$ are alternating.

With $\mathfrak{a}_0 = 0$, the ideal $\mathfrak{a} = \mathfrak{a}_M = \sum_k \mathfrak{a}_k$ forms a homogeneous ideal of $T(M)$ and is generated by all $x \otimes x$, so that

$$\Lambda(M) := T(M)/\mathfrak{a} = \sum_{k=0}^{\infty} T^k(M)/\mathfrak{a}_k \tag{2.10}$$

is a graded module whose elements are alternating.

Theorem 2.2. *Every linear map $f : M \to N$ from an R-module M to another R-module N is extended uniquely to an algebra homomorphism*

$$T(f) : T(M) \to N. \tag{2.11}$$

If N is an algebra, then

$$T(f)(x_1 \otimes \cdots \otimes x_k) = f(x_1) \cdots f(x_k). \tag{2.12}$$

Proof. Since $T(M)$ is a graded module, it suffices to consider its $(k+1)$th component $T^k(M)$ on which we should have

$$T(f)(x_1 \otimes \cdots \otimes x_k) = f(x_1, \cdots, x_k), \tag{2.13}$$

whence $T(f)$ is unique. The left-hand side of (2.13) is to be the right-hand side of (2.12) if N is an algebra. Since (2.12) is a linear map and $T(f)$ exists. Moreover, by the definition of the product in N, it is an algebra homomorphism. ∎

modules	linear map, alg homom
(0) $N = T(N)$ (tensor algebra)	f
(i) $N = A$ (algebra)	$f(x_1, \cdots, x_k) = f(x_1) \cdots f(x_k)$
(ii) $M = M^k$	$f(x_1, \cdots, x_k)$
(iii) $M = M_1 \oplus M_2, N = \Lambda(M_1) \otimes \Lambda(M_2)$	f in (2.15)

Table 2. Special cases

Corollary 2.1. (i) *Every linear map $f : M \to N$ is extended to a graded algebra homomoprphism $f : T(M) \to T(N)$. This induces an algebra homomorphism*

$$\Lambda(f) : \Lambda(M) \to \Lambda(N)$$

given by

$$\Lambda(f)(x_1 \wedge \cdots \wedge x_n) = f(x_1) \wedge \cdots \wedge f(x_n).$$

(ii) *A k-ple linear map $f : M^k \to N$ viewed as a linear map $T(f) : T^k(M) \to$ $\to N$ is alternating if and only if there exists a linear map $\bar{T}(f) : T^k(M) \to N$ such that*

$$f(x_1, \cdots, x_k) = \bar{T}(f)(x_1 \wedge \cdots \wedge x_k). \tag{2.14}$$

(iii) *The linear map $f : M_1 \oplus M_2 \to \Lambda(M_1) \otimes \Lambda(M_2)$ given by*

$$f(x + y) = x \otimes 1 + 1 \otimes y \tag{2.15}$$

is extended to a graded algebra homomoprphism $T(f) : T(M_1 \oplus M_2) \to \Lambda(M_1) \otimes \Lambda(M_2)$, which in turn induces a graded algebra homomoprphism $\bar{T}(f) : \Lambda(M_1 \oplus M_2) \to \Lambda(M_1) \otimes \Lambda(M_2)$ satisfying

$$\bar{T}(f)(x_1 \wedge \cdots \wedge x_k \wedge y_1 \wedge \cdots \wedge y_l) = (x_1 \wedge \cdots \wedge x_k) \otimes (y_1 \wedge \cdots \wedge y_l), \quad x_i \in M_1, y_j \in M_2. \tag{2.16}$$

Proof. The cases correspond to (i)–(iii) in Table 1. (i) follows from Theorem 2.2 on viewing $f : M \to N$ as $f : M \to T(N)$. The second assertion follows since $T(f)$ maps $\mathfrak{a}_M$ to $\mathfrak{a}_N$.

(ii) follows from universality. By Theorem 2.2 we may view f as a linear map $T(f) : T^k(M) \to N$. Universality means $T(f) = \bar{T}(f)(\pi_\sim)$, which implies (2.14) in view of (2.9).

