THE NUMERICAL SOLUTION OF THIRD ORDER BOUNDARY VALUE PROBLEMS IN PDES BY THE FINITE DIFFERENCE METHOD

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Abstract. In this article, we have proposed a finite difference method for the numerical solution of third order nonlinear boundary value problem in partial differential equation. To derive a proposed method, we will replace differential terms in the problem by the finite difference approximations. Hence the continuous third order boundary value problem is transformed into a system of algebraic equations. The solution of the problem is the solutions of the system of equations at the discrete points. Numerical experiments are performed to test the efficiency and accuracy of the proposed method on model problems.

1. Introduction

Over the last few years, to describe in more accurate and realistic, a physical phenomena were formulated into mathematical models. The mathematical model developed for the study of the dynamics of the soil moisture and subsoil waters [5], propagation of acoustic waves in relaxing media [10, 14] leads to a

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third order hyperbolic partial differential equations and corresponding boundary value problems. In this article we consider following third order hyperbolic partial differential equations and corresponding boundary value problems,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^3 u}{\partial x^3} + f(x, t, u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}), \quad a < x < b, \quad 0 \le t < T, \tag{1.1}$$

subject to initial-boundary conditions

$$u(x, 0) = g(x), \quad u_t(x, 0) = \alpha(x), \quad u(a, t) = \bar{g}(t),$$

$$\frac{\partial u(a, t)}{\partial x} = \beta(t) \quad \text{and} \quad \frac{\partial u(b, t)}{\partial x} = \bar{\beta}(t),$$

where $g(x)$, $\alpha(x), \bar{g}(t)$, $\beta(t), \bar{\beta}(t)$ are either real and continuous function of its argument or $g(x), \alpha(x), \bar{g}(t), \beta(t)$, $\bar{\beta}(t)$ are real constant i.e. independent of its argument. Also, let us assume that forcing function $f(x, t, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2})$ is smooth in $[a, b], t \ge 0$.

To find an analytically closed form of the solution of the higher order partial differential equations and corresponding boundary value problems is an important and interesting area of research. Because of the inability in finding a closed form analytical solution even when it exists [15], in recent years, numerical methods have been developed and applied for the numerical solution of higher order PDEs and corresponding boundary value problems in natural sciences [11, 3]. A numerical technique and study on the higher order of hyperbolic differential equations can be found in the literature [4, 1].

In this article we will not consider any specific assumption on forcing function $f(x, t, u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2})$ in problem (1.1) to ensure the existence and uniqueness of the solution of the problem. However, there is so much work devoted to the existence and uniqueness of solution of considered boundary value problems reported in [2, 12, 13] and reference therein. We assume the existence and uniqueness of the solution. Thus, the emphasis will be on the development of finite difference method for the approximate numerical solution of the problem (1.1).

The outline of this article is as follows. In Section 2, we applied the well known FDM approximation for the numerical solution of problem (1.1). In Section 3, we discuss the derivation of the proposed method. In section 4, the numerical results are given to establish efficiency and accuracy of the proposed method on two model problems. Finally a discussion and conclusion are presented in Section 5.

2. Development of finite difference method

In this section we propose a finite difference method for the numerical solution of the problem (1.1). We substitute rectangular domain $[a,b] \times [0,T]$ by a discrete set of mesh points and we wish to determine the numerical solution of the problem (1.1) at these discrete mesh points. Let $h = (b-a)/N$ and k be the step size respectively in the x and t directions of the Cartesian coordinate system parallel to coordinate axes. Thus we have generated mesh points (x_i, t_j); $x_i = a + ih, i = 0, 1, ..., N$ and $t_j = jk, j = 0, 1, ...$ Let us denote the numerical approximation of $u(x,t)$ at mesh point (x_i, t_j) by $u_{i,j}$ for $i = 0, 1, ..., N$ and $j = 0, 1, ...$ Let us denote approximation of the theoretical value of the forcing function $f(x, t, u(x), \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2})$ at mesh point (x_i, t_j) as $f_{i,j}$, $i = 0, 1, 2, ..., N,$ $j = 0, 1, ...$ Thus, using these finite difference, the problem (1.1) reduced to the following discrete problem at node (x_i, t_j),

$$u^j_{tti} - f^j_i = u^j_{xxxi} \quad i = 0, 1, \ldots, N, \quad j = 0, 1, \ldots \tag{2.1}$$

where $u_{tt} = \frac{\partial^2 u}{\partial t^2}$, $u_{xxx} = \frac{\partial^3 u}{\partial x^3}$ and subject to the initial boundary conditions

$$u^0_i = g(x_i), \quad u^0_{ti} = \alpha(x_i), \quad u^j_0 = \bar{g}^j_0, \quad u^j_{x0} = \beta(t_j) \quad \text{and} \quad u^j_{xN} = \bar{\beta}(t_j)$$

