ON THE MIDDLE DIVISORS OF AN INTEGER

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Dedicated to the memory of Professor János Galambos

Abstract. Given a positive integer n, let $\rho_1(n) = \max\{d \mid n : d \le \sqrt{n}\}$ and $\rho_2(n) = \min\{d \mid n : d \ge \sqrt{n}\}$ stand for the middle divisors of n. We obtain improvements and new estimates for sums involving these two functions.

1. Introduction

Given a positive integer n, we define the numbers $\rho_1(n)$ and $\rho_2(n)$ as

$$\rho_1(n) := \max\{d \mid n : d \le \sqrt{n}\}$$
$$\rho_2(n) := \min\{d \mid n : d \ge \sqrt{n}\}$$

and call them the *middle divisors* of n. It is clear that $\rho_1(n)\rho_2(n) = n$ and also that if n is not a perfect square, then $\rho_1(n) < \rho_2(n)$.

In 1976, Tenenbaum [5] proved that

$$(1.1) \qquad \sum_{n \le x} \rho_2(n) = \frac{\pi^2}{12} \frac{x^2}{\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right)$$

and that, given any $\varepsilon > 0$, there exists $x_0 = x_0(\varepsilon)$ such that for all $x \geq x_0$,

$$\frac{x^{3/2}}{(\log x)^{\delta+\varepsilon}} < \sum_{n \leq x} \rho_1(n) \ll \frac{x^{3/2}}{(\log x)^{\delta} (\log \log x)^{1/2}},$$

where

$$(1.2) \qquad \delta = 1 - \frac{1 + \log \log 2}{\log 2} \approx 0.086071.$$

More recently, Ford [1] showed that

$$(1.3) \qquad \sum_{n \leq x} \rho_1(n) \asymp \frac{x^{3/2}}{(\log x)^{\delta} (\log \log x)^{3/2}}.$$

Here, we provide a refinement and a generalisation of (1.1) as well as a generalisation of (1.3), and we then use these results to obtain estimates for $\sum_{n \leq x} \rho_2(n)/\rho_1(n)^r$, for every fixed real $r > -1$, and for $\sum_{n \leq x} \rho_1(n)/\rho_2(n)$, thereby improving an earlier estimate by Roesler [4] in the case of the second sum.

2. Main theorems

Theorem 1. *Let $a > 0$ be a real number. Then, for each positive integer k,*

$$\sum_{n \leq x} \rho_2(n)^a = c_0 \frac{x^{a+1}}{\log x} + c_1 \frac{x^{a+1}}{\log^2 x} + \cdots + c_{k-1} \frac{x^{a+1}}{\log^k x} + O\left(\frac{x^{a+1}}{\log^{k+1} x}\right)$$

where, for $\ell = 0, 1, \ldots, k - 1$,

$$c_\ell = c_\ell(a) = \frac{\ell!}{(a+1)^{\ell+1}} \sum_{j=0}^{\ell} \frac{(a+1)^j (-1)^j \zeta^{(j)}(a+1)}{j!}$$

with ζ standing for the Riemann zeta function.

Theorem 2. *Let $a > 0$ be a real number and let δ be as in (1.2). Then,*

$$(2.1) \qquad \sum_{n \leq x} \rho_1(n)^a \asymp \frac{x^{\frac{a+2}{2}}}{(\log x)^{\delta} (\log \log x)^{3/2}}.$$

Theorem 3. *Given any integer $k \geq 1$ and any real number $r > -1$, we have*

$$\sum_{n \leq x} \frac{\rho_2(n)}{\rho_1(n)^r} = e_0 \frac{x^2}{\log x} + e_1 \frac{x^2}{\log^2 x} + \cdots + e_{k-1} \frac{x^2}{\log^k x} + O\left(\frac{x^2}{\log^{k+1} x}\right)$$

where $e_0 = \dfrac{\zeta(r+2)}{2}$ *and for each* $1 \le \ell \le k-1$,

$$e_\ell = \left(\frac{r+2}{2}\right) c_\ell + \sum_{\nu=0}^{\ell-1} \frac{rc_\nu}{2} \prod_{m=\nu}^{\ell-1} \left(\frac{m+1}{2}\right),$$

with, for each $\nu = 0, 1, \ldots, \ell$,

$$c_\nu = \frac{\nu!}{(r+2)^{\nu+1}} \sum_{j=0}^{\nu} \frac{(r+2)^j (-1)^j \zeta^{(j)}(r+2)}{j!}.$$

