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PARTIAL BEST APPROXIMATIONS AND MAGNITUDE OF DOUBLE VILENKIN–FOURIER COEFFICIENTS

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Abstract. We give estimates for the magnitude of double Vilenkin–Fourier coefficients of functions from generalized Hölder spaces, some p-fluctuational spaces and bounded Λ-Γ-φ-fluctuation spaces. For Hölder and p-fluctuational spaces we establish the sharpness of these estimates. Also we establish relation between full and partial best approximations and Watari–Efimov type inequality concerning partial best approximation and partial modulus of continuity.

1. Inroduction

Vilenkin systems were defined in 1947 by N. Ya. Vilenkin [15] as the character systems $\chi = \{\chi_n\}_{n=0}^{\infty}$ of compact abelian groups G with the second axiom of countability. He introduced a notion of function of bounded variation using the ordering of these groups and proved the estimate $O(1/n), n \in \mathbb{N}$, for the n-th Fourier coefficient c_n of such function with respect to χ, see [15, §3.22]. He also estimated $|c_n|$ by the uniform modulus of continuity [15, §3.3]. N. Fine [3]

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et al considered Walsh, Chrestenson systems and the other ones which are de-facto the particular cases of Vilenkin systems as function systems on $[0,1]$ or $[0,1)$. Fine estimated Walsh–Fourier coefficients in terms of uniform and L^1-modulus of continuity [3, Theorems 4 and 5]. As well he obtained the analogue of the Vilenkin's result in the case of Walsh system for the usual function of bounded variation (see [3, Theorem 6]). The main results concerning estimates of Fourier-Walsh coefficients may be found in the monograph of F. Schipp, W. R. Wade and P. Simon [12, Ch. 2, sect.2.1]. Also in sections 2.2 and 2.3 from [12] the best order of decreasing to zero of Fourier–Walsh coefficients is discussed in the case of classical continuous and absolutely continuous functions. In 2008 B. L. Ghodadra and J. R. Patadia [6] proved several theorems concerning Walsh–Fourier coefficients which are analogues of the corresponding trigonometric results due to M. Schramm and D. Waterman [13]. Recent results on Walsh–Fourier coefficients are obtained by K. N. Darji and R. G. Vyas [2, 19]. The result of [19] using double Walsh system and Λ_1-Λ_2-φ-variation (cf. with the notion of Λ-Γ-φ-fluctuation in Sect.2) is generalized in our Theorem 5.3.

C. W. Onneweer and D. Waterman [10] used the notion of bounded fluctuation and its generalization to study the uniform convergence of Vilenkin–Fourier series on groups. These notions may be used also for the function defined on $[0,1)$. For example, in [16, 17] p-fluctuation and fluctuational moduli of continuity were applied to study the absolute convergence of Vilenkin–Fourier series. The main aim of the present paper is to obtain the upper estimates for Vilenkin–Fourier coefficients of the functions from generalized Hölder classes and of functions with the finite generalized fluctuation. Also we study the sharpness of these estimates in the cases of Hölder spaces and p-fluctuational spaces.

The method used in section 5 for bounds of Vilenkin–Fourier coefficients presuppose that the generating sequence of Vilenkin system is bounded. There is a great difference between such systems with bounded and unbounded generating sequences. In the last case the $(C,1)$-means of Vilenkin–Fourier series for a continuous function may be divergent in different senses (see more in [5]). We note two recent papers of G. Gát and U. Goginava [4] and [5], where the convergence of double Vilenkin–Fourier sums and double Vilenkin–Fejér means is studied in the unbounded case.

Conversely, the results of section 4 are valid in both bounded and unbounded case. There are well known C. Watari – A.V. Efimov results (see [12, Ch. 5, Theorem 2] in the case of Walsh system and [7, § 10.5] in the general case) about equivalence of best approximation by Vilenkin polynomials and modulus of continuity defined by generalized translation (see the next section). So called partial best approximations for functions of several variables (see, e.g., [14, sect. 2.2.6]) are applied in direct and inverse theorems of approximation (see, e.g., [14, sect. 6.3]). We prove an analogue of Watari–Efimov theorem for partial best approximations and modulus of continuity connected with Vilenkin system and estimate the full best approximation by partial ones.

2. Definitions

Let $\mathbf{P} = \{p_n\}_{n=1}^{\infty}$ be a sequence of natural numbers such that $p_n \geq 2$ for all $n \in \mathbb{N} = \{1, 2, \dots\}$. We set by definition $m_0 = 1$, $m_n = p_n m_{n-1}$, $n \in \mathbb{N}$. Each $x \in [0, 1)$ has an expansion

$$x = \sum_{n=1}^{\infty} x_n/m_n, \quad x_n \in \mathbb{Z} \cap [0, p_n). \tag{2.1}$$

The expansion (2.1) is unique if for $x = k/m_j$, $k, j \in \mathbb{N}$, $0 < k < m_j$, we take the series with a finite number of $x_n \neq 0$. If $k \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$ is written in the form $k = \sum_{j=1}^{\infty} k_j m_{j-1}$, $k_j \in \mathbb{Z} \cap [0, p_j)$, and x has the expansion (2.1), then by definition $\chi_k(x) = \exp\left(2\pi i \sum_{j=1}^{\infty} x_j k_j/p_j\right)$. The system $\{\chi_k\}_{k=0}^{\infty}$ is orthonormal and complete in $L^1[0, 1)$ (see [7, §1.5 and 2.8]). It is usually called a Vilenkin system.