Let us define following approximations,

$$\overline{u}^j_{ti} = \frac{u^{j+1}_i - u^{j-1}_i}{2k}, \tag{2.2}$$

$$\overline{u}^j_{xi} = \begin{cases} \dfrac{u^j_{i+1} - u^j_{i-1}}{2h}, & 1 \le i \le N-1 \\ u^j_{xi}, & i = N, \end{cases} \tag{2.3}$$

$$\overline{u}^j_{xxi} = \begin{cases} \dfrac{u^j_{i+1} - 2u^j_i + u^j_{i-1}}{h^2}, & 1 \le i \le N-1 \\ \dfrac{-u^j_{i-2} + 8u^j_{i-1} - 7u^j_i + 6hu^j_{xi}}{2h^2}, & i = N \end{cases} \tag{2.4}$$

and

$$\overline{f}^j_i = f(x_i, t_j, \overline{u}^j_{ti}, \overline{u}^j_{xi}, \overline{u}^j_{xxi}). \tag{2.5}$$

Hence, following the ideas in [6, 7, 8, 9], we propose following finite difference method for the numerical solution of the (2.1),

$$(2.6)\qquad -\overline{f}_i^j + \frac{1}{k^2}\begin{cases} u_i^{j+1} - u_i^{j-1} - 2ku_{ti}^{j-1}, & j=1 \\ u_i^{j+1} - 2u_i^j + u^{j-1}, & j=2,\ldots \end{cases} =$$

$$= \frac{1}{150h^3}\begin{cases} 50(8u_{i-1}^j - 9u_i^j + u_{i+2}^j + 6hu_{x,i-1}^j), & i=1 \\ 6(u_{i-1}^j + 18u_i^j - 33u_{i+1}^j + 14u_{i+2}^j + 6hu_{x,i-2}^j), & i=2 \\ 75(-u_{i-2}^j + 2u_{i-1}^j - 2u_{i+1}^j + u_{i+2}^j), & 3 \le i \le N-2 \\ 50(-u_{i-2}^j + 9u_i^j - 8u_{i+1}^j + 6hu_{x,i+1}^j), & i=N-1 \\ 150(u_{i-3}^j - 6u_{i-2}^j + 15u_{i-1}^j - 10u_i^j + 6hu_{x,i+1}^j), & i=N,\ j=1,\ldots \end{cases}$$

We have obtained an explicit finite difference method. Thus we have a system of equations (2.6) at each mesh point $(x_i, t_j) : i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots$ of the discrete domain. This system can be written in matrix form and can be solved by iterative methods. The solution of the system of equations (2.6) is the solution of the problem (1.1) in the discrete domain.

3. Derivation of the difference method

In this section we outline the derivation and development of the proposed finite difference method. Let us write $u_{xxxi}^j, \quad i=1$ as a linear combination of solution and the derivative of the solution of the problem (1.1),

$$h^3 u_{xxxi}^j = a_0 u_{i-1}^j + a_1 u_i^j + a_2 u_{i+1}^j + a_3 u_{i+2}^j + b_0 h u_{xi}^j,$$

where $a_0, \ldots, b_0$ are constant. To determine these constants, we expand each term in the above expression in a Taylor series about a mesh point (x_i, t_j) and compare the coefficients of $h^p, p = 0, \ldots, 4$ in the both sides of the expression. So we obtained a system of linear equations in $a_0, \ldots, b_0$. Solving the system of equations, we got

$$(a_0, a_1, a_2, a_3, b_0) = \frac{1}{3}(8, -9, 0, 1, 6).$$

Hence we have

$$(3.1)\qquad h^3 u_{xxxi}^j = 8u_{i-1}^j - 9u_i^j + u_{i+2}^j + 6hu_{xi}^j.$$

Thus using (3.1) in (2.1), we have

$$3h^3(u_{tti}^j - f_i^j) = 8u_{i-1}^j - 9u_i^j + u_{i+2}^j + 6hu_{xi}^j, \quad i=1.$$

Using approximations (2.2)–(2.4), it is easy to prove that $\overline{f}_i^j$ provides an $O(k^2+$ $+h^2)$ approximation for the f_i^j. Substituting the a second order difference approximation for the derivative term u_{tti}^j i.e.