Remark. Interestingly, as a consequence of Theorem 3,

$$T_r(x) := \sum_{n \le x} \frac{\rho_2(n)}{\rho_1(n)^r} \sim \frac{\zeta(r+2)}{2} \frac{x^2}{\log x} \qquad \text{as } x \to \infty,$$

implying that all sums $T_r(x)$ are of the same order, independently of the chosen number $r > -1$. For instance, although it may at first appear counterintuitive, we do have that $\displaystyle\sum_{n \le x} \rho_2(n)\sqrt{\rho_1(n)} \asymp \sum_{n \le x} \frac{\rho_2(n)}{\sqrt{\rho_1(n)}}$.

Theorem 4. *With δ as in* (1.2), *we have*

$$\sum_{n \le x} \frac{\rho_1(n)}{\rho_2(n)} \asymp \frac{x}{(\log x)^\delta (\log \log x)^{3/2}}.$$

3.   Preliminary results

Let $\pi(x)$ stand for the number of primes not exceeding x and let

$$\mathrm{Li}(x) := \int_2^x \frac{dt}{\log t}.$$

We will be using the prime number theorem with an error term which is sufficient for our purposes, namely the original one found by de la Vallée Poussin [6] in 1899.

Proposition 1. (Prime number theorem.) *There exists a positive constant C such that*

$$\pi(x) - \mathrm{Li}(x) = O\left(x \exp\left\{-C\sqrt{\log x}\right\}\right).$$

Lemma 1. *Assume that $n \le x$ with $\rho_2(n) > x^{2/3}$. Then, $\rho_2(n)$ is a prime.*

Proof. Since $\rho_2(n) > x^{2/3}$, we have that $\rho_1(n) < x^{1/3}$. Set $m = \rho_2(n)$. It is clear that both $\rho_1(m)$ and $\rho_2(m)$ are divisors of n. Hence, in order to prove that $\rho_2(n)$ is prime, it is sufficient to prove that $\rho_2(m) = m$. Now, since $\rho_2(m) \geq \sqrt{m} = \sqrt{\rho_2(n)} > x^{1/3} > \rho_1(n)$, it follows hat $\rho_1(n) < \rho_2(m) \leq \rho_2(n)$, which implies, by the definition of $\rho_1(n)$ and $\rho_2(n)$ that $\rho_2(m) = \rho_2(n) = m$, thus proving our claim. ∎

The following result is not new. We include it here for the sake of completeness.

Lemma 2. *Given any fixed real number* $a > 0$,

$$(3.1) \qquad S(x) = S_a(x) := \sum_{p \leq x} p^a = \int_2^x \frac{t^a}{\log t} dt + O\left(\frac{x^{a+1}}{e^{C\sqrt{\log x}}}\right).$$

Proof. Using partial summation with $A(x) = \sum_{n \leq x} a(n) = \pi(x)$ and $\varphi(t) = t^a$, we have

$$(3.2) \qquad S(x) = x^a \pi(x) - \int_2^x at^{a-1}\pi(t)\,dt.$$

Using Proposition 1, it follows from (3.2) and integration by parts that

$$S(x) = x^a\pi(x) - a\int_2^x t^{a-1}\left(\mathrm{Li}(t) + O(te^{-C\sqrt{\log t}})\right)dt =$$

$$= x^a\pi(x) - a\int_2^x t^{a-1}\mathrm{Li}(t)\,dt + O\left(\int_2^x t^a e^{-C\sqrt{\log t}}\,dt\right) =$$

$$= x^a\pi(x) - a\left(\frac{t^a}{a}\mathrm{Li}(t)\Big|_2^x - \int_2^x \frac{t^a}{a}\frac{1}{\log t}\,dt\right) + O\left(\frac{x^{a+1}}{e^{C\sqrt{\log x}}}\right) =$$

$$(3.3) \qquad = x^a\pi(x) - x^a\mathrm{Li}(x) + \int_2^x \frac{t^a}{\log t}\,dt + O\left(\frac{x^{a+1}}{e^{C\sqrt{\log x}}}\right).$$

