

A NOTE ON A RESULT OF B. BOJAN

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Communicated by Imre Kátai

(Received March 27, 2019; accepted June 21, 2019)

Abstract. We determine all solutions of those functions $f : \mathbb{N} \rightarrow \mathbb{C}$ for which $f(n^2 + m^2 + k) = f^2(n) + f^2(m) + K$ is satisfied for all positive integers n, m , where a non-negative integer k and $K \in \mathbb{C}$ are given. This improves the result of B. Bojan.

1. Introduction

Let, as usual, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{C}$ be the set positive integers, integers and complex numbers, respectively.

In 2014 B. Bojan [1] determined all solutions of those $f : \mathbb{N} \rightarrow \mathbb{C}$ for which

$$f(n^2 + m^2) = f^2(n) + f^2(m) \quad \text{for every } n, m \in \mathbb{N},$$

namely

$$f(n) \in \{0, \pm \frac{1}{2}, \pm n\}.$$

Our purpose in this note is to improve the result of B. Bojan [1].

Theorem. Assume that a non-negative integer k , a complex number K and an arithmetical function $f : \mathbb{N} \rightarrow \mathbb{C}$ satisfy the equation

$$f(n^2 + m^2 + k) = f^2(n) + f^2(m) + K \quad \text{for every } n, m \in \mathbb{N}.$$

Key words and phrases: Arithmetical functions, functional equation.

2010 Mathematics Subject Classification: 11A07, 11A25, 11N25, 11N64.

<https://doi.org/10.71352/ac.49.285>

Then one of the following assertions holds:

- a) $f(n) = \pm\chi_2(n)$, $f(n^2 + m^2 + k) = \chi_2(n) + \chi_2(m) - 1$ and $2 \nmid k$, $K = -1$,
- b) $f(n) = \pm\chi_2(n+1)$, $f(n^2 + m^2 + k) = \chi_2(n+1) + \chi_2(m+1) - 1$ and $2 \mid k$, $K = -1$,
- c) $f(n) = \frac{\varepsilon(n)}{4}(1 - \sqrt{-8K+1})$,
- d) $f(n) = \frac{\varepsilon(n)}{4}(1 + \sqrt{-8K+1})$,
- e) $f(n) = \frac{\varepsilon(n)}{4}a(n)\sqrt{-8\chi_3(n)+9}$ and $K = -\frac{7}{8}$ for $k \equiv 1 \pmod{3}$,
- f) $f(n) = \varepsilon(n)b(n)\chi_3(n)$ and $K = -1$ for $k \equiv 2 \pmod{3}$,
- g) $f(n) = \frac{\varepsilon(n)}{4}b(n)\sqrt{-8\chi_3(n)+9}$ and $K = -\frac{3}{8}$ for $k \equiv 0 \pmod{3}$.
- h) $f(n) = \varepsilon(n)n$,

where $\chi_2(n) \pmod{2}$, $\chi_3(n) \pmod{3}$ are principal Dirichlet characters, $\varepsilon(n) \in \{-1, 1\}$ and $\varepsilon(n^2 + m^2 + k) = 1$ for every $n, m \in \mathbb{N}$, $a(n)$, $b(n)$ are periodic sequences $\pmod{3}$ with $a(1) = 1$, $a(2) = a(3) = -1$, $b(1) = b(3) = 1$, $b(2) = -1$.

2. Lemmas

Assume that a non-negative integer k , a complex number K and an arithmetical function $f : \mathbb{N} \rightarrow \mathbb{C}$ satisfy the equation

$$(2.1) \quad f(n^2 + m^2 + k) = f^2(n) + f^2(m) + K \quad \text{for every } n, m \in \mathbb{N}.$$

Lemma 1. *Let $F(n) := f^2(n)$. Then*

$$(2.2) \quad \begin{aligned} F(\ell + 12) &= F(\ell + 9) + F(\ell + 8) + F(\ell + 7) - \\ &\quad - F(\ell + 5) - F(\ell + 4) - F(\ell + 3) + F(\ell) \end{aligned}$$

holds for every $\ell \in \mathbb{N}$ and

$$(2.3) \quad \begin{cases} F(7) = 2F(5) - F(1), \\ F(8) = 2F(5) + F(4) - 2F(1), \\ F(9) = F(6) + 2F(5) - F(2) - F(1), \end{cases}$$

$$(2.3) \quad \begin{cases} F(10) = F(6) + 3F(5) - F(3) - 2F(1), \\ F(11) = F(6) + 4F(5) - F(3) - F(2) - 2F(1), \\ F(12) = F(6) + 4F(5) + F(4) - F(2) - 4F(1). \end{cases}$$

