Annales Univ. Sci. Budapest., Sect. Comp. 49 (2019) 187–198

ON THE CONVERGENCE OF FEJÉR MEANS OF SOME SUBSEQUENCES OF PARTIAL SUMS OF WALSH–FOURIER SERIES

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Communicated by Ferenc Weisz

(Received March 31, 2019; accepted June 29, 2019)

Abstract. The aim of this paper is to improve a result of the author (see [7]). That is, to prove some a.e. convergence results of Fejér means of subsequences of partial sums of Walsh–Fourier series of integrable functions. We prove for sequences a satisfying the condition $a(n+1) \geq \left(1 + \frac{1}{(n+1)^\delta}\right) a(n)$ for some $0 < \delta < 1/2$ for every $n \in \mathbb{N}$, that the $(C, 1)$ means of the partial sums $S_{a(n)}f$ converge to f a.e. Thus, this a.e. convergence result also holds for lacunary sequences a and this was earlier verified in [7].

1. Introduction

It is of prior interest in the theory of Fourier series that how to reconstruct the function from the partial sums of its Fourier series. Just to mention two examples with respect to this issue: Billard proved [3] the theorem of Carleson for the Walsh–Paley system, that is, for each square integrable function we have the almost everywhere convergence $S_n f \to f$ and Fine proved [6] the Fejér–Lebesgue theorem, that is for each integrable function we have the almost everywhere convergence of Fejér means $\sigma_n f \to f$.

Key words and phrases: Cesàro means, subsequences of partial sums, Walsh–Fourier series, a.e. convergence.
2010 Mathematics Subject Classification: 42C10.
Research is supported by the project EFOP-3.6.1-16-2016-00022 supported by the European Union, co-financed by the European Social Fund.
https://doi.org/10.71352/ac.49.187

It is also of main interest that what can be said - with respect to this reconstruction issue - if we have only a subsequence of the partial sums. In 1936 Zalcwasser [17] asked how "rare" can the sequence of integers $a(n)$ be such that

$$\frac{1}{N}\sum_{n=1}^{N} S_{a(n)}f \to f. \tag{1.1}$$

This problem with respect to the trigonometric system was completely solved for continuous functions (uniform convergence) in [13, 16, 1, 5]. That is, if the sequence a is convex, then the condition $\sup_n n^{-1/2}\log a(n) < +\infty$ is necessary and sufficient for the uniform convergence for every continuous function. For the time being, this issue with respect to the Walsh–Paley system has not been solved. Only, a sufficient condition is known, which is the same as in the trigonometric case. The paper about this is written by Glukhov [9]. Besides, one can read the multi-dimensional case also by Glukhov [10].

With respect to convergence almost everywhere, and integrable functions the situation is much more complicated. In 1936 Zalcwasser [17] proved the a.e. relation $\frac{1}{N}\sum_{j=1}^{N} S_{j^2}f \to f$ for each integrable function f. In his paper Salem [13, page 394] writes that this theorem of Zalcwasser is extended to j^3 and j^4 but there is no citation in [13] about it. Belinksky proved [2] for the trigonometric system the existence of a sequence $a(n) \sim \exp(\sqrt[3]{k})$ such that the relation (1.1) holds a.e. for every integrable function. In this paper Belinksky also conjectured that if the sequence a is convex, then the condition $\sup_n n^{-1/2}\log a(n) < +\infty$ is necessary and sufficient again. So, that would be the answer for the problem of Zalcwasser [17] in this point of view (trigonometric system, a.e. convergence and L^1 functions).

In paper [7] - among others - we proved that this is not the case for the Walsh–Paley system and also not the case for the trigonometric system (see [8]). See below Corollary 2.1. On the other hand, differences between the behaviour of the Walsh–Paley and the trigonometric system is not so surprising. Because of the following. Let $v(n) := \sum_{i=0}^{\infty} |n_i - n_{i+1}|$, $(n = \sum_{i=0}^{\infty} n_i 2^i)$ be the variation of the natural number n expanded in the number system based 2. It is a well-known result in the literature that for each sequence a tending strictly monotone increasing to plus infinity with the property $\sup_n v(a(n)) < +\infty$ we have the a.e. convergence $S_{a(n)}f \to f$ for all integrable function f. Is it also a necessary condition? This question of Balashov was answered by Konyagin [11] in the negative. He gave an example. That is, a sequence a with property $\sup_n v(a(n)) = +\infty$ and he proved that $S_{a(n)}f \to f$ a.e. for all integrable function f.