Using Proposition 1 one more time, we have that

$$x^a\pi(x) - x^a\mathrm{Li}(x) = x^a(\pi(x) - \mathrm{Li}(x)) = O\left(\frac{x^{a+1}}{e^{C\sqrt{\log x}}}\right),$$

which substituted in (3.3) completes the proof of (3.1). ∎

Lemma 3. *Let $a > 0$ be an arbitrary real number. Then,*

$$(3.4) \qquad \sum_{\sqrt{x} < p \leq x} p^a \left\lfloor \frac{x}{p} \right\rfloor = \int_{\sqrt{x}}^{x} \frac{t^a}{\log t} \left\lfloor \frac{x}{t} \right\rfloor dt + O\left(\frac{x^{a+1}}{e^{\frac{C}{2}\sqrt{\log x}}} \right).$$

Proof. We follow an approach used by Naslund [3] to estimate a similar sum. Let B be a positive integer. Then,

$$\sum_{x/B < p \leq x} p^a \left\lfloor \frac{x}{p} \right\rfloor = \sum_{n \leq B-1} n \sum_{x/(n+1) < p \leq x/n} p^a =$$

$$= \sum_{n \leq B-1} n(S(x/n) - S(x/(n+1))) =$$

$$= S(x) + S(x/2) + \cdots + S(x/(B-1)) - (B-1)S(x/B) =$$

$$= \sum_{n \leq B-1} (S(x/n) - S(x/B)).$$

Using Lemma 2 in this last estimate, we obtain, provided that $B \geq x^{1/4}$,

$$\sum_{x/B < p \leq x} p^a \left\lfloor \frac{x}{p} \right\rfloor = \sum_{n \leq B-1} \int_{x/B}^{x/n} \frac{t^a}{\log t} dt + O\left(\sum_{n \leq B-1} \frac{(x/n)^{a+1}}{e^{C\sqrt{\log(x/n)}}} \right) =$$

$$= \int_{x/B}^{x} \frac{t^a}{\log t} \left\lfloor \frac{x}{t} \right\rfloor dt + O\left(\frac{x^{a+1}}{e^{C\frac{1}{2}\sqrt{\log x}}} \sum_{n=1}^{\infty} \frac{1}{n^{a+1}} \right).$$

Choosing $B = \lfloor \sqrt{x} \rfloor$ allows us to write this last equation as

$$(3.5) \qquad \sum_{\sqrt{x} < p \leq x} p^a \left\lfloor \frac{x}{p} \right\rfloor = \int_{\sqrt{x}}^{x} \frac{t^a}{\log t} \left\lfloor \frac{x}{t} \right\rfloor dt + O\left(\frac{x^{a+1}}{e^{\frac{C}{2}\sqrt{\log x}}} \right),$$

thereby completing the proof of (3.4). ∎

Lemma 4. *Let $a > 0$ be an arbitrary real number. Then,*

$$(3.6) \qquad \sum_{p \leq x} p^a \left\lfloor \frac{x}{p} \right\rfloor = \int_{2}^{x} \frac{t^a}{\log t} \left\lfloor \frac{x}{t} \right\rfloor dt + O\left(\frac{x^{a+1}}{e^{\frac{C}{2}\sqrt{\log x}}} \right).$$

Proof. Since the two quantities $\sum_{p \leq \sqrt{x}} p^a \left\lfloor \frac{x}{p} \right\rfloor$ and $\int_{2}^{\sqrt{x}} \frac{t^a}{\log t} \left\lfloor \frac{x}{t} \right\rfloor dt$ are each of smaller order than the error term appearing in (3.5), we may indeed conclude from (3.5) that (3.6) holds. ∎

