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ON ADDITIVE ARITHMETICAL FUNCTIONS WITH VALUES IN TOPOLOGICAL GROUPS IV.

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Abstract. We prove that if G is an additively written Abelian topological group with the translation invariant metric ρ and

$$\frac{1}{\log x}\sum_{n\leq x}\frac{\rho(\psi(n+1),\varphi(n))}{n}\to 0 \quad (x\to\infty),$$

where $\psi,\varphi:\mathbb{N}\to G$ are completely additive functions, then $\varphi(n)=\psi(n)$ $(\forall n\in\mathbb{N})$, and the extension $\varphi:\mathbb{R}_x\to G$ is a continuous homomorphism, where $\mathbb{R}_x$ is the multiplicative group of positive real numbers.
We also prove that if

$$\frac{1}{\log x}\sum_{n\leq x}\frac{\rho(\psi([\sqrt{2}n]),\varphi(n)+A)}{n}\to 0 \quad (x\to\infty),$$

then $\varphi(n)=\psi(n)$ $(\forall n\in\mathbb{N})$, and the extension $\varphi=\psi:\mathbb{R}_x\to G$ is a continuous homomorphism, with $\psi(\sqrt{2})=A$.

1. Notation

We shall use the following standard notation: $\mathbb{N}$ = natural numbers, $\mathbb{Q}_x$ = multiplicative group of positive rationals, $\mathbb{Q}$ = additive group of rationals, $\mathbb{R}$ = field of real numbers, $\mathbb{R}_x$ multiplicative group of positive real

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numbers, $\mathbb{C}$ = field of complex numbers, $\mathbb{T}$ = one dimensional circle group (torus). Let us consider them in the usual topology.

Let G be an Abelian group. A mapping $\varphi : \mathbb{N} \to G$ is completely additive, if

$$\varphi(nm) = \varphi(n) + \varphi(m) \quad (\forall n, m \in \mathbb{N}).$$

Let $\mathcal{A}_G^*$ be the set of completely additive functions.

If G is considered as a multiplicative (commutative) group, then the mapping $V : \mathbb{N} \to G$ satisfying the relation

$$V(nm) = V(n)V(m) \quad (\forall n, m \in \mathbb{N})$$

is called a completely multiplicative function. $\mathcal{M}_G^*$ denotes the set of these functions.

We can extend the domain of φ and V to $\mathbb{Q}_x$ by the relations

$$\varphi\left(\frac{m}{n}\right) = \varphi(m) - \varphi(n) \quad \text{and} \quad V\left(\frac{m}{n}\right) = V(m)(V(n))^{-1}$$

uniquely.

Furthermore, the relations

$$\varphi(rs) = \varphi(r) + \varphi(s) \quad (\forall r, s \in \mathbb{Q}_x)$$

and

$$V(rs) = V(r)V(s) \quad (\forall r, s \in \mathbb{Q}_x)$$

hold.

2. Preliminary results

Let $\mathcal{M}_{\mathbb{T}}^*$ be the set of those completely multiplicative functions for which $f : \mathbb{N} \to \mathbb{T}$.

Lemma 1. ([8], [9]) If $f \in \mathcal{M}_{\mathbb{T}}^*$, $\Delta f(n) = f(n+1) - f(n) \to 0 \;\; (n \to \infty)$, then $f(n) = n^{i\tau}$, $(\tau \in \mathbb{R})$ for every $n \in \mathbb{N}$.

Lemma 2. ([6], [7]) If $f \in \mathcal{M}_{\mathbb{T}}^*$ and either

$$\frac{1}{x}\sum_{n \le x} |\Delta f(n)| \to 0 \quad \text{as} \quad x \to \infty$$

or

$$\frac{1}{\log x}\sum_{n\le x}\frac{|\Delta f(n)|}{n}\to 0 \quad \text{as} \quad x\to\infty,$$

then $f(n)=n^{i\tau}$, $(\tau\in\mathbb{R})$ for every $n\in\mathbb{N}$.

From the result of [4], we have

Lemma 3. *If $f,g\in\mathcal{M}_{\mathbb{T}}^*$ and either*

$$\frac{1}{x}\sum_{n\le x}|g(n+1)-f(n)|\to 0 \quad \textit{as} \quad x\to\infty$$

or

$$\frac{1}{\log x}\sum_{n\le x}\frac{|g(n+1)-f(n)|}{n}\to 0 \quad \textit{as} \quad x\to\infty,$$

then $f(n)=g(n)=n^{i\tau}$, $(\tau\in\mathbb{R})$ for every $n\in\mathbb{N}$.