In this paper we prove (see Theorem 2.1) that for each sequence of natural numbers a satisfying a property of growth (2.1) below and for each integrable function f the relation (2.2) holds a.e. This may also be interesting in the following point of view. If the sequence a is lacunar, then the a.e. relation $S_{a(n)}f \to f$ holds for all functions f in the Hardy space H. The trigonometric and the Walsh–Paley case can be found in [18] (trigonometric case) and [12] (Walsh–Paley case). But, the space H is a proper subspace of L^1. Therefore, it is of interest to investigate relation (2.2) below for L^1 functions and rare sequences a. We also remark that very recently, the author proved [8] Theorem 2.1 for the trigonometric system - with different methods.

Next, we give a brief introduction to the theory of the Wals–Fourier series.

Let $\mathbb{P}$ denote the set of positive integers, $\mathbb{N} := \mathbb{P} \cup \{0\}$, and $Q := [0, 1)$. Denote the Lebesgue measure of any set $E \subset Q$ by $|E|$ and sometimes by mes E. Denote the $L^p(Q)$ norm of any function $f : Q \to \mathbb{C}$ by $\|f\|_p$ $(1 \le p \le \infty)$.

Denote the dyadic expansion of $n \in \mathbb{N}$ and $x \in Q$ by $n = \sum_{j=0}^{\infty} n_j 2^j$ and $x = \sum_{j=0}^{\infty} x_j 2^{-j-1}$ (in the case of $x = \frac{k}{2^m}$ $k, m \in \mathbb{N}$ choose the expansion which terminates in zeros). n_i, x_i are the i-th coordinates of n, x, respectively. Set $e_i := 1/2^{i+1} \in Q$, the i-th coordinate of e_i is 1, the rest are zeros $(i \in \mathbb{N})$. Define the dyadic addition $+$ as

$$x + y = \sum_{j=0}^{\infty} |x_j - y_j| 2^{-j-1}.$$

The sets $I_n(x) := \{y \in Q : y_0 = x_0, ..., y_{n-1} = x_{n-1}\}$ for $x \in Q$, $I_n := I_n(0)$ for $n \in \mathbb{P}$ and $I_0(x) := Q$ are the dyadic intervals of Q. Denote by $\mathcal{I} := \{I_n(x) : : x \in Q, n \in \mathbb{N}\}$ the set of the dyadic intervals on Q. $\mathcal{A}_n$ the σ algebra generated by the sets $I_n(x)\,(x \in Q)$ and E_n the conditional expectation operator with respect to $\mathcal{A}_n$ $(n \in \mathbb{N})$.

For $n \in \mathbb{P}$ denote by $|n| := \max(j \in \mathbb{N} : n_j \neq 0)$, that is, $2^{|n|} \le n < 2^{|n|+1}$. The Rademacher functions are defined as:

$$r_n(x) := (-1)^{x_n} \quad (x \in Q,\, n \in \mathbb{N}).$$

The Walsh–Paley system is defined as the sequence of Walsh–Paley functions:

$$\omega_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = (-1)^{\sum_{k=0}^{|n|} n_k x_k}, \quad (x \in Q,\, n \in \mathbb{N}).$$

That is, $\omega := (\omega_n, n \in \mathbb{N})$. Consider the Dirichlet and Fejér kernel functions:

$$D_n := \sum_{k=0}^{n-1} \omega_k,\;\; K_n := \frac{1}{n} \sum_{k=1}^{n} D_k, (n \in \mathbb{P}),\;\; D_0, K_0 := 0.$$

The Fourier coefficients, the n-th partial sum of the Fourier series, the n-th $(C,1)$ mean of $f \in L^1(Q)$:

$$\hat{f}(n) := \int_Q f(x)\omega_n(x)\,dx \ (n \in \mathbb{N}),$$

$$S_n f(y) := \sum_{k=0}^{n-1} \hat{f}(k)\omega_k(y) = \int_Q f(x+y)D_n(x)\,dx,$$

$$\sigma_n f(y) := \frac{1}{n}\sum_{k=1}^{n} S_k f(y) = \int_Q f(x+y)K_n(x)\,dx, \quad (n \in \mathbb{P}, y \in Q).$$

2. The reults

In [7] we proved an a.e. convergence relation with respect to the Cesàro means of a subsequence of the sequence of the partial sums of the Wals–Fourier series of integrable functions, where the sequence of the indices was a lacunar one. More precisely, $a(n+1) \geq a(n)q$ for some $q > 1$.