Lemma 5. *For all $s > 1$ and for each integer $k \geq 1$,*

$$\zeta^{(k)}(s) = (-1)^k \sum_{n=1}^{\infty} \frac{\log^k n}{n^s}.$$

Proof.   Differentiating k times with respect to s both sides of equation $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ yields the result.   ∎

Lemma 6. *Let $a > 0$ be an arbitrary real number. Then, for each integer $k \geq 1$,*

$$\int_{2}^{x} \frac{t^a \lfloor x/t \rfloor}{\log t} dt = c_0 \frac{x^{a+1}}{\log x} + c_1 \frac{x^{a+1}}{\log^2 x} + \cdots + c_{k-1} \frac{x^{a+1}}{\log^k x} + O\left(\frac{x^{a+1}}{\log^{k+1} x}\right),$$

where

$$c_\ell = c_\ell(a) = \frac{\ell!}{(a+1)^{\ell+1}} \sum_{j=0}^{\ell} \frac{(a+1)^j (-1)^j \zeta^{(j)}(a+1)}{j!}.$$

Proof. We use the same technique that Naslund [2] used to estimate a similar integral. With the change of variable $t = x/u$, we obtain

$$\nu_a(x) \quad := \quad \int_{2}^{x} \frac{t^a \lfloor x/t \rfloor}{\log t} dt = x^{a+1} \int_{1}^{x/2} \frac{\lfloor u \rfloor}{u^{a+2} \log\left(\frac{x}{u}\right)} du =$$

$$= \quad \frac{x^{a+1}}{\log x} \int_{1}^{x/2} \frac{\lfloor u \rfloor}{u^{a+2}} \left(1 - \frac{\log u}{\log x}\right)^{-1} du.$$

Since $1 \leq u \leq x/2$, we have $\dfrac{\log u}{\log x} < 1$. We can therefore write that for each integer $k \geq 1$,

$$\left(1 - \frac{\log u}{\log x}\right)^{-1} = 1 + \frac{\log u}{\log x} + \cdots + \left(\frac{\log u}{\log x}\right)^{k-1} + \left(\frac{\log u}{\log x}\right)^{k} \left(1 - \frac{\log u}{\log x}\right)^{-1}.$$

From this, it follows that

$$\nu_a(x) \quad = \quad \frac{x^{a+1}}{\log x} \sum_{\ell=0}^{k-1} \frac{1}{\log^\ell x} \int_{1}^{x/2} \frac{\lfloor u \rfloor}{u^{a+2}} \log^\ell u \, du +$$

$$+ \quad \frac{x^{a+1}}{\log^{k+1} x} \int_{1}^{x/2} \frac{\lfloor u \rfloor}{u^{a+2}} \log^{k+1} u \left(1 - \frac{\log u}{\log x}\right)^{-1} du.$$

Since the integral $\displaystyle\int\limits_{1}^{x/2} \frac{\lfloor u\rfloor}{u^{a+2}} \log^{k+1} u \left(1 - \frac{\log u}{\log x}\right)^{-1} du$ converges, we have that

$$\nu_a(x) = \frac{x^{a+1}}{\log x} \sum_{\ell=0}^{k-1} \frac{1}{\log^\ell x} \int\limits_{1}^{x/2} \frac{\lfloor u\rfloor}{u^{a+2}} \log^\ell u \, du + O\left(\frac{x^{a+1}}{\log^{k+1} x}\right) =$$

$$= \frac{x^{a+1}}{\log x} \sum_{\ell=0}^{k-1} \frac{1}{\log^\ell x} \left(\int\limits_{1}^{\infty} \frac{\lfloor u\rfloor}{u^{a+2}} \log^\ell u \, du - \int\limits_{x/2}^{\infty} \frac{\lfloor u\rfloor}{u^{a+2}} \log^\ell u \, du\right) +$$

(3.7) $$+O\left(\frac{x^{a+1}}{\log^{k+1} x}\right).$$

On the other hand, since

$$\int\limits_{x/2}^{\infty} \frac{\lfloor u\rfloor}{u^{a+2}} \log^\ell u \, du \leq \int\limits_{x/2}^{\infty} \frac{\log^\ell u}{u^{a+1}} du = O\left(\frac{\log^\ell x}{x^a}\right),$$

it follows from (3.7) that

$$\nu_a(x) = \frac{x^{a+1}}{\log x} \sum_{\ell=0}^{k-1} \frac{c_\ell}{\log^\ell x} + O\left(\frac{x^{a+1}}{\log^{k+1} x}\right),$$

where $c_\ell = \displaystyle\int\limits_{1}^{\infty} \frac{\lfloor u\rfloor}{u^{a+2}} \log^\ell u \, du$.

