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ON SOME UNIFORMLY DISTRIBUTED FUNCTIONS ON THE SET OF SHIFTED PRIMES

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Abstract. The uniformly distributed functions on the set of shifted primes is defined and a theorem is proved.

1. Introduction

Let, as usual, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ be the set of positive integers, integers and real numbers, respectively.

A positive arithmetical function h is said to be uniformly distributed, if

$$\frac{1}{x}\sum_{h(n)\leq x} 1 \to A, \quad \text{where} \quad A > 0.$$

(see P. Erdős [4])

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There are several papers on this topics [1]–[8]. It is quite natural to extend the notation of uniformity for subsets of integers. Let $\mathcal{B}$ be an infinite sequence of integers,

$$\mathcal{B}(x) := \#\{b \le x \mid b \in \mathcal{B}\}.$$

We say that a function $h : \mathcal{B} \to (0, \infty)$ is uniformly distributed on $\mathcal{B}$, if

$$\lim_{x\to\infty} \frac{1}{\mathcal{B}(x)} \sum_{\substack{h(m)\le x \\ m\in\mathcal{B}}} 1 = A, \quad \text{where} \quad A > 0.$$

In this short paper we shall consider the case when $\mathcal{B}$ is the set of shifted primes.

2. Formulation of the theorem

Let $\mathcal{P}$ be the set of primes, g be a multiplicative function, $g(n) > 0$ for every $n \in \mathbb{N}$. Let $f(n) = \log g(n)$.

Conditions:

(C1): There exists a constant $B > 0$ for which

(2.1) $$\frac{1}{(\log\log n)^B} \le g(n) \le (\log\log n)^B \quad \text{if} \quad n > n_0,\ n_0 \in \mathbb{N}.$$

(C2):

(2.2) $$f(q^r)(\log q^r)^C \to 0 \quad (q^r \to \infty,\ q \in \mathcal{P})$$

holds for every fixed C.

Let

$$S(x) := \#\{p \in \mathcal{P} \mid (p+1)g(p+1) \le x\}.$$

Theorem 1. *Under the conditions* $(C1)$ *and* $(C2)$*, we have*

$$\lim_{x\to\infty} \frac{S(x)}{\pi(x)} = C_g,$$

where

$$C_g = \prod_{q\in\mathcal{P}} \xi_q,$$

$$\xi_2 = \sum_{\alpha=1}^{\infty} \frac{1}{2^\alpha g(2^\alpha)}, \quad \xi_q = \left(1 - \frac{1}{q-1}\right)\left(1 + \sum_{l=1}^{\infty} \frac{q-1}{(q-2)q^l g(q^l)}\right) \quad \textit{if} \quad q > 2.$$

Here q runs over the set of primes.

Remarks. **1)** The conditions (C1) and (C2) hold in particular for $g(n) = \left(\frac{\varphi(n)}{n}\right)^\lambda, \left(\frac{\sigma(n)}{n}\right)^\lambda$, where φ is Euler's totient function and σ the sum of divisors functions, with $\lambda \in \mathbb{R}$, and for many other functions.

2) We shall use the Bombieri–Vinogradov inequality. A weaker version is enough for our purpose. With the same method we could prove a more general theorem.

Theorem 2. *Let $a_1, \dots, a_k$ be distinct non zero integers, $g_1, \dots, g_k$ be multiplicative functions for which the conditions $(C1)$ and $(C2)$ hold. Let*

$$s(n) = g_1(n + a_1) \cdots g_k(n + a_k).$$

Then

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{ps(p)\le x} 1 = C_s, \quad C_s \ne 0.$$

We shall not prove this assertion.

3) We are unable to prove that

$$\lim_{x\to\infty} \frac{1}{\pi\left(\frac{x}{\log x}\right)} \sum_{p\tau(p+1)\le x} 1 = C, \quad C \ne 0, \tag{2.3}$$

where $\tau(n)$ is the number of divisors of n.

3. Lemma

Let $\mathrm{Li}\,(x) := \int_2^x \frac{dt}{\log t}$ and $\pi(u, m, l) := \#\{p \le u \mid p \equiv l \pmod m\}$.

Lemma 1. (Bombieri–Vinogradov) *Let $\delta > 0$ be fixed, A be an arbitrary positive constant. Then*

$$\sum_{m\le X^{1/2-\delta}} \max_{(l,m)=1} \max_{u\le X} \left|\pi(u, m, l) - \frac{\mathrm{Li}\,(u)}{\varphi(m)}\right| \le C(\delta) \frac{X}{(\log X)^A}. \tag{3.1}$$

(See Theorem 17.6 in H. Iwaniec and E. Kowalski [6]).

4. Proof of Theorem 1

I. Let

$$y=\frac{1}{16}\log x \quad \text{and} \quad Q=\prod_{\substack{2<\pi\le y\\ \pi\in\mathcal{P}}}\pi.$$

Let $\mathcal{P}_{\alpha,D}$ be the set of those primes p for which $p+1\equiv 0 \pmod{2^\alpha D}$, and $\left(\frac{p+1}{2^\alpha D},2Q\right)=1$.