The system $\{\chi_k(x)\chi_l(y)\}_{k,l=0}^{\infty}$ is also orthonormal and complete in $L^1[0, 1)^2$. For $f \in L^1[0, 1)^2$ we define Vilenkin–Fourier coefficients and rectangular Vilenkin–Fourier sums by

$$\widehat{f}(m, n) = \int_0^1 \int_0^1 f(x, y)\overline{\chi_m(x)\chi_n(y)}\, dx\, dy, \quad m, n \in \mathbb{Z}_+,$$

and

$$S_{m,n}(f)(x, y) = \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \widehat{f}(j, k)\chi_j(x)\chi_k(y), \quad m, n \in \mathbb{N}.$$

Let $G(\mathbf{P})$ be the group consisting of all sequences of type $\tilde{x} = (x_1, x_2, \dots)$, $x_j \in \mathbb{Z} \cap [0, p_j)$, with addition $\tilde{x} \oplus \tilde{y} = \tilde{z}$, where $z_j = x_j + y_j (\text{mod } p_j)$, $j \in \mathbb{N}$. The inverse operation $\tilde{x} \ominus \tilde{y}$ is defined similarly. The mapping $\lambda_{\mathbf{P}}(\tilde{x}) = \sum_{j=1}^{\infty} x_j/m_j$ is not bijective since the elements of the type

$$x = k/m_l, \quad k, l \in \mathbb{N}, \quad k < m_l, \tag{2.2}$$

have two different prototypes. Let us introduce the inverse mapping $\lambda_{\mathbf{P}}^{-1}$. For x with the representation (2.2) we set $x_j = [m_j x] (\text{mod } p_j)$, $j \in \mathbb{N}$. Then

$\lambda_{\mathbf{P}}^{-1}(x) = (x_1, \ldots, x_l, 0, 0, \ldots)$. For other $x \in [0,1)$ there exists the unique element $\tilde{x} \in G(\mathbf{P})$ with the property $\lambda_{\mathbf{P}}(\tilde{x}) = x$ and $\lambda_{\mathbf{P}}^{-1}(x) = \tilde{x}$. Let us define a generalized distance $\rho(x,y) = \lambda_{\mathbf{P}}(\lambda_{\mathbf{P}}^{-1}(x) \ominus \lambda_{\mathbf{P}}^{-1}(y))$ and an addition $x \oplus y = \lambda_{\mathbf{P}}(\lambda_{\mathbf{P}}^{-1}(x) \oplus \lambda_{\mathbf{P}}^{-1}(y))$ on $[0,1)$. The result of this addition is not defined if $\lambda_{\mathbf{P}}^{-1}(x) \oplus \lambda_{\mathbf{P}}^{-1}(y) = \tilde{z}$ where $z_j = p_j - 1$ for all j starting from some number. It is easy to see that $\cdot \oplus y$ is defined a.e. on $[0,1]$ if y is fixed. Moreover, $x \oplus 1/m_{k+1}$, $k \in \mathbb{Z}_+$, is defined for all $x \in [0,1)$. Finally, we note that $\rho(x \oplus 1/m_{k+1}, x) < 1/m_k$.

It is also known that

$$\chi_n(x \oplus y) = \chi_n(x)\chi_n(y), \quad \chi_n(x \ominus y) = \chi_n(x)\overline{\chi_n(y)}, \quad n \in \mathbb{Z}_+, \tag{2.3}$$

for almost all $y \in [0,1)$ if $x \in [0,1)$ is fixed.

For $1 \le p < \infty$ and $f \in L^p[0,1)^2$ we define two discrete moduli of continuity

$$\omega^*_{rs}(f)_p = \sup\{\|f(\cdot \oplus h, \cdot \oplus \tau) - f(\cdot,\cdot)\|_p : h \in [0, m_r^{-1}), \tau \in [0, m_s^{-1})\}, \quad r, s \in \mathbb{Z}_+,$$

and

$$\omega_{rs}(f)_p = \sup\{\|f(\cdot \oplus h, \cdot \oplus \tau) - f(\cdot \oplus h, \cdot) - f(\cdot, \cdot \oplus \tau) + f(\cdot,\cdot)\|_p : h \in U_r, \tau \in U_s\},$$

where $\|f\|_p = \left(\int_0^1 \int_0^1 |f(x,y)|^p \, dx \, dy\right)^{1/p}$, $r, s \in \mathbb{Z}_+$, $U_r = [0, m_r^{-1})$. A measurable function $f(x,y)$ belongs to $C^*[0,1)^2 = C^*(\mathbf{P}, [0,1)^2)$ if for any $\varepsilon > 0$ there exist $\delta_1 > 0$, $\delta_2 > 0$ such that for any $(x_1, y_1), (x_2, y_2) \in [0,1)^2$ such that $\rho(x_1, x_2) < \delta_1$ and $\rho(y_1, y_2) < \delta_2$ the inequality $|f(x_1, y_1) - f(x_2, y_2)| < \varepsilon$ is fulfilled. The space $C^*[0,1)^2$ with uniform norm $\|f\|_\infty = \sup_{x,y \in [0,1)} |f(x,y)|$ is a Banach space. Also we may consider this space as the completion of the space of polynomials with respect to the system $\{\chi_k(x)\chi_l(y)\}_{k,l=0}^\infty$. We define two discrete moduli of continuity in $C^*[0,1)^2$ for $r, s \in \mathbb{Z}_+$ as follows

$$\omega^*_{rs}(f)_\infty = \sup\{|f(x_1,y_1) - f(x_2,y_2)| : \rho(x_1,x_2) < m_r^{-1}, \rho(y_1,y_2) < m_s^{-1}\},$$

and

$$\omega_{rs}(f)_\infty = \sup\{|f(x_1,y_1) - f(x_2,y_1) - f(x_1,y_2) + f(x_2,y_2)|\},$$

where sup is taken over all x_1, x_2, y_1, y_2 with properties $\rho(x_1, x_2) < m_r^{-1}$, $\rho(y_1, y_2) < m_s^{-1}$. If $\mathcal{P}_{m,n} = \{f \in L^1[0,1)^2 : \widehat{f}(j,k) = 0, j \in [0, m-1], k \in [0, n-1]\}$, $f \in L^p[0,1)^2$, $1 \le p < \infty$, or $f \in C^*[0,1)^2$ and $p = \infty$, then $E_{mn}(f)_p = \inf\{\|f - g\|_p : g \in \mathcal{P}_{mn}\}$ is the best approximation of orders m, n for f. If $\omega = \{\omega_n\}_{n=0}^\infty$ and $\beta = \{\beta_n\}_{n=0}^\infty$ are decreasing to zero sequences, one can define the space $H_p^{\omega,\beta} = \{f \in L^p[0,1)^2 : \omega_{r,s}(f)_p \le C\omega_r\beta_s, r, s \in \mathbb{Z}_+\}$ in the case $1 \le p < \infty$ and similarly in the case $f \in C^*[0,1)^2$, $p = \infty$. Here C is independent of r and s.