It remains to obtain explicit expressions for the constants c_ℓ. We have

$$c_\ell = \int\limits_{1}^{\infty} \frac{\lfloor u\rfloor}{u^{a+2}} \log^\ell u \, du = \sum_{s=1}^{\infty} s \int\limits_{s}^{s+1} \frac{\log^\ell u}{u^{a+2}} du.$$

Performing integration by parts k times yields

$$\int\limits_{s}^{s+1} \frac{\log^\ell u}{u^{a+2}} du = \sum_{i=0}^{\ell} \frac{\ell!}{(\ell - i)!(a + 1)^{i+1}} \left(\frac{\log^{\ell-i} s}{s^{a+1}} - \frac{\log^{\ell-i}(s+1)}{(s+1)^{a+1}}\right),$$

so that, using Lemma 5, we get

$$\sum_{s=1}^{\infty} s \int\limits_{s}^{s+1} \frac{\log^\ell u}{u^{a+2}} du =$$

$$= \sum_{s=1}^{\infty} \left(s \sum_{i=0}^{\ell} \frac{\ell!}{(\ell-i)!(a+1)^{i+1}} \left(\frac{\log^{\ell-i} s}{s^{a+1}} - \frac{\log^{\ell-i}(s+1)}{(s+1)^{a+1}} \right) \right) =$$

$$= \sum_{i=0}^{\ell} \frac{\ell!}{(\ell-i)!(a+1)^{i+1}} \left(\sum_{s=1}^{\infty} s \left(\frac{\log^{\ell-i} s}{s^{a+1}} - \frac{\log^{\ell-i}(s+1)}{(s+1)^{a+1}} \right) \right) =$$

$$= \sum_{i=0}^{\ell} \frac{\ell!}{(\ell-i)!(a+1)^{i+1}} \left(\sum_{s=1}^{\infty} \frac{\log^{\ell-i} s}{s^{a+1}} \right) =$$

$$= \sum_{i=0}^{\ell} \frac{\ell!}{(\ell-i)!} \frac{(-1)^{\ell-i} \zeta^{(\ell-i)}(a+1)}{(a+1)^{i+1}}.$$

Setting $j = \ell-i$, we conclude that $c_\ell = \dfrac{\ell!}{(a+1)^{\ell+1}} \displaystyle\sum_{j=0}^{\ell} \dfrac{(a+1)^j (-1)^j \zeta^{(j)}(a+1)}{j!}$,

thus completing the proof of Lemma 6.   ∎

Let $H(x, y, z)$ stand for the number of positive integers $n \le x$ having a divisor in the interval $(y, z]$.

Theorem A. (Ford [1], Théorème 1(v)) *Let x, y, z be real numbers all strictly positive. If $x > 100000$, $100 \le y \le z - 1$, $y \le \sqrt{x}$ and $2y \le z \le y^2$, then*

$$H(x, y, z) \asymp xu^\delta \left(\log \frac{2}{u} \right)^{-3/2},$$

where u is defined implicitly by $z = y^{1+u}$ and where δ is the constant defined in (1.2).