Proof. We infer from (2.1) and the next relations

$$(2n+1)^2 + (n-2)^2 = (2n-1)^2 + (n+2)^2, \quad (2n+3)^2 + (n-6)^2 = (2n-3)^2 + (n+6)^2$$

that

$$f[(2n+1)^2 + (n-2)^2 + k] = f[(2n-1)^2 + (n+2)^2 + k]$$

and

$$f[(2n+3)^2 + (n-6)^2 + k] = f[(2n-3)^2 + (n+6)^2 + k].$$

These imply that

$$F(2n+1) + F(n-2) = F(2n-1) + F(n+2)$$

and

$$F(2n+3) + F(n-6) = F(2n-3) + F(n+6).$$

We infer from the last relations that

$$\begin{aligned} F(n+6) - F(n-6) &= F(2n+3) - F(2n-3) = \\ &= [F(2n+3) - F(2n+1)] + [F(2n+1) - F(2n-1)] + \\ &\quad + [F(2n-1) - F(2n-3)] = \\ &= [F(n+3) - F(n-1)] + [F(n+2) - F(n-2)] + \\ &\quad + [F(n+1) - F(n-3)], \end{aligned}$$

consequently

$$\begin{aligned} F(n+6) &= F(n+3) + F(n+2) + F(n+1) - F(n-1) - \\ &\quad - F(n-2) - F(n-3) + F(n-6). \end{aligned}$$

This with $m = \ell + 6$ proves (2.2).

In the order to prove (2.3), we note from (2.1) that

$$(2.4) \quad \text{If } x^2 + y^2 = u^2 + v^2, \text{ then } F(x) + F(y) = F(u) + F(v).$$

Applications of (2.4) in the cases

$$\begin{aligned} (x, y, u, v) \in \{ & (5, 5, 7, 1), (7, 4, 8, 1), (7, 6, 9, 2), \\ & (9, 7, 11, 3), (9, 8, 12, 1), (10, 5, 11, 2) \} \end{aligned}$$

prove that (2.3) holds for $F(7), F(8), F(9), F(11), F(12)$ and $F(10)$.

Thus, (2.3) and Lemma 1 is proved. ■

Lemma 2. Assume that the function $F : \mathbb{N} \rightarrow \mathbb{C}$ satisfy (2.2) and (2.3). Let

$$\begin{aligned} A &:= \frac{1}{120} \left(F(6) + 4F(5) - F(3) - F(2) - 3F(1) \right), \\ \Gamma_2 &:= \frac{-1}{8} \left(F(6) - 4F(5) + 4F(4) - F(3) + 3F(2) - 3F(1) \right), \\ \Gamma_3 &:= \frac{-1}{3} \left(F(6) - 2F(5) + 2F(3) - F(2) \right), \\ \Gamma_4 &:= \frac{1}{4} \left(F(6) - 2F(4) - F(3) + F(2) + F(1) \right), \\ \Gamma_5 &:= \frac{1}{5} \left(F(6) - F(5) - F(3) - F(2) + 2F(1) \right), \\ \Gamma &:= \frac{1}{4} \left(F(6) - 4F(5) + 2F(4) + 3F(3) + F(2) + F(1) \right), \\ B_k &:= \Gamma_2 \chi_2(k) + \Gamma_3 \chi_3(k) + \Gamma_4 \chi_4(k-1) + \Gamma_5 \chi_5(k) + \Gamma, \end{aligned}$$

where $\chi_2(k) \pmod{2}$, $\chi_3(k) \pmod{3}$ are the principal Dirichlet characters and $\chi_4(k) \pmod{4}$, $\chi_5(k) \pmod{5}$ are the real, non-principal Dirichlet characters, i.e.

$$\begin{aligned} \chi_2(0) &= 0, \chi_2(1) = 1, \chi_3(0) = 0, \chi_3(1) = \chi_3(2) = 1, \\ \chi_4(0) &= \chi_4(2) = 0, \chi_4(1) = 1, \chi_4(3) = -1, \\ \chi_5(2) &= \chi_5(3) = -1, \chi_5(1) = \chi_5(4) = 1. \end{aligned}$$