$$u_{tti}^j = \frac{1}{2k^2}\begin{cases} u_i^{j+1} - u_i^{j-1} - 2ku_{ti}^{j-1}, & j=1 \\ 2(u_i^{j+1} - 2u_i^j + u^{j-1}), & j=2,\dots \end{cases}$$

in the above equation. So, we have obtained our proposed finite difference method for $i=1$,

$$3h^3\left(-\overline{f}_i^j + \frac{1}{2k^2}\begin{cases} u_i^{j+1} - u_i^{j-1} - 2ku_{ti}^{j-1}, & j=1 \\ 2(u_i^{j+1} - 2u_i^j + u^{j-1}), & j=2,\dots \end{cases}\right) = \tag{3.2}$$

$$= 8u_{i-1}^j - 9u_i^j + u_{i+2}^j + 6hu_{xi}^j$$

and local truncated error term at mesh point (x_i, t_j) is

$$T_1 = \frac{h^2}{60}\left(9\frac{\partial^5 u}{\partial x^5} + 10\frac{\partial^3 u}{\partial x^3}\frac{\partial f}{\partial u_x} + 5\frac{\partial^4 u}{\partial x^4}\frac{\partial f}{\partial u_{xx}}\right) +$$

$$+\frac{k^2}{12}\left(\frac{\partial^3 u}{\partial t^3}\frac{\partial f}{\partial u_t} + \begin{cases} \frac{4}{k}\frac{\partial^3 u}{\partial t^3}, & j=1, \\ -\frac{\partial^4 u}{\partial t^4}, & j=2,\dots \end{cases}\right). \tag{3.3}$$

Thus, from (3.3), we conclude that the order of truncation error in the proposed method is at least $O(k+h^2)$. Following the same line of derivation as above, we derive other equations in (2.6) for different values of $i = 2,\dots,N$.

4. Numerical results

In this section, we have tested the computational efficiency of our proposed method (2.6) on linear and nonlinear model problems. In each model problem, we took a uniform step size h in space and k in the time direction. We have shown the maximum absolute error MAE in the solution $u(x,t)$ of the problem (1.1) for different values of N and M. For computation purpose we have used following formulas,

$$\mathrm{MAE} = \max_{1\le i\le N} |U(x_i,t) - u_{x_i,t}|,$$

where $U(x_i,t)$ and $u(x_i,t)$ are respectively exact and computed solution of problem.

All computations were performed on a Windows 7 Home Basic operating system in the GNU FORTRAN environment version 99 compiler (2.95 of gcc) on Intel Core i3-2330M, 2.20 Ghz PC. The iteration is continued until either the maximum difference between two successive iterates is less than 10^{-8} or the number of iteration reached 2×10^4.

Problem 1. The model linear problem given by

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^3 u}{\partial x^3} + 7u(x,t) + f(x,t), \quad 0 < x < 1, \quad t \geq 0,$$

subject to initial and boundary conditions

$$u(x,0) = \cos(2x), \quad u_t(x,0) = \sin(2x), \quad u(0,t) = \cos(t),$$
$$u_x(0,t) = 2\sin(t), \quad \text{and} \quad u_x(1,t) = -2\sin(2-t)\cos(t),$$

where $f(x,t)$ is calculated so that the analytical solution of the problem is $u(x,t) = \cos(2x - t)$. The MAE computed by proposed method (2.6) for different values of N and t are presented in Table 1.

		Maximum absolute error		
M	N	$t = 1.0 \times 10^{-3}$	$t = 5.0 \times 10^{-3}$	$t = 1.0 \times 10^{-2}$
4	8	.12350082(-3)	.60963631(-3)	.11898875(-2)
	16	.61929226(-4)	.29361248(-3)	.50479174(-3)
	32	.30577183(-4)	.10281801(-3)	.81241131(-4)
8	8	.18513203(-3)	.91660023(-3)	.18019676(-2)
	16	.93042850(-4)	.44941902(-3)	.82939863(-3)
	32	.46193600(-4)	.19127131(-3)	.18996000(-3)
16	8	.21600723(-3)	.10704994(-2)	.21122694(-2)
	16	.10859966(-3)	.52946806(-3)	.10073781(-2)
	32	.54061413(-4)	.24259090(-3)	.36239624(-3)

Table 1. Maximum absolute error (Problem 1)

Problem 2. The model non-linear problem given by

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^3 u}{\partial x^3} - u(x,t)(4u(x,t) + u_{xx}(x,t)) + f(x,t), \quad 0 < x < 1, \quad t \geq 0,$$

subject to initial and boundary conditions

$$u(x,0) = \sin(2x), \quad u_t(x,0) = 0, \quad u(0,t) = \cos(2t),$$
$$u_x(0,t) = 2, \quad \text{and} \quad u_x(1,t) = 2\cos(2),$$

where $f(x,t)$ is calculated so that the analytical solution of the problem is $u(x,t) = \sin(2x)\cos(2t)$. The MAE computed by method (2.6) for different values of N and t are presented in Table 2 and Table 3.