Now, we state the main theorem of this paper which improves this result:

Theorem 2.1. *Let $f \in L^1(Q)$ be a function and a be a sequence of natural numbers with the property that*

$$a(n+1) \geq \left(1 + \frac{1}{n^\delta}\right) a(n) \tag{2.1}$$

holds for $n \in \mathbb{N}$ and for some $0 < \delta < 1/2$. Then the almost everywhere relation

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} S_{a(n)} f = f \tag{2.2}$$

holds on Q.

Theorem 2.1 has the following immediate consequence and that was proved by the author in 2014:

Corollary 2.1. ([7]) *Let a be a lacunary sequence of natural numbers. Then the almost everywhere relation $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} S_{a(n)} f = f$ holds for every $f \in L^1(Q)$.*

Then we turn our attention to the process of the proof of Theorem 2.1. However, it needs some more preliminary assumptions and also some already proved lemmas (they can be found in [7]) and a new one (Lemma 2.4). The character C denotes a positive constant which may vary from line to line and can depend only on a (more precisely on the parameter δ).

Expand every positive integer n with respect to the binary number system as

$$n = 2^{n_{(\alpha)}} + \cdots + 2^{n_{(0)}} = \sum_{i=0}^{\infty} n_i 2^i,$$

where $n_{(\alpha)} > \cdots > n_{(0)} \geq 0$ are integers and $n_i \in \{0,1\}$ for $i \in \mathbb{N}$. Recall the lower integer part of the binary logarithm of such an n is denoted by $|n|$. That is, $2^{|n|} \leq n < 2^{|n|+1}$ $(n \geq 1)$. To tell the truth, α depends on n of course, but this notation will not cause any misunderstood. If it is absolutely necessary, then we use $\alpha(n)$. That is,

$$n = \sum_{i=0}^{\alpha(n)} 2^{n_{(i)}}.$$

Moreover, we also use the notations:

$$n^j = \sum_{i=j}^{\infty} n_i 2^i, \quad n^{(j)} = \sum_{i=j}^{\alpha} 2^{n_{(i)}}.$$

For $n, i \in \mathbb{N}, n \geq 1$ set the two-dimensional sequences $\lambda_{n,i}, d^1_{n,i}, d^2_{n,i}$ as:

$$\lambda_{n,i} = \begin{cases} 0, & \text{if } i \in \left\{n_{(0)}, n_{(1)}, \ldots, n_{(\alpha)}\right\} \text{ or } i \notin [n_{(0)}, n_{(\alpha)}), \\ 1, & \text{otherwise}, \end{cases}$$

$$d^1_{n,i} = \lambda_{n,i}\omega_{n^{i+1}} D_{2^i}, \quad d^2_{n,i} = \lambda_{n,i}\omega_{n^{i+1}} D_{2^{i+1}}.$$

We proved the following version of the decomposition of the Dirichlet kernel functions.

Lemma 2.1. ([7]) *Let n be a positive integer. Then*

$$D_n = D_{2^{|n|+1}} + \sum_{i=0}^{|n|-1} (d^1_{n,i} - d^2_{n,i}).$$

Set

$$d_{n,i} := d^1_{n,i} - d^2_{n,i}, \quad \tilde{D}_n := \sum_{i=0}^{|n|-1} d_{n,i}.$$