Then we have

$$(2.5) \quad F(\ell) = A\ell^2 + B_\ell \quad \text{for every } \ell \in \mathbb{N}.$$

Proof. The proof is very similar to that of Lemma 2 in [4]. Here we omit the proof. ■

3. Proof of the Theorem

Assume that a non-negative integer k , a complex number K and an arithmetical function $f : \mathbb{N} \rightarrow \mathbb{C}$ satisfy the equation (2.1). Let $F(n) := f^2(n)$ for every $n \in \mathbb{N}$. From (2.1) and Lemma 2, we have

$$(3.1) \quad F(n^2 + m^2 + k) = (F(n) + F(m) + K)^2$$

and

$$(3.2) \quad F(\ell) = f^2(\ell) = A\ell^2 + B_\ell \quad \text{for every } \ell \in \mathbb{N},$$

where

$$B_\ell := \Gamma_2\chi_2(\ell) + \Gamma_3\chi_3(\ell) + \Gamma_4\chi_4(\ell - 1) + \Gamma_5\chi_5(\ell) + \Gamma.$$

Lemma 3. *We have*

$$A \in \{0, 1\}.$$

Proof. We obtain from (3.1) and (3.2) that

$$(3.3) \quad A(n^2 + m^2 + k)^2 + B_{n^2+m^2+k} = \left(A(n^2 + m^2) + B_n + B_m + K \right)^2$$

for every $n, m \in \mathbb{N}$. Since

$$|B_\ell| \leq |\Gamma_2| + |\Gamma_3| + |\Gamma_4| + |\Gamma_5| + |\Gamma| \quad \text{for every } \ell \in \mathbb{N}$$

and

$$n^2 + m^2 + k \rightarrow \infty \quad \text{as } n, m \rightarrow \infty,$$

we infer from (3.3) that

$$\begin{aligned} A &= \lim_{n, m \rightarrow \infty} \left[A \left(1 + \frac{k}{n^2 + m^2 + k} \right)^2 + \frac{B_{n^2+m^2+k}}{(n^2 + m^2 + k)^2} \right] = \\ &= \lim_{n, m \rightarrow \infty} \left[\left(A + \frac{-k + B_n + B_m + K}{n^2 + m^2 + k} \right)^2 \right] = A^2. \end{aligned}$$

Therefore, we have $A \in \{0, 1\}$. Lemma 3 is proved. ■

Lemma 4. *Assume that $A = 1$, i.e.*

$$F(\ell) = f^2(\ell) = \ell^2 + B_\ell \quad \text{for every } \ell \in \mathbb{N},$$

then

$$B_m = 0, \quad f(m) = \pm m \quad \text{and} \quad f(n^2 + m^2 + k) = n^2 + m^2 + k$$

for every $n, m \in \mathbb{N}$.

Proof. From (3.3) we obtain that

$$\begin{aligned} (3.4) \quad & (n^2 + m^2 + k)^2 + B_{n^2+m^2+k} - \\ & - \left((n^2 + m^2) + B_n + B_m + K \right)^2 = \\ & = 2(k - B_n - B_m - K)n^2 + W(n, m) = 0, \end{aligned}$$

holds for every $n, m \in \mathbb{N}$, where

$$(3.5) \quad W(n, m) := B_{n^2+m^2+k} + (m^2 + k)^2 - (m^2 + B_n + B_m + K)^2.$$

Now let $m \in \mathbb{N}$ be fixed, $n \in \mathbb{N}$, $n \equiv a \pmod{60}$ with some $a \in \mathbb{N}$, $0 \leq a < 60$. Since the sequence $\{B_\ell\}_0^\infty$ is periodic $\pmod{60}$, we have

$$|W(n, m)| = |W(a, m)| < \infty$$

and so we obtain from (3.4) that

$$k - B_a - K - B_m = \lim_{\substack{n \rightarrow \infty \\ n \equiv a \pmod{60}}} \frac{-W(n, m)}{2n^2} = 0$$

and

$$W(a, m) = 0$$

hold for each $a, m \in \mathbb{N}$. It follows from these that

$$B_m = k - B_a - K = c \quad \text{for every } m \in \mathbb{N}$$

and

$$W(a, m) := c + (m^2 + k)^2 - (m^2 + 2c + K)^2 = 0 \quad \text{for every } m \in \mathbb{N},$$

where $c \in \mathbb{C}$ is some fixed constant. These show that

$$(3.6) \quad k - K = 2c \quad \text{and} \quad c + k^2 - (2c + K)^2 = 0.$$

Finally, we obtain from the last relations that

$$k = 2c + K \quad \text{and} \quad 0 = c + k^2 - (2c + K)^2 = c + k^2 - k^2 = c.$$