Further we consider the sets of intervals $I_j^n = [(j-1)/m_n, j/m_n)$, $n \in \mathbb{Z}_+$, $j \in [1, m_n] \cap \mathbb{Z}$. If f is bounded on $[0,1)$, its oscillations $osc(f, I_j^n) = \sup\{|f(x) - f(y)| : x, y \in I_j^n\}$ are finite. Correspondingly, for a bounded on $[0,1)^2$ function f we consider

$$osc(f, I_j^r \times I_k^s) = \sup\{|f(x,y) - f(u,y) - f(x,v) + f(u,v)| \,:\, x,u \in I_j^r,\, y,v \in I_k^s\}.$$

Let $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ and $\Gamma = \{\gamma_n\}_{n=1}^{\infty}$ be a nondecreasing sequence of positive numbers such that $\lim_{n\to\infty} \Lambda_n = \lim_{n\to\infty} \Gamma_n = \infty$, where $\Lambda_n := \sum_{i=1}^n \lambda_i^{-1}$, $\Gamma_n = \sum_{i=1}^n \gamma_i^{-1}$, W_n be the set of all rearrangements of $\{1, 2, \ldots, m_n\}$.

Let $\varphi(x)$ be a convex function on $[0,\infty)$ such that

$$\varphi(0) = 0, \qquad \lim_{x\to 0+0} \varphi(x)/x = 0, \qquad \lim_{x\to +\infty} \varphi(x)/x = +\infty.$$

Then $\varphi(x)$ is strictly increasing and continuous on $[0,\infty)$ (see [8, §1]). Such functions φ are called N-functions.

Let Λ, Γ and W_n be as above, φ be a N-function. A bounded on $[0,1)^2$ measurable function f belongs to the class $\Lambda\Gamma Fl_{\varphi}[0,1)^2$ of bounded Λ-Γ-φ-fluctuation functions if

$$Fl_{\Lambda,\Gamma,\varphi}(f) = \sup_{r,s\in\mathbb{Z}_+} \sup_{\{l_j\},\{k_i\}} \left\{ \sum_{j=1}^{m_r} \sum_{i=1}^{m_s} \frac{\varphi(osc(f, I_{l_j}^r \times I_{k_i}^s))}{\lambda_j \gamma_i} \right\},$$

where $\{l_j\}_{j=1}^{m_r} \in W_r$, $\{k_i\}_{i=1}^{m_s} \in W_s$, is finite. Similar variational definition for $\varphi(t) = t$ was suggested by D. Waterman [20].

In the case $\varphi(t) = t^p$, $1 < p < \infty$, we can give the following version of above definition. For $r, s \in \mathbb{Z}_+$ we define a p-fluctuational modulus of continuity

$$V_p^{[2]}(f)_{r,s} = \sup_{k\ge r} \sup_{l\ge s} \left(\sum_{i=1}^{m_k} \sum_{j=1}^{m_l} osc^p(f, I_i^k \times I_j^l) \right)^{1/p}.$$

If $Fl_p^{[2]}(f) = V_p^{[2]}(f)_{0,0} < \infty$, then $f \in Fl_p[0,1)^2$. The space $Fl_p[0,1)^2$ is Banach with the norm $\|f\|_{Fl_p} = \|f\|_\infty + Fl_p^{[2]}(f)$ (see a similar proof in [16]). If $\lim_{r,s\to\infty} V_p^{[2]}(f)_{r,s} = 0$, i.e. for any $\varepsilon > 0$ there exist M, N such that for $r \ge M$, $s \ge N$ the inequality $V_p^{[2]}(f)_{r,s} < \varepsilon$ holds, then f belongs to the space $CFl_p[0,1)^2$. For $p = 1$ we introduce $Fl_1^{[2]}(f)$ and $Fl_1[0,1)^2$ as above.

Let us consider the partial moduli of smoothness in the case $1 \le p < \infty$

$$\omega_r^{(1)}(f)_p = \sup_{h\in I_1^r} \left(\int_0^1 \int_0^1 |f(x \oplus h, y) - f(x,y)|^p \, dy\, dx \right)^{1/p},$$

$$\omega_s^{(2)}(f)_p = \sup_{h\in I_1^s}\left(\int_0^1\int_0^1 |f(x,y\oplus h)-f(x,y)|^p\,dx\,dy\right)^{1/p}, \quad r,s\in\mathbb{Z}_+.$$

For $p=\infty$ and $f\in C^*[0,1)^2$ we consider for $r,s\in\mathbb{Z}_+$

$$\omega_r^{(1)}(f)_\infty = \sup_{y\in[0,1)} \sup\{|f(x_1,y)-f(x_2,y)| : x_1,x_2\in[0,1), \rho(x_1,x_2)<m_r^{-1}\},$$

$$\omega_r^{(2)}(f)_\infty = \sup_{x\in[0,1)} \sup\{|f(x,y_1)-f(x,y_2)| : y_1,y_2\in[0,1), \rho(y_1,y_2)<m_r^{-1}\}.$$

Correspondingly, we may consider partial p-fluctuational moduli of smoothness

$$V_p^{(1)}(f)_n = \sup_{y\in[0,1)}\sup_{k\geq n}\left(\sum_{i=1}^{m_k} osc^p(f(\cdot,y),I_i^k)\right)^{1/p}, \quad n\in\mathbb{Z}_+,$$

and, similarly, $V_p^{(2)}(f)_n$. We say that a bounded on $[0,1)^2$ function f belongs to $PVFl_p[0,1)^2$ (here PV means "partial variables"), if both quantities $Fl_p^{(1)}(f) = = V_p^{(1)}(f)_0$ and $Fl_p^{(2)}(f) = V_p^{(2)}(f)_0$ are finite. Also we introduce partial best approximation

$$E_{k,\infty}(f)_p = \inf\left\{\left\|f(x,y)-\sum_{j=0}^{k-1} a_j(y)\chi_j(x)\right\|_p\right\}, \quad k\in\mathbb{N},$$

where inf is taken over all measurable functions a_j on $[0,1)$ and f belongs to $L^p[0,1)^2$ for $1\leq p<\infty$ or $f\in C^*[0,1)^2$ for $p=\infty$. The quantity $E_{\infty,l}(f)_p$, $l\in\mathbb{N}$, is defined in a similar way. For $f\in L^1[0,1)^2$ and $k,l\in\mathbb{N}$ we consider

$$S_{k,\infty}(f)(x,y) = \sum_{j=0}^{k-1}\int_0^1 f(u,y)\overline{\chi_j(u)}\,du\chi_j(x)$$

and $S_{\infty,l}(f)(x,y)$ in a similar way.