We shall consider only those D's all prime factors of which divide Q.

II. Let $w=2(\log\log x)^B$. From (C1) we have

$$\text{if } \ (p+1)g(p+1)\le x, \ \text{then} \quad p\le 2xw,$$

$$\text{if } \ (p+1)g(p+1)> x, \ \text{then} \quad p>\frac{x}{2w}.$$

III. Let

$$\Omega_Q(p+1)=\sum_{\substack{p^r\mid p+1\\ q\mid Q}}1.$$

Then, with a suitable constant c,

$$\begin{aligned}\sum_{p\le 2xw}\Omega_Q(p+1)&\le\sum_{q\mid Q}\sum_{r\ge 1}\pi(xw,q^r,-1)\le\\ &\le\frac{cxw}{\log x}\sum_{\substack{r\ge 1\\ q\mid Q}}\frac{1}{\varphi(q^r)}\le\frac{3Cxw\log\log y}{\log x}.\end{aligned}\tag{4.1}$$

Here we used the Brun–Titchmarsh inequality. Thus

$$\#\left\{p\le xw \mid \Omega_Q(p+1)>w(\log\log y)^2\right\}=o(\pi(x)).\tag{4.2}$$

Let $\mathcal{R}^{(1)}$ be the set of primes listed in (4.2). For the other primes p,

$$\begin{aligned}\prod_{\substack{p^r\mid p+1\\ q\mid Q}}q^r&\le y^{w(\log\log y)^2}=\exp(w(\log y)(\log\log y)^2)\le\\ &\le\exp\left(c_2(\log\log y)^{B+1}\right)=:T_x.\end{aligned}\tag{4.3}$$

Similarly,

$$\pi\left(xw,2^\nu,-1\right)\le\frac{cxw}{2^\nu\log x} \quad \text{if} \quad 2^\nu\le\sqrt{x}.\tag{4.4}$$

Let ν_x be such an integer for which $2^{\nu_x} \le w^2 < 2^{\nu_x+1}$. Let

$$\mathcal{R}^{(2)} = \{p \mid p \le xw,\ 2^{\nu_x} \mid p+1\}.$$

Then

$$\mathcal{R}^{(2)} = o\big(\pi(x)\big).$$

IV. Let

$$f_y(p+1) = \sum_{\substack{q^r \ge y \\ q^r \mid p+1}} f(q^r).$$

By using the Brun–Titchmarsh inequality, we have

$$\sum_{p \le xw} |f_y(p+1)| \le \frac{cxw}{\log x} \sum_{y < q^r \le \sqrt{x}} \frac{|f(q^r)|}{q} + O(x^{\frac{4}{5}}) \le \frac{cxw}{\log x} \frac{1}{(\log\log x)^C},$$

where C is an arbitrary constant and c depends on C. Consequently

$$\frac{1}{\pi(x)} \#\{p \le xw \mid |f_y(p+1)| > (\log\log x)^{-C'}\} \to 0 \quad \text{as} \quad x \to \infty. \tag{4.5}$$

C' is an arbitrary large constant.

Let $\mathcal{R}^{(3)}$ be the set of prime counted in (4.5). Then $\#\mathcal{R}^{(3)} = o(\pi(x))$.

V. Let

$$\Pi\big(X, 2^\alpha D\big) = \#\big\{p \le X \mid p \in \mathcal{P}_{\alpha,D}\big\}. \tag{4.6}$$

It is clear that

$$\Pi\big(X, 2^\alpha D\big) = \sum_{\delta \mid 2Q} \mu(\delta) \pi\big(X, 2^\alpha D\delta, -1\big) \tag{4.7}$$

We are interested only on those D which are smaller than T_x and those α which are less than ν_x.

From Lemma 1 we can deduce that

$$\Pi\big(X, 2^\alpha D\big) = \big(\mathrm{Li}\,(X)\big) \sum_{\delta \mid 2Q} \frac{\mu(\delta)}{\varphi(2^\alpha D\delta)} + O\left(\frac{X}{(\log X)^A}\right),$$

where A is arbitrary large. Thus

$$\Pi\big(X, 2^\alpha D\big) = \big(\mathrm{Li}\,(X)\big) \frac{\varrho(D)}{2^\alpha D} S_Q,$$

where ϱ is the multiplicative function defined on prime powers π^l by

(4.8)
$$\varrho\left(\pi^{\ell}\right)=\varrho(\pi)=\begin{cases}\dfrac{\pi-1}{\pi-2} & \text{if } \pi \mid Q\\ 1 & \text{if } \pi\nmid Q,\end{cases}$$