Thus we have

$$B_m = c = 0 \quad \text{and} \quad F(m) = m^2 \quad \text{for every } m \in \mathbb{N}$$

and from (3.6) we have $k = K$. Thus, in the case $A = 1$, we proved that

$$f^2(m) = m^2, f(n) = \pm m$$

and

$$f(n^2 + m^2 + k) = f^2(n) + f^2(m) + K = n^2 + m^2 + k$$

for every $n, m \in \mathbb{N}$. Lemma 4 is proved. ■

Lemma 5. *Assume that $A = 0$. Then*

$$(3.7) \quad \Gamma_4 = 0,$$

where Γ_4 is defined in Lemma 2.

Proof. We infer from the fact $A = 0$ and from (3.1), (3.2) that

$$(3.8) \quad B_{n^2+m^2+k} = (B_n + B_m + K)^2 \quad \text{for every } n, m \in \mathbb{N}.$$

Since $\{B_\ell\}_0^\infty$ is periodic (mod 60), therefore

$$B_{2^2+2^2+k} = B_{8^2+2^2+k} \quad \text{and} \quad B_{2^2+2^2+k} = B_{8^2+8^2+k},$$

which with (3.8) imply that

$$(B_2 + B_2 + K)^2 = (B_8 + B_2 + K)^2 \quad \text{and} \quad (B_2 + B_2 + K)^2 = (B_8 + B_8 + K)^2.$$

Consequently

$$(3.9) \quad (B_2 - B_8)(3B_2 + B_8 + 2K) = 0$$

and

$$(3.10) \quad 4(B_2 - B_8)(B_2 + B_8 + K) = 0.$$

Since

$$B_2 = \Gamma_3 + \Gamma_4 - \Gamma_5 + \Gamma \quad \text{and} \quad B_8 = \Gamma_3 - \Gamma_4 - \Gamma_5 + \Gamma,$$

we have $\Gamma_4 = 0$ if $B_2 = B_8$. Assume now that $B_2 - B_8 \neq 0$. Then we infer from (3.9) and (3.10) that

$$3B_2 + B_8 + 2K = 0 \quad \text{and} \quad B_2 + B_8 + K = 0,$$

consequently

$$B_2 = B_8.$$

This contradicts to the assumption $B_2 - B_8 \neq 0$. Thus, $\Gamma_4 = 0$ follows and the proof of Lemma 5 is completed. ■

Lemma 6. Assume that $A = 0$, $\Gamma_4 = 0$ and $\Gamma_2 \neq 0$. Then $K = -1$, furthermore either

$$2 \nmid k, \quad f(n) = \pm \chi_2(n) \quad \text{and} \quad f(n^2 + m^2 + k) = \chi_2(n) + \chi_2(m) - 1$$

or

$$2 \mid k, \quad f(n) = \pm \chi_2(n+1) \quad \text{and} \quad f(n^2 + m^2 + k) = \chi_2(n+1) + \chi_2(m+1) - 1$$

for every $n, m \in \mathbb{N}$.

Proof. Assume that $A = 0$, $\Gamma_4 = 0$ and $\Gamma_2 \neq 0$. Then

$$(3.11) \quad B_k := \Gamma_2 \chi_2(k) + \Gamma_3 \chi_3(k) + \Gamma_5 \chi_5(k) + \Gamma \quad \text{for every } k \in \mathbb{N},$$

and

$$(3.12) \quad B_n = B_m \quad \text{if } n \equiv m \pmod{30}.$$

It is clear from (3.12) that if $n^2 + m^2 \equiv u^2 + v^2 \pmod{30}$, then

$$n^2 + m^2 + k \equiv u^2 + v^2 + k \pmod{30} \quad \text{and} \quad B_{n^2+m^2+k} = B_{u^2+v^2+k}.$$

Let

$$(3.13) \quad \begin{aligned} E(n, m, u, v) &:= (B_n + B_m + K)^2 - (B_u + B_v + K)^2 = \\ &= (B_n + B_m - B_u - B_v)(B_n + B_m + B_u + B_v + 2K). \end{aligned}$$

We obtain from the above relations and from (3.8) that

$$(3.14) \quad \text{If } n^2 + m^2 \equiv u^2 + v^2 \pmod{30}, \text{ then } E(n, m, u, v) = 0.$$

With the help of a computer and Maple program, we find those positive integers n, m, u, v such that $n^2 + m^2 \equiv u^2 + v^2 \pmod{30}$, therefore we obtain the equations $E(n, m, u, v) = 0$.