Further C and C_i are constants different in distinct cases.

3. Auxiliary lemmas

Lemma 3.1. *Let $m_k\leq n<m_{k+1}$, $k\in\mathbb{Z}_+$. Then $\chi_n(x)$ is constant on all I_j^{k+1}, $1\leq j\leq m_{k+1}$ and for any $1\leq j\leq m_k$ the equality $\int_{I_j^k}\chi_n(x)\,dx=0$ holds.*

For the proof of Lemma 3.1 see [7, §1.5, (1.5.13) and (1.5.18)].

Lemma 3.2. (i) *Let* $1 < p < \infty$, $f \in PVFl_p[0,1)^2$. *Then*

$$\omega_n^{(i)}(f)_p \le m_n^{-1/p} V_p^{(i)}(f)_n, \quad n \in \mathbb{Z}_+, \quad i = 1, 2.$$

(ii) *Let* $f \in CFl_p[0,1)^2$, $1 < p < \infty$. *Then for* $r, s \in \mathbb{Z}_+$ *we have*

$$\omega_{r,s}(f)_p \le (m_r m_s)^{-1/p} V_p^{[2]}(f)_{r,s}.$$

Proof. Part (i) of Lemma 3.2 is established similar to [17, Lemma 9]. For the part (ii) we use the notation $\Delta f(x,y;h,t) = f(x \oplus h, y \oplus t) - f(x \oplus h, y) - f(x, y \oplus t) + f(x,y)$ and the fact that $x \in I_k^r$, $h \in I_1^r$ implies $x \oplus h \in I_k^r$ for a.e. x to obtain

$$\omega_{r,s}(f)_p = \sup_{h \in I_1^r, t \in I_1^s} \left(\int_0^1 \int_0^1 |\Delta f(x,y;h,t)|^p \, dx \, dy \right)^{1/p} =$$

$$= \sup_{h \in I_1^r, t \in I_1^s} \left(\sum_{k=1}^{m_r} \sum_{l=1}^{m_s} \int_{I_k^r} \int_{I_l^s} |\Delta f(x,y;h,t)|^p \, dy \, dx \right)^{1/p} \le$$

$$\le \left(\sum_{k=1}^{m_r} \sum_{l=1}^{m_s} osc^p(f, I_k^r \times I_l^s) m_r^{-1} m_s^{-1} \right)^{1/p} \le (m_r m_s)^{-1/p} V_p^{[2]}(f)_{r,s}. \qquad \blacksquare$$

Remark 3.1. For $f \in Fl_1[0,1)^2$ and $r, s \in \mathbb{Z}_+$ by the same method as in the proof of Lemma 2 we see that $\omega_{r,s}(f)_1 \le (m_r m_s)^{-1} Fl_1^{[2]}(f)$.

Lemma 3.3 is proved in the case $p_i \equiv 2$ for double Walsh system by F. Móricz in [9] and for multiple Vilenkin system in [18].

Lemma 3.3. *Let* $1 \le p < \infty$, $f \in L^p[0,1)^2$, *or* $p = \infty$, $f \in C^*[0,1)^2$, $r, s \in \mathbb{Z}_+$. *Then*

$$2^{-1} \omega_{r,s}^*(f)_p \le E_{m_r, m_s}(f)_p \le \|f - S_{m_r, m_s}(f)\|_p \le \omega_{r,s}^*(f)_p.$$

Lemma 3.4. *Let* $r, s \in \mathbb{Z}_+$, $1 \le j \le m_r$, $1 \le i \le m_s$, $x \in I_j^r$, $y \in I_i^s$. *Then*

$$S_{m_r,\infty}(f)(x,y) = m_r \int_{I_j^r} f(u,y) \, du, \quad S_{\infty, m_s}(f)(x,y) = m_s \int_{I_i^s} f(x,v) \, dv.$$

The proof of Lemma 3.4 is similar to the one-dimensional case and uses the formula $\sum_{j=0}^{m_r - 1} \chi_j(x) = m_r X_{I_1^r}(x)$, where X_E is the indicator of a set E (see [7, § 1.5, (1.5.21)]).

Lemma 3.5. *Let* $r, s \in \mathbb{Z}_+$, $1 \le p < \infty$ *and* $f \in L^p[0,1)^2$ *or* $p = \infty$ *and* $f \in C^*[0,1)^2$. *Then*

$$\|S_{m_r,\infty}(f)\|_p \le \|f\|_p, \quad \|S_{\infty.m_s}(f)\|_p \le \|f\|_p.$$

Proof. By Lemma 3.4 and Hölder inequality we have for $1 \le p < \infty$

$$\int_0^1 \int_0^1 |S_{m_r,\infty}(f)(x,y)|^p \, dx \, dy = \int_0^1 \sum_{j=1}^{m_r} \int_{I_j^r} \left| m_r \int_{I_j^r} f(u,y) \, du \right|^p dx \, dy \le$$

$$\le m_r^p \int_0^1 \sum_{j=1}^{m_r} \int_{I_j^r} \int_{I_j^r} |f(u,y)|^p \, du (m_r^{-1})^{p(1-1/p)} \, dx \, dy =$$

$$= \int_0^1 \sum_{j=1}^{m_r} \int_{I_j^r} |f(u,y)|^p \, du \, dy = \|f\|_p^p$$

In the case $p = \infty$ we easily obtain from Lemma 3.4

$$\|S_{m_r,\infty}(f)\|_\infty \le m_r \max_{1 \le j \le m_r} \int_{I_j^r} \|f\|_\infty \, du = \|f\|_\infty.$$

The second inequality of Lemma 3.5 may be proved in a similar manner. ■

Lemma 3.6. *If* $1 \le p < \infty$ *and* $f \in L^p[0,1)^2$ *or* $p = \infty$ *and* $f \in C^*[0,1)^2$, $k \in [m_r, m_{r+1})$, $r \in \mathbb{Z}_+$, $l \in [m_s, m_{s+1})$, $s \in \mathbb{Z}_+$, *then*

$$|\widehat{f}(k,l)| \le E_{k,l}(f)_p \le \omega^*_{r,s}(f)_p.$$

Proof. Let $g \in \mathcal{P}_{kl}$ be such that $\|f - g\|_p = E_{k,l}(f)_p$. Then by the Hölder inequality and orthogonality of $\{\chi_i\}_{i=0}^\infty$ we have

$$|\widehat{f}(k,l)| = \left| \int_0^1 \int_0^1 (f(x,y) - g(x,y)) \overline{\chi_k(x)\chi_l(y)} \, dx \, dy \right| \le \|f - g\|_p = E_{k,l}(f)_p.$$

