A REMARK ON UNIFORMLY DISTRIBUTED FUNCTIONS

Karl-Heinz Indlekofer* (Paderborn, Germany)
Imre Kátai* (Budapest, Hungary)
Oleg I. Klesov* (Kiev, Ukraine)
Bui Minh Phong* (Budapest, Hungary)

Dedicated to the memory of Professor János Galambos

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Abstract. A sufficient condition is given for which a function is uniformly distributed.

1. Introduction

Let, as usual, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ be the set of positive integers, integers and real numbers, respectively.

Let $g(n) \in (0, \infty)$,

$$S(x) := \sum_{ng(n) \le x} 1. \tag{1.1}$$

Our purpose in this short paper is to give quite general condition for $g(n)$ which guarantees that

$$\lim_{x \to \infty} \frac{S(x)}{x} = C, \tag{1.2}$$

where C is a positive constant.

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Theorem. *Assume that g has a continuous limit distribution F, for which*

$$\int_0^\infty \frac{F(u)}{u^2}\,du < \infty \tag{1.3}$$

and that

$$F_x(u) = \frac{1}{x}\sharp\{n \le x | g(n) < u\} = F(u) + O\left(\frac{1}{(\log\log x)^2}\right)$$

uniformly in $u \in \mathbb{R}$. Let furthermore

$$g(n)\log\log n \ge C\ (>0) \quad \text{if} \quad n \ge 10. \tag{1.4}$$

Then (1.2) holds with $C = \int_0^\infty \frac{F(u)}{u^2}du$.

2. Proof of the theorem

From (1.3) we deduce that $\lim_{u\to 0}\frac{F(u)}{u} = 0$.

We have

$$\frac{1}{x}\sum_{n\le x}\frac{1}{g(n)} = \int_0^\infty \frac{F_x(u)}{u^2}du \to \int_0^\infty \frac{F(u)}{u^2}du \quad \text{as} \quad x \to \infty.$$

Let $\epsilon > 0$. Then

$$F_x(u) \le \frac{1}{x}\sum_{\substack{n\le x\\ g(n)\le u}}\frac{u}{g(n)} = u\cdot\frac{1}{x}\sum_{\substack{n\le x\\ \frac{1}{g(n)}\ge\frac{1}{u}}}\frac{1}{g(n)} \le u\epsilon,$$

whenever $u \le u_0(\epsilon)$, and so $F_x(u) = o(u)$ as $u \to 0$.

Thus

$$\lim_{u\to 0}\frac{1}{u}\sup_{x\ge 1}\left(\frac{1}{x}\sum_{\substack{n\le x\\ \frac{1}{g(n)}\ge\frac{1}{u}}}\frac{1}{g(u)}\right) = \lim_{u\to 0}\sup_{x\ge 1}\frac{F_x(u)}{u} = 0,$$

from which it follows that

$$\lim_{u\to 0}\frac{F(u)}{u} = 0.$$

It is clear that

$$\sup_{\delta_x\le\delta<1}\left|\frac{1}{\delta x}[(x+\delta x)F_{x+\delta x}(u) - xF_x(u)] - F(u)\right| \to 0 \quad \text{as} \quad x \to \infty, \tag{2.1}$$

where $\delta_x \to 0$, appropriately. That is, there exists such a δ_x tending to zero, for which (2.1) holds.

From (2.1) we obtain

$$\sup_{u\in\mathbb{R}} \sup_{\delta_x \le v<1} \left| \frac{1}{vx} \sharp\{n \in [x, x+vx], g(n) < u\} - F(u) \right| \to 0 \quad \text{as} \quad x \to \infty. \tag{2.2}$$

Let

$$\mathcal{B}_1 = \{n \in \mathbb{N} | g(n) < 1\} \quad \text{and} \quad \mathcal{B}_2 = \{n \in \mathbb{N} | g(n) \ge 1\},$$

$$\Sigma_1 = \sum_{\substack{ng(n)\le x \\ n\in\mathcal{B}_1}} 1 \quad \text{and} \quad \Sigma_2 = \sum_{\substack{ng(n)\le x \\ n\in\mathcal{B}_2}} 1.$$

Estimation of Σ_2. If $F(1) = 1$, then $\Sigma_2 = o(x)$. Assume that $F(1) < 1$. Let $y_\nu = 1 + \nu\delta$ $(\nu = 0, \dots, \kappa_\delta)$, where $\kappa_\delta\delta \to \infty$, arbitrarily slowly. Let

$$I_\nu = [y_\nu, y_{\nu+1}).$$

We have

$$\begin{aligned}
\frac{1}{x}\Sigma_2 &= \sum_{\nu=0}^{\kappa_\delta} \frac{1}{x} \sum_{\substack{g(n)\in I_\nu \\ ng(n)<x}} 1 + O\left(\frac{1}{x} \sum_{n<\frac{x}{y_{\kappa_\delta}}} 1 \right) = \\
&= \sum_{\nu=0}^{\kappa_\delta} \frac{1}{x} \Sigma^{(\nu)} + O\left(\frac{1}{y_{\kappa_\delta}} \right).
\end{aligned}$$

Since

$$\Sigma^{(\nu)} = \sum_{\substack{g(n)\in I_\nu \\ ng(n)<x}} \le \sum_{\substack{g(n)\in I_\nu \\ n<\frac{x}{y_\nu}}} 1 \quad \text{and} \quad \Sigma^{(\nu)} \ge \sum_{\substack{g(n)\in I_\nu \\ n<\frac{x}{y_{\nu+1}}}} 1,$$

we obtain

$$\frac{1}{x}\Sigma_2 = \sum_{\nu=0}^{\kappa_\delta} \frac{F(y_{\nu+1}) - F(y_\nu)}{y_\nu} + O\left(\delta \sum_{\nu=0}^{\kappa_\delta} \frac{F(y_{\nu+1}) - F(y_\nu)}{y_\nu y_{\nu+1}} \right) + o_x(1).$$

Let us observe that the first error term is $O(\delta)$.