That is,

$$\begin{aligned}
D_n &= D_{2^{|n|+1}} + \tilde{D}_n, \\
\tilde{S}_n f &:= f * \tilde{D}_n, \\
\sigma_N f &= \frac{1}{N}\sum_{n=1}^{N} S_{a(n)} f = \frac{1}{N}\sum_{n=1}^{N} S_{2^{A(n)+1}} f + \frac{1}{N}\sum_{n=1}^{N} \tilde{S}_{a(n)} f =: \\
&=: \sigma_N^0 f + \tilde{\sigma}_N f,
\end{aligned}$$

where $|a(n)| = A(n)$ is the integer part of the binary logarithm of $a(n)$ and $g * h$ is the dyadic convolution of functions g and h. We claim to prove that operator $\sup_N |\sigma_N|$ is of weak type (L^1, L^1). It is well-known that the power of two indexed partial sums converge to the function almost everywhere (see e.g. [15]) and therefore we note that it is trivial to see that operator $\sup_N |\sigma_N^0|$ is of weak type (L^1, L^1).

The following lemma also plays a role in this paper.

Lemma 2.2. ([7]) *Let $f, g \in L^1, 1 \le n \in \mathbb{N}, i, j \in \{0, 1, \ldots, |n|-1\}$. Then*

$$\langle \phi(f * d_{n,i}), \psi(g * d_{n,j}) \rangle = 0, \tag{2.3}$$

for all $i \neq j$, where ϕ is an $\mathcal{A}_i$ and ψ is an $\mathcal{A}_j$ measurable function. Moreover, if $1 \le m, n \in \mathbb{N}, i, j \in \mathbb{N}, i < |m|, j < |n|$ and $|m| \neq |n|$, then we have again

$$\langle \phi(f * d_{m,i}), \psi(g * d_{n,j}) \rangle = 0. \tag{2.4}$$

For each real $\lambda > 0$ (there is no connection between the real λ and the zero-one sequence $\lambda_{n,i}$) define the following stopping time [15, 4]

$$\nu_\lambda(x) := \inf\Big\{ n \in \mathbb{N} : E_n(|f|)(x) > \lambda \Big\} \quad (\inf \emptyset = +\infty).$$

Denote by 1_X the characteristic function of the set X.

Next, we refer the following inequality with respect to the kernels $d_{n,i}$.

Lemma 2.3. ([7]) *Let $f \in L^1(Q), 1 \le n \in \mathbb{N}, \lambda > 0$. Then we have*

$$\sum_{i=0}^{|n|-1} \|1_{\{\nu_\lambda > i\}} (f * d_{n,i})\|_2^2 \le C\lambda \|f\|_1.$$

Besides, it will be important in the proof of Theorem 2.1 that by orthogonality lemma, that is by Lemma 2.2 and by Lemma 2.3 (this can also be found in paper [7]) we have

$$
\begin{aligned}
(2.5)\qquad &\left\|\sum_{n=1}^{N}\sum_{i=0}^{A(n)-1} 1_{\{\nu_\lambda>i\}}(f*d_{a(n),i})\right\|_2^2 = \\
&=\sum_{n=1}^{N}\sum_{i=0}^{A(n)-1} \left\|1_{\{\nu_\lambda>i\}}(f*d_{a(n),i})\right\|_2^2 \le CN\lambda\|f\|_1
\end{aligned}
$$

for lacunary sequences a with parameter not less then 2. That is, in the case when $a(n+1)\ge qa(n)$ with some $q\ge 2$ for every $n\in\mathbb{N}$.

Introduce the following operator

$$T_Nf := \frac{1}{N}\sum_{n=1}^{N}\sum_{i=0}^{A(n)-1} 1_{\{\nu_\lambda>i\}}(f*d_{a(n),i}).$$

Let $T^*f := \sup_N|T_Nf|$. In the sequel, we prove the following lemma which is the main tool in the proof of Theorem 2.1.

Lemma 2.4. *Suppose that sequence a satisfies the growth condition (2.1). Then the operator T^* is of weak type (L^1, L^1).*

Proof. For a number $N\in\mathbb{N}$ let $K=\lfloor\sqrt{N}\rfloor$. That is, $K^2\le N<(K+1)^2$. Besides, for (the fixed) $0<\delta<1/2$ let $K_0:=\lfloor K^{2\delta}\rfloor$. For every natural number $1\le n\le N$ there is a unique pair of natural numbers (j,b) such that $n=(j-1)K_0+b$, where $1\le j\le N/K_0+1$ and $0\le b<K_0$.