The second inequality of Lemma follows from the decreasing of $E_{k,l}(f)_p$ in k and l and Lemma 3.3. ■

4. Approximation theorems

Theorem 4.1. *Let* $1 \le p < \infty$ *and* $f \in L^p[0,1)^2$ *or* $f \in C^*[0,1)^2$ *and* $p = \infty$. *Then the inequalities*

$$E_{m_r,\infty}(f)_p \le \|f - S_{m_r,\infty}(f)\|_p \le \omega_r^{(1)}(f)_p \le 2E_{m_r,\infty}(f)_p \tag{4.1}$$

and

$$E_{\infty,m_s}(f)_p \le \|f - S_{\infty,m_s}(f)\|_p \le \omega_s^{(2)}(f)_p \le 2E_{\infty,m_s}(f)_p \tag{4.2}$$

are valid for all $r, s \in \mathbb{Z}_+$.

Proof. The left inequality from (4.1) is obvious. By Lemma 4 and Hölder inequality we have in the case $1 \le p < \infty$

$$\|f - S_{m_r,\infty}(f)\|_p = \left(\sum_{j=1}^{m_r} \int_0^1 \int_{I_j^r} |f(x,y) - S_{m_r,\infty}(f)(x,y)|^p \, dx\, dy\right)^{1/p} \le$$

$$\le \left(\int_0^1 \sum_{j=1}^{m_r} \int_{I_j^r} \left(m_r \int_{I_j^r} |f(x,y) - f(u,y)|\, du\right)^p dx\, dy\right)^{1/p} \le$$

$$\le \left(\int_0^1 \sum_{j=1}^{m_r} \int_{I_j^r} m_r^p \int_{I_j^r} |f(x,y) - f(u,y)|^p \, du\, m_r^{1-p}\, dx\, dy\right)^{1/p} =$$

$$= \left(\int_0^1 \sum_{j=1}^{m_r} \int_{I_j^r} \int_{I_1^r} |f(x,y) - f(x\oplus h,y)|^p \, dh\, m_r\, dx\, dy\right)^{1/p} =$$

$$= \left(\int_0^1 \int_{I_1^r} \int_0^1 |f(x,y) - f(x\oplus h,y)|^p \, dx\, m_r\, dh\, dy\right)^{1/p} \le \omega_r^{(1)}(f)_p. \tag{4.3}$$

In the case $p = \infty$ we use the inequality $|f(x,y) - f(u,y)| \le \omega_r^{(1)}(f)_\infty$ for $x, u \in I_j^r$. Thus, the second inequality from (4.1) is proved. Finally, let $\varepsilon > 0$ and g_r be a polynomial of type $\sum_{j=0}^{m_r-1} a_j(y)\chi_j(x)$, where a_j be measurable on $[0,1)$ and $\|f - g_r\|_p < E_{m_r,\infty}(f)_p + \varepsilon$. If $h \in I_1^r$, then by Lemma 3.1

$g_r(x,y) = g_r(x \oplus h, y)$ for almost all $x \in [0,1)$. Therefore, for $1 \le p < \infty$ we obtain

$$\omega_r^{(1)}(f)_p = \sup_{h \in I_1^r} \left(\int_0^1 \int_0^1 |f(x \oplus h, y) - f(x,y)|^p \right)^{1/p} =$$

$$= \sup_{h \in I_1^r} \|f(x \oplus h, y) - g_r(x \oplus h, y) - f(x,y) + g_r(x,y)\|_p \le 2\|f - g_r\|_p.$$

Since $2\|f - g_r\|_p < 2E_{m_r,\infty}(f)_p + 2\varepsilon$ and $\varepsilon > 0$ is arbitrary, we obtain the last inequality from (4.1) in the case $p \in [1,\infty)$. The case $p = \infty$ is similar to the proved one. The inequality (4.2) is established in the same manner. ■

Theorem 4.2. *Let* $1 \le p < \infty$ *and* $f \in L^p[0,1)^2$ *or* $p = \infty$ *and* $f \in C^*[0,1)^2$, $r, s \in \mathbb{Z}_+$. *Then*

$$E_{m_r,m_s}(f)_p \le 2E_{m_r,\infty}(f)_p + 3E_{\infty,m_s}(f)_p.$$

Proof. Let $\varepsilon > 0$ and g_s be a polynomial of type $\sum_{j=0}^{m_s-1} b_j(x)\chi_j(y)$ such that $b_j(x)$ be measurable on $[0,1)$ and $\|f - g_s\|_p < E_{\infty,m_s}(f)_p + \varepsilon$. By virtue of Theorem 4.1 and properties of $S_{m_r,\infty}$ we have

$$E_{m_r,m_s}(f)_p \le E_{m_r,m_s}(f - g_s)_p + E_{m_r,m_s}(g_s)_p \le$$

$$\le \|f - g_s\|_p + \|g_s - S_{m_r,\infty}(g_s)\|_p \le E_{\infty,m_s}(f)_p + \varepsilon + 2E_{m_r,\infty}(g_s)_p \le$$

$$\le E_{\infty,m_s}(f)_p + \varepsilon + 2E_{m_r,\infty}(g_s - f)_p + 2E_{m_r,\infty}(f)_p \le E_{\infty,m_s}(f)_p + \varepsilon +$$

$$+ 2\|g_s - f\|_p + 2E_{m_r,\infty}(f)_p \le 3E_{\infty,m_s}(f)_p + 2E_{m_r,\infty}(f)_p + 3\varepsilon. \tag{4.4}$$

Since $\varepsilon > 0$ is arbitrary, we obtain the inequality of Theorem. ■

Remark 4.1. Similar to (4.4) we can obtain $E_{m_r,m_s}(f)_p \le 3E_{m_r,\infty}(f)_p + +2E_{\infty,m_s}(f)_p$. Summing this inequality with the inequality of Theorem 4.2 yields

$$E_{m_r,m_s}(f)_p \le \frac{5}{2}(E_{m_r,\infty}(f)_p + E_{\infty,m_s}(f)_p).$$

It is interesting to find the best constant in last inequality.