It is clear from (3.11) that

$$\begin{cases} B_1 = \Gamma_2 + \Gamma_3 + \Gamma_5 + \Gamma, \\ B_2 = \Gamma_3 - \Gamma_5 + \Gamma, \\ B_3 = \Gamma_2 - \Gamma_5 + \Gamma, \\ B_4 = \Gamma_3 + \Gamma_5 + \Gamma, \\ B_5 = \Gamma_2 + \Gamma_3 + \Gamma, \\ B_6 = \Gamma_5 + \Gamma. \end{cases}$$

By using (3.13), we infer from these and from the following relations

$$\begin{cases} E(1, 1, 4, 4) = 4\Gamma_2(2\Gamma + 2\Gamma_5 + 2\Gamma_3 + \Gamma_2 + K) = 0, \\ E(1, 5, 4, 10) = 4\Gamma_2(2\Gamma + \Gamma_5 + 2\Gamma_3 + \Gamma_2 + K) = 0, \\ E(1, 3, 2, 6) = 4\Gamma_2(2\Gamma + \Gamma_3 + \Gamma_2 + K) = 0 \end{cases}$$

that

$$\begin{cases} 2\Gamma + 2\Gamma_5 + 2\Gamma_3 + \Gamma_2 + K = 0, \\ 2\Gamma + \Gamma_5 + 2\Gamma_3 + \Gamma_2 + K = 0, \\ 2\Gamma + \Gamma_3 + \Gamma_2 + K = 0. \end{cases}$$

Solve these equations to get

$$\Gamma_3 = 0, \Gamma_5 = 0 \quad \text{and} \quad \Gamma_2 = -(2\Gamma + K).$$

Therefore the sequence $B_n = \Gamma_2 \chi_2(n) + \Gamma$ is periodic (mod 2) and

$$B_1 = \Gamma_2 + \Gamma = -(\Gamma + K) \quad \text{and} \quad B_2 = \Gamma.$$

On the other hand, we obtain from (3.8) that

$$B_k = (2\Gamma + K)^2 \quad \text{and} \quad B_{k+1} = 0.$$

Thus, two possibilities exist: either

$$(a) \quad B_1 = B_k \quad \text{and} \quad B_2 = B_{k+1} = 0$$

or

$$(b) \quad B_1 = B_{k+1} = 0 \quad \text{and} \quad B_2 = B_k.$$

Case (a). Assume that $B_k = B_1$ and $B_{k+1} = B_2$. Then $k \equiv 1 \pmod{2}$ and

$$\begin{cases} B_k - B_1 = (2\Gamma + K)^2 + (\Gamma + K) = 0, \\ B_{k+1} - B_2 = 0 - \Gamma = 0. \end{cases}$$

Since $\Gamma_2 = -(2\Gamma + K) \neq 0$, the above system implies that $\Gamma = 0$ and $K = -1$. Consequently

$$B_1 = 1, B_2 = 0 \quad \text{and} \quad F(n) = B_n = \chi_2(n)$$

and

$$f(n^2 + m^2 + k) = \chi_2(n) + \chi_2(m) - 1, \quad k \equiv 1 \pmod{2}.$$

Case (b). Assume that $B_k = B_2$ and $B_{k+1} = B_1$. Then

$$\begin{cases} B_k - B_2 = (2\Gamma + K)^2 - \Gamma = 0, \\ B_{k+1} - B_1 = 0 + (\Gamma + K) = 0. \end{cases}$$

These imply that $\Gamma = -K$ and $0 = (2\Gamma + K)^2 - \Gamma = K^2 + K$. Since

$$\Gamma_2 = -(2\Gamma + K) = -(-2K + K) = K \neq 0,$$

the relation $K^2 + K = 0$ gives $K = -1$, and so $\Gamma = -K = 1$, $\Gamma_2 = -(2\Gamma + K) = -(2 - 1) = -1$. Thus, we have proved that

$$B_1 = -(\Gamma + K) = 0, B_2 = \Gamma = 1 \quad \text{and} \quad F(n) = B_n = \chi_2(n + 1).$$