The first sum on the right can be rewritten as

$$\begin{aligned}
&-\frac{F(y_0)}{y_0} + F(y_1)\left(\frac{1}{y_0} - \frac{1}{y_1} \right) + F(y_2)\left(\frac{1}{y_1} - \frac{1}{y_2} \right) + \cdots = \\
&= -F(1) + \delta \sum_{h=1}^{\kappa_\delta} \frac{F(1+\delta h)}{\big(1+\delta(h-1)\big)(1+\delta h)} + o_x(1).
\end{aligned}$$

Consequently,

$$\lim_{x\to\infty} \frac{1}{x}\Sigma_2 = -F(1) + \int_1^\infty \frac{F(u)}{u^2}du.$$

Estimation of Σ_1. If $F(1) = 0$, then $\lim_{x\to\infty} \frac{1}{x}\Sigma_1 = 0$.

Assume that $F(1) > 0$. Let

$$M = [\log\log x] + 1 \quad \text{and} \quad \mathcal{H}_k = \left[\frac{k}{M}, \frac{k+1}{M}\right].$$

Since

$$\sum_{\substack{g(n)\in\mathcal{H}_k \\ ng(n)\le x}} 1$$

is in between

$$\sum_{\substack{n\le \frac{xM}{k+1} \\ n\in\mathcal{H}_k}} 1 \quad \text{and} \quad \sum_{\substack{n\le \frac{xM}{k} \\ n\in\mathcal{H}_k}} 1,$$

it is at most

$$\left(F\left(\frac{k+1}{M}\right) - F\left(\frac{k}{M}\right)\right)\frac{xM}{k} + O\left(\frac{x}{(\log\log x)^2}\right)$$

and at least

$$\left(F\left(\frac{k+1}{M}\right) - F\left(\frac{k}{M}\right)\right)\frac{xM}{k+1} + O\left(\frac{x}{(\log\log x)^2}\right),$$

therefore

$$\frac{1}{x}\Sigma_1 = \frac{1}{x}\sum_{\substack{g(n)<\frac{1}{M} \\ ng(n)\le x}} 1 + \sum_{k=1}^{M-1}\left(F\left(\frac{k+1}{M}\right) - F\left(\frac{k}{M}\right)\right)\frac{M}{k} +$$
$$+ O(1)\sum_{k=1}^{M-1}\left(F\left(\frac{k+1}{M}\right) - F\left(\frac{k}{M}\right)\right)\frac{M}{k^2} + o_x(1).$$

The first sum on the right hand side is empty, whereas the second sum can be rewritten as

$$-F\left(\frac{1}{M}\right)M + M\sum_{k=2}^{M-1}\frac{F(\frac{k}{M})}{(k-1)k} + \frac{F(1)M}{M-1} = \int_0^1 \frac{F(u)}{u^2}du + F(1) + o_x(1).$$

Finally one can see that the last error term is

$$<< M \sum_{k=2}^{M} \frac{F(\frac{k}{M})}{k^3} = \frac{1}{M} \sum_{k=2}^{M} \frac{1}{k} \frac{F(\frac{k}{M})}{(\frac{k}{M})^2} = o_x(1).$$

Consequently,

$$\lim_{x\to\infty} \frac{1}{x}\Sigma_1 = \int_0^1 \frac{F(u)}{u^2} du + F(1),$$

thereby completing the proof of the theorem. ■

3. Remarks

1) The conditions stated for $g(n)$ are satisfied for many functions, namely for

$$g(n) = \left(\frac{\varphi(n+a_1)}{n+a_1}\right)^{k_1} \cdots \left(\frac{\varphi(n+a_\ell)}{n+a_\ell}\right)^{k_\ell},$$

where $a_1, \cdots, a_\ell \in \mathbb{Z}, k_1, \cdots, k_\ell \in \mathbb{Z}$, and for

$$g(n) = \prod_{j=1}^{\ell} \left(\frac{\sigma(n+a_j)}{n+a_j}\right)^{k_j}.$$

Here φ is the Euler function and σ is the sum of divisors function.

2) We note that according to the theorem of H. G. Diamond [1]

$$\frac{1}{x} \sum_{\frac{\varphi(n)}{n}<a} = F(a) + O\left(\frac{1}{\log x}\right),$$

where F is a distribution function.

References

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K.-H. Indlekofer
Faculty of Computer Science
Electrical Engineering and Mathematics
University of Paderborn
D-33098 Paderborn, Warburger Str. 100
Germany
k-heinz@math.uni-paderborn.de

O.I. Klesov
Department of Mathematical Analysis and Probability Theory
National Technical University of Ukraine (KPI)
Peremogy Avenue, 37
02056 Kiev
Ukraine
klesov@matan.kpi.ua

I. Kátai and B.M. Phong
Department of Computer Algebra
Faculty of Informatics
Eötvös Loránd University
H-1117 Budapest, Pázmány Péter sétány 1/C
Hungary
katai@compalg.inf.elte.hu
bui@compalg.inf.elte.hu