(iii) By Theorem 2.2, f is extended to $T(f)$. By (2.12), we have for $z = x + y \in M_1 \oplus M_2$, $x \in M_1$, $y \in M_2$

$$\begin{aligned} T(f)(z \otimes z) &= f(z)f(z) = (x \otimes 1 + 1 \otimes y)(x \otimes 1 + 1 \otimes y) \\ &= (x \otimes 1)(1 \otimes y) + (1 \otimes y)(x \otimes 1) = 0 \end{aligned}$$

by alternating property of the product in $\Lambda(M_1) \otimes \Lambda(M_2)$. Hence by universality, $T(f)$ induces $\bar{T}(f)$ satisfying (2.16). ■

Lemma 2.1.

$$\Lambda(M_1 \oplus M_2) \simeq M_1 \otimes M_2. \tag{2.17}$$

Proof. The injection

$$\iota_j : M_j \to M_1 \oplus M_2, \quad j = 1, 2$$

induces the algebra homomorphism

$$\Lambda(\iota_j) : \Lambda(M_j) \to \Lambda(M_1 \oplus M_2).$$

Hence the algebra homomorpshim $g : \Lambda(M_j) \otimes \Lambda(M_2) \to \Lambda(M_1 \oplus M_2)$ is induced such that

$$g((x_1 \wedge \cdots \wedge x_k) \otimes (y_1 \wedge \cdots \wedge y_l)) = (x_1 \wedge \cdots \wedge x_k) \wedge (y_1 \wedge \cdots \wedge y_l).$$

Since this is the inverse map of $\bar{T}(f)$, the isomorphism (2.17) is given by $\bar{T}(f)$. ■

Theorem 2.3. *Suppose M is a free R-module with basis $\{u_1, \cdots, u_n\}$: $M = Ru_1 \oplus \cdots \oplus Ru_n$. Then for $k > n$, $\Lambda^k(M) = 0$ and for $k \le n$, $\Lambda^k(M)$ is a free R-module of rank $\binom{n}{k}$ with basis $\{u_{i_1}, \cdots, u_{i_k}\}$, $i_1 < \cdots < i_k$. $\Lambda^n(M)$ is a free R-module of rank 2^n*

Proof. By Lemma 2.1,

$$\Lambda(M) \simeq \Lambda(Ru_1) \otimes \cdots \otimes \Lambda(Ru_n).$$

Since $\Lambda(Ru_j) = R1 \oplus Ru_j$, the result follows. ■

Remark 2.1. The differential forms can be concisely introduced by the above. Let $R = C^r(\Omega)$ and $M = Ru_1 \oplus \cdots \oplus Ru_n$ be a free R-module of rank n. Then the k-ple tensor product $T^k(M)$ is $\{\omega^k = \sum_I a_I u_I\}$ with $I = \{i_1, \cdots, i_k\} \subset \subset \{1, \cdots, n\}$. With $\mathfrak{a}_k$ as above and $\mathrm{d}x_j$, the space of k-forms is

$$\Lambda^k(M) = T^k(M)/\mathfrak{a}_k = \left\{ \sum_I a_I \mathrm{d}x_I \right\}, \tag{2.18}$$

where $\mathrm{d}x_j$'s have alternating properties.

3. Field theory

The Cartan–Bourbaki proof of the celebrated Tychonoff theorem [2], [5], [8] depends on multiple use of Zorn's lemma (or the Axiom of choice, its equivalent). Tsukada [7] is the first who gave a proof which uses Zorn's lemma only once. In field theory construction of an algebraic closure is one of the most fundamental stuff and there are proofs depending on multiple use of Zorn's lemma. Our aim in this subsection is to give a slightly simplified proof given in [4] which used Zorn's lemma once in the form of maximization.

Lemma 3.1. *Let L_i/K_i be field extensions $i = 1, 2$ and let $\varkappa : K_1 \to K_2$ be an (injective) isomorphism. There there exists an extension filed M/L_2 and an isomorphism $\lambda : L_1 \to M$ which extends $\varkappa$ such that $M = \lambda(L_1) \cdot L_2$.*

Proof. Let $K = \varkappa(K_1)(\subset K_2) \subset L_2$. Define the action of K on L_1 by

$$a(\alpha_1) = \varkappa^{-1}(a)\alpha_1, \quad a \in K, \quad \alpha_1 \in L_1.$$

Then L_1 becomes an algebra over K. Then we form the tensor product $A := L_1 \otimes_K L_2$. Since A is a ring containing 1, it follows by Proposition 1.5 that there exists a maximal ideal $\mathfrak{a}$. Then by Example 1.6, $M = A/\mathfrak{a}$ is a field. By the natural projection $\pi = \pi_{\mathfrak{m}} : A \to M$ form a map $\lambda_2; L_2 \to M$; $\alpha_2 \to \pi(1 \otimes \alpha_2)$ which turns to be an injective isomorphism and we regard L_2 as a subfield $\lambda_2(L_2) \subset M$. On the other hand, $\lambda_1; L_1 \to M; \alpha_1 \to \pi(\alpha_1 \otimes 1)$ is an isomorphism and for $a \in K$, $\lambda_1(a) = \pi(a \otimes 1) = \pi(1 \otimes a)$ which is a by the above embedding. Hence λ_1 is a K-isomorphism. ∎

3.1. Inductive limit

We recall basic facts about the inductive and projective limits.