Corollary 4.1. (i) *Let* $1 \le p < \infty$ *and* $f \in L^p[0,1)^2$ *or* $p = \infty$ *and* $f \in \in C^*[0,1)^2$. *Then*

$$E_{m_r,m_s}(f)_p \le 3(\omega_r^{(1)}(f)_p + \omega_s^{(2)}(f)_p), \quad r, s \in \mathbb{Z}_+.$$

(ii) *Let* $1 < p < \infty$ *and* $f \in PVFl_p[0,1)^2$. *Then*

$$E_{m_r,m_s}(f)_p \le 3(m_r^{-1/p}V_p^{(1)}(f)_r + m_s^{-1/p}V_p^{(2)}(f)_s), \quad r, s \in \mathbb{Z}_+.$$

Proof. Part (i) of Corollary 4.1 may be deduced from Theorems 4.1 and 4.2, while the part (ii) follows from (1) and Lemma 3.2 (i). ■

Corollary 4.2. (i) *Under conditions of Corollary* 4.1 (i) *we have for* $k \in \in [m_r, m_{r+1})$, $r \in \mathbb{Z}_+$, $l \in [m_s, m_{s+1})$, $s \in \mathbb{Z}_+$,

$$|\widehat{f}(k,l)| \leq 3(\omega_r^{(1)}(f)_p + \omega_s^{(2)}(f)_p).$$

(ii) *Under conditions of Corollary* 4.1 (ii) *we have for* $k \in [m_r, m_{r+1})$, $r \in \mathbb{Z}_+$, $l \in [m_s, m_{s+1})$, $s \in \mathbb{Z}_+$,

$$|\widehat{f}(k,l)| \leq 3(m_r^{-1/p} Fl_p^{(1)}(f) + m_s^{-1/p} Fl_p^{(2)}(f)).$$

Both assertions of Corollary 4.2 follows from Corollary 4.1 and Lemma 3.6.

5. Estimates of Fourier coefficients

Theorem 5.1. *Let* $\mathbf{P} = \{p_n\}_{n=1}^{\infty}$ *is bounded by* N*. Then for* $k \in [m_r, m_{r+1})$, $l \in [m_s, m_{s+1})$, $r, s \in \mathbb{Z}_+$, *we have*

$$|\widehat{f}(k,l)| \leq C \int_0^1 \int_0^1 |\Delta_{r,s} f(x,y)|\, dx\, dy,$$

where $\Delta_{r,s} f(x,y) = f(x \oplus 1/m_{r+1}, y \oplus m_{s+1}^{-1}) - f(x \oplus m_{r+1}^{-1}, y) - f(x, y \oplus m_{s+1}^{-1}) + f(x,y)$ *and* C *depends on* N.

Proof. It is known that for $f_{a,b}(x,y) = f(x \oplus a, y \oplus b)$ the equality $\widehat{f_{a,b}}(k,l) = = \chi_k(a)\chi_l(b)\widehat{f}(k,l)$ holds. Therefore,

$$\int_0^1 \int_0^1 \Delta_{r,s} f(x,y) \overline{\chi_k(x)\chi_l(y)}\, dx\, dy = = (\chi_k(m_{r+1}^{-1}) - 1)(\chi_l(m_{s+1}^{-1}) - 1)\widehat{f}(k,l). \tag{5.1}$$

Since $k \in [m_r, m_{r+1})$, we can write $k = am_r + k'$, where $k' \in [0, m_r) \cap \mathbb{Z}$ and $a \in [1, p_{r+1} - 1] \cap \mathbb{Z}$. By Lemma 3.1 and definition we have

$$\chi_k(m_{r+1}^{-1}) = \chi_{am_r}(m_{r+1}^{-1})\chi_{k'}(m_{r+1}^{-1}) = \chi_{am_r}(m_{r+1}^{-1})\chi_{k'}(0) = \exp(2\pi i a/p_{r+1})$$

and

$$|\chi_k(m_{r+1}^{-1})-1| \geq |\exp(2\pi i/p_{r+1})-1| = 2\sin(\pi/p_{k+1}) \geq 2\sin(\pi/N).$$

Similar inequality is valid for $|\chi_l(m_{s+1}^{-1})-1|$. From (5.1) we obtain

$$|\widehat{f}(k,l)| \leq (2\sin(\pi/N))^{-2}\int_0^1\int_0^1 |\Delta_{r,s}f(x,y)|\,dx\,dy. \qquad \blacksquare$$

Theorem 5.2. *Let* $\mathbf{P}$ *be bounded,* $\omega=\{\omega_n\}_{n=0}^{\infty}$ *and* $\beta=\{\beta_n\}_{n=1}^{\infty}$ *be decreasing to zero sequences,* $1\leq p\leq\infty$.

(i) *If* $f(x,y)\in H_p^{\omega,\beta}$, $k\in[m_r,m_{r+1})$, $l\in[m_s,m_{s+1})$, $r,s\in\mathbb{Z}_+$, *then* $|\widehat{f}(k,l)|\leq C\omega_r\beta_s$, *where* C *is independent of* k *and* l.

(ii) *Let* ω *and* β *satisfy the Bary condition* (see, e.g, [1, Introductory Material, § 4, (4.1)])

$$\sum_{k=n}^{\infty}\omega_k = O(\omega_n), \quad \sum_{k=n}^{\infty}\beta_k = O(\beta_n), \quad n\in\mathbb{Z}_+. \tag{5.2}$$

Then there exists a function $f_{\omega,\beta}\in H_p^{\omega,\beta}$ *such that* $|\widehat{f_{\omega,\beta}}(m_r,m_s)|=\omega_r\beta_s$.

Proof. (i) Since $\|\cdot\|_1\leq\|\cdot\|_p\leq\|\cdot\|_\infty$, we obtain

$$\omega_{r,s}(f)_1\leq\omega_{r,s}(f)_p\leq\omega_{r,s}(f)_\infty, \quad r,s\in\mathbb{Z}_+. \tag{5.3}$$

From Theorem 5.1 we deduce that for $k\in[m_r,m_{r+1})$, $l\in[m_s,m_{s+1})$, $r,s\in\in\mathbb{Z}_+$, the inequality

$$|\widehat{f}(k,l)|\leq C_1\omega_{r,s}(f)_1\leq C_1\omega_{r,s}(f)_p\leq C_2\omega_r\beta_s$$

holds.