Thus all assertions of Lemma 6 are proved. ■

Lemma 7. Assume that $A = 0$, $\Gamma_4 = 0$ and $\Gamma_2 = 0$. Then one of the following assertions holds:

- (A) $f(n) = \varepsilon(n) \frac{1 \pm \sqrt{-8K+1}}{4}$,
- (B) $f(n) = \frac{\varepsilon(n)}{4} a(n) \sqrt{-8\chi_3(n)+9}$ and $K = -\frac{7}{8}$ for $k \equiv 1 \pmod{3}$,
- (C) $f(n) = \varepsilon(n) b(n) \chi_3(n)$ and $K = -1$ for $k \equiv 2 \pmod{3}$,
- (D) $f(n) = \frac{\varepsilon(n)}{4} b(n) \sqrt{-8\chi_3(n)+9}$ and $K = -\frac{3}{8}$ for $k \equiv 0 \pmod{3}$,

where $\chi_3(n) \pmod{3}$ is a principal Dirichlet character, $\varepsilon(n) \in \{-1, 1\}$ and $\varepsilon(n^2 + m^2 + k) = 1$ for every $n, m \in \mathbb{N}$, $a(n)$, $b(n)$ are periodic sequences $\pmod{3}$ with $a(1) = 1$, $a(2) = a(3) = -1$, $b(1) = b(3) = 1$, $b(2) = -1$.

Proof. Assume that $\Gamma_2 = 0$ and $\Gamma_4 = 0$. Then the sequence

$$B_k = \Gamma_3 \chi_3(k) + \Gamma_5 \chi_5(k) + \Gamma \quad \text{for every } k \in \mathbb{N}$$

is periodic $\pmod{15}$. The elements of sequence are:

$$(3.15) \quad \begin{cases} B_1 = B_4 = B_{11} = B_{14} &= \Gamma_3 + \Gamma_5 + \Gamma \\ B_2 = B_7 = B_8 = B_{13} &= \Gamma_3 - \Gamma_5 + \Gamma \\ B_3 = B_{12} &= -\Gamma_5 + \Gamma \\ B_5 = B_{10} &= \Gamma_3 + \Gamma \\ B_6 = B_9 &= \Gamma_5 + \Gamma \\ B_{15} &= \Gamma. \end{cases}$$

By using (3.8) and (3.15) we have

$$(3.16) \quad \begin{cases} B_{k+1} &= B_{1^2+15^2+k} &= (\Gamma_3 + \Gamma_5 + 2\Gamma + K)^2 \\ B_{k+2} &= B_{1^2+1^2+k} &= (2\Gamma_3 + 2\Gamma_5 + 2\Gamma + K)^2 \\ B_{k+3} &= B_{3^2+3^2+k} &= (-2\Gamma_5 + 2\Gamma + K)^2 \\ B_{k+4} &= B_{2^2+15^2+k} &= (\Gamma_3 - \Gamma_5 + 2\Gamma + K)^2 \\ B_{k+5} &= B_{1^2+2^2+k} &= (2\Gamma_3 + 2\Gamma + K)^2 \\ B_{k+6} &= B_{15^2+6^2+k} &= (\Gamma_5 + 2\Gamma + K)^2 \end{cases}$$

Since the sequence $\{B_\ell\}_0^\infty$ is periodic $\pmod{15}$, we have

$$B_{16} = B_1, B_{17} = B_2, B_{18} = B_3, B_{19} = B_4, B_{20} = B_5.$$

Let I be the set of 15 different 6 consecutive elements of sequence $\{B_\ell\}_0^\infty$, i.e

$$I := \{(B_1, B_2, B_3, B_4, B_5, B_6), (B_2, B_3, B_4, B_5, B_6, B_7), \dots, (B_{15}, B_1, B_2, B_3, B_4, B_5)\}.$$

It is obvious that

$$(B_{k+1}, B_{k+2}, B_{k+3}, B_{k+4}, B_{k+5}, B_{k+6}) \in I,$$

which with (3.15) and (3.16) give 15 system of equations. By using the computations with Maple, we solve these systems and we have 4 solutions:

$$\begin{aligned} \text{(A)} \quad & \Gamma_3 = 0, \Gamma_5 = 0 \quad \text{and} \quad \Gamma = \frac{-4K + 1 \pm \sqrt{-8K + 1}}{8} \quad \text{for every } n \in \mathbb{N}, \\ \text{(B)} \quad & \Gamma_3 = -\frac{1}{2}, \Gamma_5 = 0, \Gamma = \frac{9}{16} \quad \text{and} \quad K = -\frac{7}{8} \quad \text{for } k \equiv 1 \pmod{3}, \\ \text{(C)} \quad & \Gamma_3 = 1, \Gamma_5 = 0, \Gamma = 0 \quad \text{and} \quad K = -1 \quad \text{for } k \equiv 2 \pmod{3}, \\ \text{(D)} \quad & \Gamma_3 = -\frac{1}{2}, \Gamma_5 = 0, \Gamma = \frac{9}{16} \quad \text{and} \quad K = -\frac{3}{8} \quad \text{for } k \equiv 0 \pmod{3}. \end{aligned}$$

In the case (A) we infer from

$$\Gamma = \frac{-4K + 1 \pm \sqrt{-8K + 1}}{8} = \left(\frac{1 \pm \sqrt{-8K + 1}}{4} \right)^2,$$

and

$$f^2(n) = B_n = \Gamma = \left(\frac{1 \pm \sqrt{-8K + 1}}{4} \right)^2$$

that

$$f(n) = \pm \left(\frac{1 \pm \sqrt{-8K + 1}}{4} \right),$$

and so

$$f(n^2 + m^2 + k) = f^2(n) + f^2(m) + K = 2\Gamma + K = \frac{1 \pm \sqrt{-8K + 1}}{4}.$$

Thus in the cases (A)–(D) we have

$$\begin{aligned} \text{(A)} \quad & f(n) = \varepsilon(n) \frac{1 \pm \sqrt{-8K + 1}}{4}, \\ \text{(B)} \quad & f(n) = \frac{\varepsilon(n)}{4} a(n) \sqrt{-8\chi_3(n) + 9} \quad \text{and} \quad K = -\frac{7}{8} \quad \text{for } k \equiv 1 \pmod{3}, \\ \text{(C)} \quad & f(n) = \varepsilon(n) b(n) \chi_3(n) \quad \text{and} \quad K = -1 \quad \text{for } k \equiv 2 \pmod{3}, \\ \text{(D)} \quad & f(n) = \frac{\varepsilon(n)}{4} b(n) \sqrt{-8\chi_3(n) + 9} \quad \text{and} \quad K = -\frac{3}{8} \quad \text{for } k \equiv 0 \pmod{3}. \end{aligned}$$

Thus, Lemma 7 is proved. ■

Proof of the Theorem. Our theorem follows from Lemma 4, Lemma 6 and Lemma 7. ■

Remarks. I. Kátai and B. M. Phong posed the following conjecture:

Conjecture. (I. Kátai and B. M. Phong [2]) *Assume that the number $D \in \mathbb{N}$ and the arithmetical function $f : \mathbb{N} \rightarrow \mathbb{C}$ satisfy the equation*

$$f(n^2 + Dm^2) = f^2(n) + Df^2(m) \quad \text{for every } n, m \in \mathbb{N}.$$

Then one of the following assertions holds:

- a) $f(n) = 0$ for every $n \in \mathbb{N}$,
- b) $f(n) = \frac{\epsilon(n)}{D+1}$ for every $n \in \mathbb{N}$,
- c) $f(n) = \epsilon(n)n$ for every $n \in \mathbb{N}$,

where $E := \{n^2 + Dm^2 | n, m \in \mathbb{N}\}$, $\epsilon(n) = 1$ if $n \in E$ and $\epsilon(n) \in \{-1, 1\}$ if $n \in \mathbb{N} \setminus E$.

This conjecture is proved in our paper [4]. Some special cases of this result have been proven in [3], [5], [6] and [7].

In a next article we determine all functions $f, g : \mathbb{N} \rightarrow \mathbb{C}$ for which for which $f(n^2 + Dm^2 + k) = g^2(n) + Dg^2(m) + K$ is satisfied for all $n, m \in \mathbb{N}$, where a non-negative integer k , $D \in \mathbb{N}$ and $K \in \mathbb{C}$ are given.

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