Lemma 4. ([5]) If $f,g\in\mathcal{M}_{\mathbb{T}}^*$ and either

$$\frac{1}{\log x}\sum_{n\le x}\frac{|g([\sqrt{2}n])-Af(n)|}{n}\to 0 \quad \text{as} \quad x\to\infty$$

then $f(n)=g(n)=n^{i\tau}$, $(\tau\in\mathbb{R})$ for every $n\in\mathbb{N}$, furthermore $A=g(\sqrt{2})=$ $=2^{\frac{i\tau}{2}}$.

Let now G be an Abelian topological group, $\varphi:\mathbb{Q}_x\to G$ be a homomorphism. We shall say that φ is continuous at the point 1, if $r_\nu\in\mathbb{Q}_x$, $r_\nu\to 1$ implies that

$$\varphi(r_\nu)\to 0.$$

Let $\mathbb{R}_x$ be the multiplicative group of positive real numbers.

Lemma 5. ([1], [2]) Let G be an additively written closed Abelian topological group, $\varphi:\mathbb{Q}_x\to G$ be a homomorphism that is continuous at the point 1. Then its domain can be extended to $\mathbb{R}_x$ by the relation

$$\varphi(\alpha):=\lim_{\substack{r_\nu\to\alpha\\ r_\nu\in\mathbb{Q}_x}}\varphi(r_\nu).$$

uniquely. The so obtained $\varphi:\mathbb{R}_x\to G$ is a continuous homomorphism, consequently

$$\varphi(\alpha\beta)=\varphi(\alpha)+\varphi(\beta)\quad \forall\alpha,\beta\in\mathbb{R}_x.$$

Lemma 6. ([3]) Let G be an additively written Abelian topological group with the translation invariant metric ρ. Let $\varphi \in \mathcal{A}_G^*$ such that

$$\frac{1}{\log x}\sum_{n\le x}\frac{\rho(\varphi(n+1),\varphi(n))}{n}\to 0 \quad (x\to\infty).$$

Then the extension $\varphi:\mathbb{R}_x\to G$ is a continuous homomorphism.

3. The theorems

Theorem 1. *Let G be an additively written Abelian topological group with the translation invariant metric ρ. Let $\psi,\varphi\in\mathcal{A}_G^*$ such that*

$$\frac{1}{\log x}\sum_{n\le x}\frac{\rho(\psi(n+1),\varphi(n))}{n}\to 0 \quad (x\to\infty). \tag{3.1}$$

Then $\varphi(n)=\psi(n)$ $(n\in\mathbb{N})$, and the extension $\varphi:\mathbb{R}_x\to G$ is a continuous homomorphism.

Theorem 2. *Let G be as in Theorem 1. Assume that $\psi,\varphi\in\mathcal{A}_G^*$ and*

$$\frac{1}{\log x}\sum_{n\le x}\frac{\rho(\psi([\sqrt{2}n]),\varphi(n)+A)}{n}\to 0 \quad (x\to\infty).$$

Then $\varphi(n)=\psi(n)$ $(n\in\mathbb{N})$, and the extension $\varphi:\mathbb{R}_x\to G$ is a continuous homomorphism. Furthermore $A=\psi(\sqrt{2})$.

Proof of Theorem 1. It is easy consequence of Lemma 3 and Lemma 6.

Let $\chi:G\to\mathbb{T}$ be any continuous character. Let

$$U(n)=\chi(\psi(n)) \quad\text{and}\quad V(n)=\chi(\varphi(n)).$$

Since χ is a continuous character, we have

$$|U(n+1)-V(n)|\le C\rho(\psi(n+1),\varphi(n)),$$

and so, by (3.1)

$$\frac{1}{\log x}\sum_{n\le x}\frac{|U(n+1)-V(n)|)}{n}\to 0 \quad (x\to\infty),$$

which with Lemma 3 implies that $U(n) = V(n)$, and so $\chi(\varphi(n)) = \chi(\psi(n))$ holds for every continuous character. Thus $\varphi(n) = \psi(n) \quad (\forall n \in \mathbb{N})$. Lemma 6 implies the theorem. ■

Proof of Theorem 2. Let χ, ψ, φ, U be as above. Then

$$|U([\sqrt{2}n]) - \chi(A)V(n)| \leq C\rho(\psi([\sqrt{2}n]), \varphi(n) + A),$$

and so

$$\frac{1}{\log x} \sum_{n \leq x} \frac{|U([\sqrt{2}n]) - \chi(A)V(n)|}{n} \to 0.$$

From Lemma 4 we obtain that

$$U(n) = V(n) = n^{i\tau} \quad \text{and} \quad \chi(A) = U(\sqrt{2}) = (\sqrt{2})^{i\tau}.$$

Repeating the argument used in [3] our theorem follows. ■

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