Definition 3.1. A commutative diagram $\{X_\mu, \rho^\mu_\nu\}$ with a directed set M as its type which consists of sets X_μ and maps ρ^μ_ν is called an *inductive system* or a direct system. I.e., suppose for each $\mu \in \mathrm{M}$, there corresponds a set X_μ and for every pair (μ, ν) with $\mu \le \nu$, there exists a map

$$\rho_\nu^\mu : X_\mu \to X_\nu \tag{3.1}$$

satisfying

$$\rho_\mu^\mu = I_{X_\mu}, \quad \rho_\nu^\mu \circ \rho_\mu^\lambda = \rho_\nu^\lambda \quad (\lambda \le \mu \le \nu) \tag{3.2}$$

Then in the disjoint union $\dot{\bigcup} X_\mu$ we introduce the equivalence relation: two elements $x_{\mu_1} \in X_{\mu_1}$ and $x_{\mu_2} \in X_{\mu_2}$ are equivalent

$$x_{\mu_1} \sim x_{\mu_2} \tag{3.3}$$

if for some $\nu \ge \mu_1, \mu_2$

$$\rho_\nu^{\mu_1} x_{\mu_1} = \rho_\nu^{\mu_2} x_{\mu_2} \tag{3.4}$$

holds true. We denote the canonical projection π by ρ^μ, $\rho^\mu : X_\mu \to X_\mu \bmod \sim$.

We call the set of all equivalence classes $\bigcup X_\mu / \sim$ the **inductive limit** (or direct limit) denoted

$$\varinjlim X_\mu = \bigcup_{\mu \in \mathrm{M}} \rho^\mu(X_\mu). \tag{3.5}$$

Theorem 3.1. *Suppose the set $\{f^\mu : X_\mu \to X | \mu \in \mathrm{M}\}$ satisfies the condition*

$$f^\mu \circ \rho_\nu^\mu = f^\nu \qquad (\mu \le \nu). \tag{3.6}$$

Then there exists a unique $f : \varinjlim X_\mu \to X$ such that

$$f \circ \rho^\mu = f^\mu. \tag{3.7}$$

(1.3) reads $f(\rho^\mu(x_\mu)) = f^\mu(x_\mu)$.

Corollary 3.1. *For direct systems (X_μ, ρ_ν^μ), (Y_μ, σ_ν^μ), the set of mappings $\{f^\mu : X_\mu \to Y_\mu | \mu \in \mathrm{M}\}$ satisfying the condition is called a morphism:*

$$f^\nu \circ \rho_\nu^\mu = \sigma_\nu^\mu \circ f^\mu \quad (\mu \le \nu). \tag{3.8}$$

For a morphism, there exists a unique $f^\infty : \varinjlim X_\mu \to \varinjlim Y_\mu$ such that

$$f^\infty \circ \rho^\mu = \sigma^\mu \circ f^\mu \quad (\mu \in \mathrm{M}). \tag{3.9}$$

Example 3.2. Let X be a topological space [resp. the complex plane]. Let $\mathcal{U}(x)$ denote the fundamental system of neighborhoods of x or a subsystem thereof. Defining the order $V \leqslant U$ by $U \subset V$, $\mathcal{U}(x)$ becomes a directed set. The space $C(U)$ [resp. $A(U)$] of all complex-valued continuous [resp. analytic] functions on U forms an Abelian group with respect to addition. For $V \leqslant U$ let $\rho_V^U = \iota_U : C(U) \to C(V)$ be the restriction map of the domain, then $\{C(U)\}$ forms a direct system. The inductive limit $\varinjlim C(U)$ [resp. $\lim_{\to} A(U)$] is the space of all (equivalence classes of) functions which are regarded as the same if they coincide in some (small enough) neighborhoods. $\varinjlim C(U)$ [resp. $\varinjlim A(U)$] is called the *germ* of continuous [resp. analytic] functions at the point x.