(ii) Let us consider the function

$$f_{\omega,\beta}(x,y)=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\omega_r\beta_s\chi_{m_r}(x)\chi_{m_s}(y). \tag{5.4}$$

Since ω and β satisfy the Bary condition (5.2), in particular, we have

$$\sum_{r=0}^{\infty}\omega_r<\infty, \quad \sum_{s=0}^{\infty}\beta_s<\infty, \quad \text{and morover} \quad \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\omega_r\beta_s<\infty.$$

Thus, the right-hand side of (5.4) converges absolutely and uniformly to $\mathbf{P}$-continuous function $f_{\omega,\beta}$. If $\rho(x_1,x_2)<m_r^{-1}$, $\rho(y_1,y_2)<m_r^{-1}$, then

$$|(\chi_{m_k}(x_1)-\chi_{m_k}(x_2))(\chi_{m_l}(y_1)-\chi_{m_l}(y_2))|=0$$

for $k < r$ or $l < s$ by Lemma 3.1. Therefore, (5.4) implies

$$|f_{\omega,\beta}(x_1,y_1) - f_{\omega,\beta}(x_1,y_2) - f_{\omega,\beta}(x_2,y_1) + f_{\omega,\beta}(x_2,y_2)| \leq$$

$$\leq \sum_{k=r}^{\infty}\sum_{l=s}^{\infty}\omega_k\beta_l|\chi_{m_k}(x)\chi_{m_l}(y)| = \sum_{k=r}^{\infty}\omega_k\sum_{l=s}^{\infty}\beta_l \leq C_3\omega_r\beta_s$$

and by (5.3) we have $\omega_{r,s}(f_{\omega,\beta})_p \leq \omega_{r,s}(f_{\omega,\beta})_\infty \leq C_3\omega_r\beta_s$. On the other hand, $\widehat{f_{\omega,\beta}}(m_r,m_s) = \omega_r\beta_s$. ■

Corollary 5.1. (i) *Let* $\mathbf{P}$ *be bounded and* $1 \leq p < \infty$, $f \in Fl_p[0,1)^2$. *Then for* $k,l \in \mathbb{N}$ *the inequality* $|\widehat{f}(k,l)| \leq C(kl)^{-1/p}$ *holds.*

(ii) *There exists* $f_0 \in Fl_p[0,1)^2$ *such that* $|\widehat{f_0}(k,l)| = (kl)^{-1/p}$ *for* $k = m_r$, $l = m_s$, $r,s \in \mathbb{Z}_+$.

Proof. (i) By Lemma 3.2(ii) we have $\omega_{r,s}(f)_p \leq Fl_p^{[2]}(f)(m_rm_s)^{-1/p}$ and for $k \in [m_r,m_{r+1})$, $l \in [m_s,m_{s+1})$, $r,s \in \mathbb{Z}_+$, by Theorem 5.2(i) we obtain

$$|\widehat{f}(k,l)| \leq C_1(m_rm_s)^{-1/p} \leq C_1N^{2/p}(m_{r+1}m_{s+1})^{-1/p} \leq C_2(kl)^{-1/p},$$

where N is an upper bound for $\mathbf{P}$.

(ii) Let $\omega_r = \beta_r = m_r^{-1/p}$. Then $\omega = \{\omega_i\}_{i=0}^{\infty}$ and $\beta = \{\beta_i\}_{i=0}^{\infty}$ satisfy the Bary condition (5.2) and by Theorem 5.2(ii) the function

$$f_0(x,y) = f_{\omega,\beta}(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(m_rm_s)^{-1/p}\chi_{m_r}(x)\chi_{m_s}(y)$$

belongs to $H_\infty^{\omega,\beta}$. Then $osc(f_0, I_i^r \times I_j^s) \leq C_3(m_rm_s)^{-1/p}$ and for $r,s \in \mathbb{Z}_+$

$$\left(\sum_{i=1}^{m_r}\sum_{j=1}^{m_s}osc^p(f_0, I_i^r \times I_j^s)\right)^{1/p} \leq C_3\left(\sum_{i=1}^{m_r}\sum_{j=1}^{m_s}(m_rm_s)^{-1}\right)^{1/p} = C_3 < \infty.$$

Thus, $f_0 \in Fl_p[0,1)^2$ and satisfies all assertions of Corollary. ■

Corollary 5.2. (i) *Let* $\mathbf{P}$ *be bounded,* $1 < p < \infty$, $f \in CFl_p[0,1)^2$, $\omega = \{\omega_n\}_{n=0}^{\infty}$ *and* $\beta = \{\beta_n\}_{n=1}^{\infty}$ *be decreasing to zero sequences, and* $V_p^{[2]}(f)_{r,s} \leq C\omega_r\beta_s$, $r,s \in \mathbb{Z}_+$. *Then for* $k \in [m_r,m_{r+1})$, $l \in [m_s,m_{s+1})$, $r,s \in \mathbb{Z}_+$, *the relation* $|\widehat{f}(k,l)| = O((m_rm_s)^{-1/p}\omega_r\beta_s)$ *holds.*

(ii) *There exists* $f_0 \in CFl_p[0,1)^2$, $1 < p < \infty$, *such that* $V_p^{[2]}(f_0)_{r,s} \leq C\omega_r\beta_s$, $r,s \in \mathbb{Z}_+$, *and* $|\widehat{f_0}(m_r,m_s)| = (m_rm_s)^{-1/p}\omega_r\beta_s$, $r,s \in \mathbb{Z}_+$.

Proof. (i) By the proof of Theorem 5.2 and Lemma 3.2 we have

$$|\widehat{f}(k,l)| \le C_1\omega_{rs}(f)_p \le C_1(m_rm_s)^{-1/p}V_p^{[2]}(f)_{r,s} \le C_2(m_rm_s)^{-1/p}\omega_r\beta_s$$

for $k \in [m_r, m_{r+1})$, $l \in [m_s, m_{s+1})$, $r, s \in \mathbb{Z}_+$.