Theorem B. (Ford [1], Théorème 2) *For $y_0 \le y \le \sqrt{x}$, $z \ge y + 1$ and $\dfrac{x}{\log^{10} z} \le \Delta \le x$, we have*

$$H(x, y, z) - H(x - \Delta, y, z) \asymp \frac{\Delta}{x} H(x, y, z).$$

4.   Proof of Theorem 1

Using Lemma 1, we easily obtain that

$$\sum_{n \le x} \rho_2(n)^a =$$

$$= \sum_{\substack{n \le x \\ \rho_2(n) > x^{2/3}}} \rho_2(n)^a + O\left(x^{\frac{2a+3}{3}}\right) = \sum_{x^{2/3} < p \le x} p^a \sum_{\substack{n \le x \\ \rho_2(n) = p}} 1 + O\left(x^{\frac{2a+3}{3}}\right) =$$

$$= \sum_{x^{2/3} < p \le x} p^a \sum_{mp \le x} 1 + O\left(x^{\frac{2a+3}{3}}\right) = \sum_{x^{2/3} < p \le x} p^a \left\lfloor \frac{x}{p} \right\rfloor + O\left(x^{\frac{2a+3}{3}}\right) =$$

$$= \sum_{p \le x} p^a \left\lfloor \frac{x}{p} \right\rfloor - \sum_{p \le x^{2/3}} p^a \left\lfloor \frac{x}{p} \right\rfloor + O\left(x^{\frac{2a+3}{3}}\right) =$$

$$(4.1) \quad = \Sigma_1 - \Sigma_2 + O\left(x^{\frac{2a+3}{3}}\right),$$

say. From Lemma 2, we obtain that

$$(4.2) \quad \Sigma_2 = \sum_{p \le x^{2/3}} p^a \left\lfloor \frac{x}{p} \right\rfloor \le x \sum_{p \le x^{2/3}} p^{a-1} \ll x \int_2^{x^{2/3}} \frac{t^{a-1}}{\log t} dt \ll \frac{x^{\frac{2a+3}{3}}}{\log x}.$$

Hence, it follows from (4.1) and (4.2) that

$$(4.3) \quad \sum_{n \le x} \rho_2(n)^a = \sum_{p \le x} p^a \left\lfloor \frac{x}{p} \right\rfloor + O\left(x^{\frac{2a+3}{3}}\right).$$

Finally, combining the results of Lemmas 4 and 6 in (4.3), the proof of Theorem 1 is complete.

5.   Proof of Theorem 2

Observe that the relation (2.1) we need to prove is equivalent to

$$(5.1) \quad \frac{x^{\frac{a+2}{2}}}{(\log x)^\delta (\log \log x)^{3/2}} \ll \sum_{n \le x} \rho_1(n)^a \ll \frac{x^{\frac{a+2}{2}}}{(\log x)^\delta (\log \log x)^{3/2}}.$$

We will first show the first inequality in relation (5.1). We start by observing that if $x/2 < n \le x$, then n has a divisor d_1 satisfying $\frac{\sqrt{x}}{2} < d_1 \le \sqrt{x}$ if and

only if $\rho_1(n) > \frac{\sqrt{x}}{2}$. It follows from this that

$$
\sum_{n \leq x} \rho_1(n)^a \geq \sum_{\substack{x/2 < n \leq x \\ \rho_1(n) > \sqrt{x}/2}} \rho_1(n)^a > \left(\frac{\sqrt{x}}{2}\right)^a \sum_{\substack{x/2 < n \leq x \\ \rho_1(n) > \sqrt{x}/2}} 1 \geq
$$

$$
\geq \left(\frac{\sqrt{x}}{2}\right)^a \sum_{\substack{x/2 < n \leq x \\ \exists\, d_1 \mid n \\ d_1 \in (\sqrt{x}/2, \sqrt{x}]}} \geq
$$

$$
\geq \left(\frac{\sqrt{x}}{2}\right)^a \left(H\left(x, \frac{\sqrt{x}}{2}, \sqrt{x}\right) - H\left(\frac{x}{2}, \frac{\sqrt{x}}{2}, \sqrt{x}\right) \right).
$$

Using Theorem B followed by Theorem A (with $\Delta = x/2$), we find that

$$
H\left(x, \frac{\sqrt{x}}{2}, \sqrt{x}\right) - H\left(\frac{x}{2}, \frac{\sqrt{x}}{2}, \sqrt{x}\right) \asymp \frac{x/2}{x} \cdot H\left(x, \frac{\sqrt{x}}{2}, \sqrt{x}\right) \asymp
$$

$$
\asymp x \cdot \left(\frac{2 \log 2}{\log x}\right)^\delta \cdot (\log \log x)^{-3/2} \asymp
$$

$$
\asymp \frac{x}{(\log x)^\delta (\log \log x)^{3/2}}.
$$

Combining these last two estimates, it follows that

$$
\sum_{n \leq x} \rho_1(n)^a \gg \frac{x^{\frac{a+2}{2}}}{(\log x)^\delta (\log \log x)^{3/2}},
$$

thus establishing the first inequality in (5.1).