(4.9)
$$S_Q:=\prod_{\pi\mid Q}\frac{1}{\varrho(\pi)}.$$

For every $p\in\mathcal{P}_{\alpha,D}$, let

$$\kappa_y(p+1):=\frac{g(p+1)}{g(2^\alpha D)}.$$

If $p\notin\mathcal{R}^{(3)}$, then $|\kappa_y(p+1)-1|\le\dfrac{2}{(\log\log x)^{C'}}$, ($C'$ is arbitrary large), therefore the number of primes $p\in\mathcal{P}_{\alpha,D}$ for which $p\in\mathcal{P}_{\alpha,D}$ is in between

$$\Pi\left(\frac{x\pm\frac{2x}{(\log\log x)^{C'}}}{g(2^\alpha)g(D)}\Big|2^\alpha D\right).$$

VI. Let

(4.10)
$$R=\sum_{\substack{1\le\alpha\le\nu_x\\ D\le T_x}}\max_{u\le xw}\left|\Pi\big(u|2^\alpha D\big)-\frac{\varrho(D)}{2^\alpha D}S_Q\mathrm{Li}\,(u)\right|.$$

Starting from (4.7) and Lemma 1, and letting τ stand for the number of divisors function, we have

$$R\le\sum_{1\le\alpha\le\nu_x}\sum_{k\le T_xQ}\tau(k)\Big\{\max_{u\le xw}\left|\Pi(u,2^\alpha k,-1)-\frac{\mathrm{Li}\,(u)}{\varphi(2^\alpha k)}\right|+$$
$$+\max_{u\le xw}\left|\Pi(u,2^{\alpha+1}k,-1)-\frac{\mathrm{Li}\,(u)}{\varphi(2^{\alpha+1}k)}\right|\Big\}=\Sigma_1+\Sigma_2.$$

Here we observed that for odd k the number of those D,δ for which $D\delta=k$ is at most $\tau(k)$. Since δ is either odd, or $2|\delta$, therefore $D\delta=2k$ holds for at most $\tau(k)$ distinct cases. In Σ_1 we sum over those k for which $\tau(k)\le(\log x)^E$, and in Σ_2 over those k for which $\tau(k)>(\log x)^E$. Here E is an appropriate large constant. From Lemma 1,

$$\Sigma_1=O\left(\frac{x}{(\log x)^{C_1}}\right),$$

C_1 is arbitrary large.

Since

$$\Sigma_2 \le \frac{c}{(\log x)^E} \sum_{\alpha \le \nu_x} \sum_{k \le T_x Q} \frac{\tau^2(k)\mathrm{Li}\,(x)}{\varphi(2^\alpha k)} \le \frac{cx}{(\log x)^{E+1}} \sum_{k \le T_x Q} \frac{\tau^2(k)}{\varphi(k)},$$

we have

$$\sum_{k \le T_x Q} \frac{\tau^2(k)}{\varphi(k)} \le \prod_{\pi | Q} \Big(1+\frac{2^2}{\pi-1}+\frac{3^2}{\pi(\pi-1)}+\cdots\Big) \le c\exp(4\log\log y) \le c_1(\log y)^4,$$

therefore

$$\Sigma_2 = o\big(\pi(x)\big).$$

Consequently

$$R = o\big(\pi(x)\big).$$

It remains to estimate

$$U(x) := \sum_{\substack{\alpha \le \nu_x \\ D \le T_x}} \Pi\left(\frac{x}{g(2^\alpha D)}\Big| 2^\alpha D\right).$$

We have

$$U(x) = \sum_{\substack{1 \le \alpha \le \nu_x \\ D \le T_x}} \mathrm{Li}\left(\frac{x}{g(2^\alpha D)}\right) \frac{\varrho(D)}{2^\alpha D} S_Q + o\big(\pi(x)\big).$$

Since

$$\mathrm{Li}\left(\frac{x}{g(2^\alpha D)}\right) = \big(1+o_x(1)\big)\frac{\mathrm{Li}\,(x)}{g(2^\alpha D)},$$

we easily obtain that

$$\lim \frac{U(x)}{\mathrm{Li}\,(x)} = \Big(\sum_{\alpha=1}^{\infty} \frac{1}{2^\alpha g(2^\alpha)}\Big) \prod_{\substack{q\in\mathcal{P} \\ q\ne 2}} \left(1-\frac{1}{q-1}\right)\left(1+\sum_{l=1}^{\infty}\frac{q-1}{q-2}\frac{1}{q^l g(q^l)}\right).$$

The product on the right hand side is clearly convergent, since

$$1-g(q) = 1-e^{f(q)} = -f(q)+O(f^2(q)) = O\left(\frac{1}{(\log q)^2}\right).$$

Thus

$$\lim \frac{U(x)}{\mathrm{Li}\,(x)} = C_g.$$

Since

$$|S(x)-U(x)| \le \left|U\Big(x+\frac{2x}{(\log\log x)^C}\Big) - U\Big(x-\frac{2x}{(\log\log x)^C}\Big)\right| + o\big(\pi(x)\big),$$

the theorem follows. ∎

5. Final remark

Theorem 3. *Under the conditions of Theorem 1,*

$$\frac{1}{\mathrm{Li}\,(x)}\sum_{p\le x}\frac{1}{g(p+1)}\to C_g.$$

This can be proved easily, applying the method of proof of Theorem 1.

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