(ii) It is easy to see that $\{m_n^{-1/p}\omega_n\}_{n=0}^{\infty}$ and $\{m_n^{-1/p}\beta_n\}_{n=0}^{\infty}$ satisfy the Bary condition (5.2), e.g.

$$\sum_{k=n}^{\infty} m_k^{-1/p}\omega_k \le \omega_n \sum_{k=n}^{\infty} m_k^{-1/p} \le C_3\omega_n m_n^{-1/p}, \quad n \in \mathbb{Z}_+.$$

By Theorem 5.2(ii) the function

$$f_0(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(m_rm_s)^{-1/p}\omega_r\beta_s\chi_{m_r}(x)\chi_{m_s}(y)$$

satisfies the inequality $\omega_{k,l}(f_0)_\infty \le C_4\omega_k\beta_l(m_km_l)^{-1/p}$, $k, l \in \mathbb{Z}_+$. Hence,

$$V_p^{[2]}(f)_{r,s} = \sup_{k\ge r, l\ge s}\left(\sum_{i=1}^{m_k}\sum_{j=1}^{m_l} osc^p(f_0, I_i^k \times I_j^l)\right)^{1/p} \le$$

$$\le C_4 \sup_{k\ge r, l\ge s}\left(\sum_{i=1}^{m_k}\sum_{j=1}^{m_l} m_k^{-1}m_l^{-1}\omega_k^p\beta_l^p)\right)^{1/p} = C_4\omega_r\beta_s. \tag{5.5}$$

Thus, we have $f_0 \in CFl_p[0,1)^2$ and (5.5) is valid, i.e. f_0 is a required function. ∎

Theorem 5.3. *Let φ be a N-function, $\mathbf{P} = \{p_j\}_{j=1}^{\infty}$ be bounded and $\Lambda = \{\lambda_i\}_{i=1}^{\infty}$, $\Gamma = \{\gamma_i\}_{i=1}^{\infty}$ be increasing sequences of positive numbers such that $\lim_{n\to\infty}\Lambda_n = \lim_{n\to\infty}\Gamma_n = \infty$, where $\Lambda_n = \sum_{k=1}^{n}\lambda_k^{-1}$, $\Gamma_n = \sum_{k=1}^{n}\gamma_k^{-1}$. Then for $k \in [m_r, m_{r+1})$, $l \in [m_s, m_{s+1})$, $r, s \in \mathbb{Z}_+$, the inequality*

$$|\widehat{f}(k,l)| \le C(f,\mathbf{P})\varphi^{-1}(\Lambda_k^{-1}\Gamma_l^{-1})$$

holds.

Proof. Let $k \in [m_r, m_{r+1})$, $l \in [m_s, m_{s+1})$, $r, s \in \mathbb{Z}_+$, and $\mathbf{P}$ be bounded by N. Using the property $\int_0^1 g(x \oplus a)\,dx = \int_0^1 g(x)\,dx$ (see [7, § 2.1] in the case $p_i \equiv 2$) for $g \in L^1[0,1)$ and Theorem 5.1 we obtain

$$|\widehat{f}(k,l)| \le C_1\int_0^1\int_0^1 |\Delta_{r,s}f(x \oplus i/m_r, y \oplus j/m_s)|\,dx\,dy, \tag{5.6}$$

where $i \in [0, m_r) \cap \mathbb{Z}$, $j \in [0, m_s) \cap \mathbb{Z}$ and $\Delta_{r,s} f(x, y)$ is defined in Theorem 5.1. It is easy to see that $x \oplus i/m_r \oplus 1/m_{r+1}$ and $x \oplus i/m_r$ always exist and belong to the same interval I_k^r, while for different i the corresponding numbers k are also different. Denoting $\Delta_{r,s} f(x \oplus i/m_r, y \oplus j/m_s)$ by $g_{ij}(x, y)$ and applying the integral Jensen inequality we deduce from (5.6) that for $C_2 > 0$

$$\varphi(C_2|\widehat{f}(k,l)|) \le \varphi\left(C_1 C_2 \int_0^1 \int_0^1 |g_{ij}(x,y)|\,dx\,dy\right) \le \tag{5.7}$$
$$\le \int_0^1 \int_0^1 \varphi(C_1 C_2 |g_{ij}(x,y)|)\,dx\,dy.$$

Multiplying both sides of (5.7) by $\lambda_{i+1}^{-1}\gamma_{j+1}^{-1}$ and summing new inequalities over $i = 0, 1, \ldots, m_r - 1$ and $j = 0, 1, \ldots, m_s - 1$ we obtain

$$\Lambda_{m_r}\Gamma_{m_s}\varphi(C_2|\widehat{f}(k,l)|) \le \int_0^1 \int_0^1 \sum_{i=0}^{m_r-1} \sum_{j=0}^{m_s-1} \frac{\varphi(C_1 C_2 |g_{ij}(x,y)|)}{\lambda_{i+1}\gamma_{j+1}}\,dx\,dy \le$$
$$\le Fl_{\Lambda,\Gamma,\varphi}(C_1 C_2 f).$$

Since φ is convex and $\varphi(0) = 0$, for $\alpha \in (0, 1)$ we have $\varphi(\alpha x) \le \alpha\varphi(x)$. Therefore, $Fl_{\Lambda,\Gamma,\varphi}(C_1 C_2 f) \le 1$ for sufficiently small $C_2 > 0$, whence

$$C_2|\widehat{f}(k,l)| \le \varphi^{-1}(Fl_{\Lambda,\Gamma,\varphi}(C_1 C_2 f)\Lambda_{m_r}^{-1}\Gamma_{m_s}^{-1}) \le \varphi^{-1}(\Lambda_{m_r}^{-1}\Gamma_{m_s}^{-1}),$$

where C_2 depends on f. It is known that φ^{-1} is subadditive, i.e. $\varphi^{-1}(a+b) \le \le \varphi^{-1}(a) + \varphi^{-1}(b)$, $a, b \ge 0$ (see [8, § 1, (1.20)]). Also we have $\Lambda_k \le \Lambda_{m_{r+1}} \le \le N\Lambda_{m_r}$ and $\Gamma_l \le N\Gamma_{m_s}$ since $\{\lambda_i^{-1}\}_{i=1}^{\infty}$ and $\{\gamma_i^{-1}\}_{i=1}^{\infty}$ are decreasing. Thus, we obtain

$$|\widehat{f}(k,l)| \le C_2^{-1}\varphi^{-1}(\Lambda_{m_r}^{-1}\Gamma_{m_s}^{-1}) \le N^2 C_2^{-1}\varphi^{-1}(\Lambda_k^{-1}\Gamma_l^{-1}),$$

where N is a majorant for **P**. ■

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