M	N	Maximum absolute error		
		$t = 5.0 \times 10^{-3}$	$t = 1.0 \times 10^{-2}$	$t = 2.0 \times 10^{-2}$
4	8	.66012144(-5)	.26255846(-4)	.10208786(-3)
	16	.31739473(-5)	.11950731(-4)	.36373734(-4)
	32	.13485551(-5)	.25667240(-5)	.31705946(-4)
8	8	.94920397(-5)	.37759542(-4)	.14868379(-3)
	16	.46864152(-5)	.18112361(-4)	.63590705(-4)
	32	.21606684(-5)	.66459179(-5)	.55879354(-5)
16	8	.10937452(-4)	.43615699(-4)	.17288327(-3)
	16	.54463744(-5)	.21420419(-4)	.79907477(-4)
	32	.38146973(-5)	.91195107(-5)	.22023916(-4)

Table 2. Maximum absolute error (Problem 2)

Problem 3. The model non-linear problem given by

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^3 u}{\partial x^3} + u_t(x,t)(u(x,t) + u_x(x,t)) + f(x,t), \quad 0 < x < 1, \quad t \geq 0,$$

subject to initial and boundary conditions

$$u(x,0) = \cos(x), \quad u_t(x,0) = \sin(x), \quad u(0,t) = \cos(t),$$
$$u_x(0,t) = \sin(t), \quad \text{and} \quad u_x(1,t) = -\sin(1-t),$$

where $f(x,t)$ is calculated so that the analytical solution of the problem is $u(x,t) = \cos(x-t)$. The MAE computed by method (2.6) for different values of N and t are presented in Table 4.

In numerical experiment, we observed maximum absolute error increases as either t increases or k the step size in direction of t decreases. The order of accuracy in numerical experiment is approximately quadratic.

M	N	Maximum absolute error		
		$t = 5.0 \times 10^{-3}$	$t = 1.0 \times 10^{-2}$	$t = 2.0 \times 10^{-2}$
4	5	.10758638(-4)	.42915344(-4)	.17058849(-3)
	10	.52303076(-5)	.20623207(-4)	.77992678(-4)
	20	.24735928(-5)	.87693334(-5)	.17471611(-4)
	40	.89406967(-6)	.77103509(-6)	.63247979(-4)
8	5	.15050173(-4)	.60379505(-4)	.24053454(-3)
	10	.75846910(-5)	.30040741(-4)	.11640787(-3)
	20	.36954880(-5)	.13872981(-4)	.43064356(-4)
	40	.15832484(-5)	.36843121(-5)	.17008185(-3)
16	5	.17315149(-3)	.69051981(-4)	.27588010(-3)
	10	.87469816(-5)	.34838915(-4)	.13701618(-3)
	20	.43362379(-5)	.16748905(-4)	.58621168(-4)
	40	.35762787(-5)	.61579049(-5)	.18186867(-4)

Table 3. Maximum absolute error (Problem 2)

M	N	Maximum absolute error		
		$t = 1.0 \times 10^{-3}$	$t = 5.0 \times 10^{-3}$	$t = 1.0 \times 10^{-2}$
4	8	.62048435(-4)	.30463934(-3)	.58943033(-3)
	16	.30934811(-4)	.14442205(-3)	.24265051(-3)
	32	.15139580(-4)	.48995018(-4)	.44822693(-4)
8	8	.93221664(-4)	.457764446(-3)	.89102983(-3)
	16	.46372414(-4)	.22071600(-3)	.39857626(-3)
	32	.22828579(-4)	.91612339(-4)	.82612038(-4)
16	8	.10859966(-3)	.53423643(-3)	.10436773(-2)
	16	.54121017(-4)	.25999546(-3)	.48381090(-3)
	32	.26822090(-4)	.11628866(-3)	.16367435(-3)

Table 4. Maximum absolute error (Problem 3)

5. Conclusion

In this article, we have considered for the numerical solution of the hyperbolic class of third order boundary value problems PDEs. We applied the well known FDM approximation, a numerical method for the numerical solution of the considered class of the problem. The proposed finite difference method has good accuracy and uses one function value at each mesh point (x_i, t_j). We have tested the proposed method on model problems. The numerical results obtained in experiment suggest the convergence of the proposed method for the considered problems. Improving the accuracy and computational efficiency of the proposed method is a challenge. Works in these directions are in progress.

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