Example 3.3. Let $D \subset \mathbb{C}$ be a domain. We denote the germ $\varinjlim A(U)$ of analytic functions at ζ by $\boldsymbol{f}_z$. The set of all germs $\boldsymbol{f}_z$, $z \in D$ is called a *sheaf* and denoted $\mathcal{S} = \mathcal{S}_D$. Theorem 3.1 reads as follows. $(f, \zeta) = (f, \zeta, U) = X_U$ and the canonical projection is $\rho^U : U \to \boldsymbol{f}_{\boldsymbol{\zeta}}$. Given a map $f^U : X_U \to X_V$, there exists a unique map $f : \boldsymbol{f}_{\boldsymbol{\zeta}} \to X_V$ such that $f \circ \pi_U = f_V$, i.e. $f(\rho^U(x_U)) = f^V(x_U)$.

A sheaf $\mathcal{S} = \mathcal{S}_D$ on D in general is a topological space with the projection $\pi : \mathcal{S} \to D$ which is a local homeomorphism. The canonical projection ρ^U above is $\pi^{-1}|U$ restriction to U called a section.

(3.9) reads

$$f^\infty \circ \rho^U = \sigma^U \circ f^U \qquad (U \in \mathrm{M}) \tag{3.10}$$

and $f^\infty : \boldsymbol{f}_\zeta \to \boldsymbol{g}_\zeta$.

3.2. Some facts from category theory

Definition 3.2. A category $\mathcal{C}$ consists of the following three ingredients.

(i) A domain $\mathrm{Ob}(\mathcal{C})$ consisting of objects of $\mathcal{C}$.

(ii) Given two objects (M, N) of $\mathcal{C}$, there exists a set $\hom(M, N) = \hom_{\mathcal{C}}(M, N)$ of morphisms.

(iii) Given three objects (M, N, L) of $\mathcal{C}$, there exists a map $h : \hom(M, N) \times \hom(M, L) \to \hom(N, L)$ called a composition.

They satisfy the conditions.

(iv) If $(M, N) \neq (M', N')$, then $\hom(M, N) \cap h \hom(M', N') = \emptyset$.

(v) For any $M \in \mathcal{C}$ there exists an $I_M : M \to M$ satisfying

$$\begin{aligned} I_M \circ f = f \quad &\text{for} \quad f : L \to M \\ g \circ I_M = g \quad &\text{for} \quad g : M \to N. \end{aligned} \tag{3.11}$$

(iv) The composition satisfies the associative law.

Example 3.4. If a category $\mathcal{C}'$ satisfies the following conditions, then it is called a *subcategory* of $\mathcal{S}$: any $M \in \mathrm{Ob}(\mathcal{C}')$ also belongs to $\mathrm{Ob}(\mathcal{C})$. $\hom_{\mathcal{C}'}(M,N) \subset \hom_{\mathcal{C}}(M,N)$. I_M and composition in $\mathcal{C}'$ coincide with those in $\mathcal{C}$. Reversing the order of all arrows in the definition of a category, we obtain a *dual* category $\mathcal{C}^*$.

Definition 3.3. A mapping T between two categories $\mathcal{C}$ and $\mathcal{D}$ which maps $M \in \mathrm{Ob}(\mathcal{C})$ to $T(M) \in \mathrm{Ob}(\mathcal{C})$ and $f \in \hom(M,N)$ to $T(f) \in \hom(f(M), f(N)$ is called a *covariant functor* if it satisfies the following conditions

(i) $T(I_M) = I_{T(M)}$.

(ii) $T(g \circ f) = T(g) \circ T(f)$.

If in the above conditions, the morphism part is changed by $f \in \hom(M,N)$ to $T(f) \in \hom(f(N), f(M)$ and (ii) by (iii) $T(g \circ f) = T(f) \circ T(g)$, then it is called a contravariant functor. A contravariant functor $\mathcal{C} \to \mathcal{D}$ may be treated as a covariant functor $\mathcal{C} \to \mathcal{D}^*$.

Example 3.5. Let $\mathrm{Ob}(\mathcal{S})$ be a domain of any sets and let $\hom(M,N)$ be any mappings. Then $\mathcal{S}$ is a category.

Let $(\mathrm{M}, <)$ be a patially ordered set. Let $\mathrm{Ob}(\mathcal{C}) = \mathrm{M}$. If $\mu \le \nu$, then let $\hom(\mathrm{M},\mathrm{M})$ be a singleton f_ν^μ. $\mu \not\le \nu$, then $\hom(\mathrm{M},\mathrm{M}) = \emptyset$. This defines a category $(\mathrm{M}, <)$.

A covariant functor $(\mathrm{M}, <) \to \mathcal{S}$ is an inductive system. A contravariant function $(\mathrm{M}, <) \to \mathcal{S}$ is a projective system

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