In order to prove the second inequality in (5.1), first observe that if $n \leq x$, then it is obvious that $\frac{\sqrt{x}}{2^k} < \rho_1(n) \leq \frac{\sqrt{x}}{2^{k-1}}$ for some integer $k \geq 1$, and therefore that

$$
(5.2) \qquad \sum_{n \leq x} \rho_1(n)^a \leq \sum_{k \geq 1} \left(\frac{\sqrt{x}}{2^{k-1}}\right)^a H\left(x, \frac{\sqrt{x}}{2^k}, \frac{\sqrt{x}}{2^{k-1}}\right).
$$

Then, using Theorem A, we find that

$$
\sum_{k \geq 1} \left(\frac{\sqrt{x}}{2^{k-1}}\right)^a H\left(x, \frac{\sqrt{x}}{2^k}, \frac{\sqrt{x}}{2^{k-1}}\right) \ll \sum_{k \geq 1} \left(\frac{\sqrt{x}}{2^{k-1}}\right)^a x \cdot \frac{1}{(\log x)^\delta} \frac{1}{(\log \log x)^{3/2}} \ll
$$

$$
(5.3) \qquad\qquad\qquad\qquad \ll \frac{x^{\frac{a+2}{2}}}{(\log x)^\delta (\log \log x)^{3/2}}.
$$

Combining estimates (5.2) and (5.3), the second inequality in (5.1) is proved.

6.   Proof of Theorem 3

First observe that for each positive integer n, we have $\dfrac{\rho_2(n)}{\rho_1(n)^r} = \dfrac{\rho_2(n)^{r+1}}{n^r}$.
On the other hand, it follows from Theorem 1 that for each positive integer k,

$$(6.1) \qquad A(x) := \sum_{n \leq x} \rho_2(n)^{r+1} = \frac{x^{r+2}}{\log x} \sum_{\ell=0}^{k-1} \frac{c_\ell}{\log^\ell x} + \vartheta(x),$$

where

$$\vartheta(x) = O\left(\frac{x^{r+2}}{\log^{k+1} x}\right) \quad \text{and} \quad c_\ell = \frac{\ell!}{(r+2)^{\ell+1}} \sum_{j=0}^{\ell} \frac{(r+2)^j (-1)^j \zeta^{(j)}(r+2)}{j!}.$$

Hence, using (6.1) and partial summation, we obtain

$$
\begin{aligned}
\sum_{n \leq x} \frac{\rho_2(n)}{\rho_1(n)^r} &= \sum_{n \leq x} \frac{\rho_2(n)^{r+1}}{n^r} = 1 + \sum_{2 \leq n \leq x} \frac{\rho_2(n)^{r+1}}{n^r} = \\
&= 1 + \frac{A(x)-1}{x^r} + \int_2^x \frac{r}{t^{r+1}}(A(t)-1)\,dt = \\
&= \frac{A(x)}{x^r} + O(1) + \int_2^x \frac{r}{t^{r+1}} A(t)\,dt = \\
&= \frac{x^2}{\log x} \sum_{\ell=0}^{k-1} \frac{c_\ell}{\log^\ell x} + O\left(\frac{x^2}{\log^{k+1} x}\right) + \\
&\quad + r \sum_{\ell=0}^{k-1} c_\ell \int_2^x \frac{t}{\log^{\ell+1} t}\,dt + r \int_2^x \frac{\vartheta(t)}{t^{r+1}}\,dt.
\end{aligned}
$$

$$(6.2)$$

It is easily seen that

$$(6.3) \qquad r \int_2^x \frac{\vartheta(t)}{t^{r+1}}\,dt \ll \int_2^x \frac{t}{\log^{k+1} t}\,dt \ll \frac{x^2}{\log^{k+1} x}.$$

Moreover, integrating by parts, we have

$$c_{k-1} \int_2^x \frac{t}{\log^k t}\,dt = \frac{c_{k-1}}{2} \frac{x^2}{\log^k x} + O\left(\frac{x^2}{\log^{k+1} x}\right)$$

and for $0 \leq \ell \leq k - 2$,

$$c_\ell \int_2^x \frac{t}{\log^{\ell+1} t} \, dt = \frac{c_\ell}{2} \frac{x^2}{\log^{\ell+1} x} + \frac{c_\ell}{2} \sum_{i=\ell}^{k-2} \frac{x^2}{\log^{i+2} x} \prod_{m=\ell}^{i} \left(\frac{m+1}{2} \right) + O \left(\frac{x^2}{\log^{k+1} x} \right).$$

Summing on ℓ from 0 to $k - 1$, we then obtain that

$$(6.4) \qquad r \sum_{\ell=0}^{k-1} c_\ell \int_2^x \frac{t}{\log^{\ell+1} t} \, dt = \frac{x^2}{\log x} \sum_{\ell=0}^{k-1} \frac{d_\ell}{\log^\ell x} + O \left(\frac{x^2}{\log^{k+1} x} \right),$$

where $d_0 = \dfrac{r c_0}{2}$, and for $1 \leq \ell \leq k - 1$,

$$(6.5) \qquad\qquad\qquad d_\ell = \frac{r c_\ell}{2} + \sum_{\nu=0}^{\ell-1} \frac{r c_\nu}{2} \prod_{m=\nu}^{\ell-1} \left(\frac{m+1}{2} \right).$$

This is why, combining estimates (6.2), (6.3), (6.4), (6.5), we may conclude that

$$\sum_{n \leq x} \frac{\rho_2(n)}{\rho_1(n)^r} = \frac{x^2}{\log x} \sum_{\ell=0}^{k-1} \frac{e_\ell}{\log^\ell x} + O \left(\frac{x^2}{\log^{k+1} x} \right)$$

with $e_0 = \dfrac{(r+2) c_0}{2}$ and for $1 \leq \ell \leq k - 1$,

$$e_\ell = \left(\frac{r+2}{2} \right) c_\ell + \sum_{\nu=0}^{\ell-1} \frac{r c_\nu}{2} \prod_{m=\nu}^{\ell-1} \left(\frac{m+1}{2} \right).$$

7. Proof of Theorem 4

First observe that for each positive integer n, we have $\dfrac{\rho_1(n)}{\rho_2(n)} = \dfrac{\rho_1(n)^2}{n}$. Set

$$A(x) := \sum_{n \leq x} \rho_1(n)^2 \qquad \text{and} \qquad \alpha := \sum_{n < e^e} \rho_1(n)^2.$$

Then, using Theorem 1 along with partial summation, we obtain that

$$\sum_{n \le x} \frac{\rho_1(n)}{\rho_2(n)} = \sum_{n \le x} \frac{\rho_1(n)^2}{n} = O(1) + \sum_{e^e \le n \le x} \frac{\rho_1(n)^2}{n} =$$

$$= O(1) + \frac{A(x) - \alpha}{x} + \int_{e^e}^{x} \frac{1}{t^2}(A(t) - \alpha)dt =$$

$$= \frac{A(x)}{x} + O(1) + \int_{e^e}^{x} \frac{A(t)}{t^2}dt \asymp$$

$$\asymp \frac{x}{(\log x)^\delta (\log \log x)^{3/2}} + \int_{e^e}^{x} \frac{dt}{(\log t)^\delta (\log \log t)^{3/2}}.$$

Since

$$\int_{e^e}^{x} \frac{dt}{(\log t)^\delta (\log \log t)^{3/2}} \ll \frac{x}{(\log x)^\delta (\log \log x)^{3/2}},$$

the proof of Theorem 4 is complete.

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