We prove that the sequence $(a((j-1)K_0+b))$ is lacunary with parameter $q\ge 2$ for any fixed $b<K_0$ as j runs from 1 in a way that $(j-1)K_0+b$ is less than $(K+1)^2$. This observation follows, from

$$a(\beta+K_0)\ge\left(1+\frac{1}{(\beta+K_0)^\delta}\right)^{K_0}a(\beta)>\left(1+\frac{1}{(1+K)^{2\delta}}\right)^{K^{2\delta}-1}a(\beta)\ge\frac{5}{2}a(\beta)$$

for every $\beta\in\mathbb{N}$ with $\beta+K_0\le(K+1)^2$ for $K\ge k_\delta$ for some fixed k_δ because

$$\left(1+\frac{1}{(1+K)^{2\delta}}\right)^{K^{2\delta}-1}\to\exp(1)>\frac{5}{2}.$$

Consequently (2.5) can be applied for the sequence $a((j-1)K_0+b)$ (j is running). Before doing this, apply the well-known inequality between the arithmetic and quadratic means. (Suppose that $N\ge k_\delta^2$.)

$$|T_N f|^2 = \frac{2}{K^4}\left|\sum_{n=1}^{K^2}\sum_{i=0}^{A(n)-1} 1_{\{\nu_\lambda > i\}}(f * d_{a(n),i})\right|^2 +$$
$$+ \frac{2}{K^4}\left|\sum_{n=K^2+1}^{N}\sum_{i=0}^{A(n)-1} 1_{\{\nu_\lambda > i\}}(f * d_{a(n),i})\right|^2 =:$$
$$=: A_N + B_N.$$

Again, by the inequality between the arithmetic and quadratic means we have

$$A_N \leq$$
$$\leq \frac{CK_0}{K^4}\sum_{b<K_0}\sum_{L=\lfloor K^2/K_0\rfloor}^{\lfloor K^2/K_0\rfloor+1}\left|\sum_{j=1}^{L}\sum_{i=0}^{A((j-1)K_0+b)-1} 1_{\{\nu_\lambda > i\}}(f * d_{a((j-1)K_0+b),i})\right|^2 =:$$
$$=: A_K^1.$$

That is, it depends only on K, where $K^2 \leq N < (K+1)^2$. Apply (2.5).

$$\|A_K^1\|_1 \leq C\frac{K_0}{K^4}\sum_{b<K_0}\frac{K^2}{K_0}\|f\|_1\lambda \leq C\frac{1}{K^{2(1-\delta)}}\|f\|_1\lambda.$$

Fix natural numbers K, N such that $K^2 \leq N < (K+1)^2$, where $K \geq k_\delta$. Apply again the inequality between the arithmetic and quadratic means. Set for fixed K, N, b

$$J(N,K,b) := \{j \in \mathbb{N} : K^2 < (j-1)K_0 + b \leq N\}.$$

We simple denote $J(N,K,b)$ by J below (it will not cause misunderstand).

$$B_N \leq \frac{CK_0}{K^4}\sum_{b<K_0}\left|\sum_{j\in J}\sum_{i=0}^{A((j-1)K_0+b)-1} 1_{\{\nu_\lambda > i\}}(f * d_{a((j-1)K_0+b),i})\right|^2.$$

Apply (2.5) and the fact that $N - K^2 < 2K$ and therefore also the fact that the cardinality of the set J is not more than $C(N-K^2)/K_0 \leq CK/K_0$.

$$\|B_N\|_1 \leq C\frac{K_0}{K^4}\sum_{b<K_0}\frac{K}{K_0}\|f\|_1\lambda \leq C\frac{1}{K^{1+2(1-\delta)}}\|f\|_1\lambda.$$

Consequently, for $B_K^1 := \sup_{K^2\leq N<(K+1)^2} B_N$ ($k_\delta \leq K \in \mathbb{N}$ is fixed) we have

$$\|B_K^1\|_1 \leq \sum_{N=K^2}^{(K+1)^2-1} C\frac{1}{K^{1+2(1-\delta)}}\|f\|_1\lambda \leq C\frac{1}{K^{2(1-\delta)}}\|f\|_1\lambda.$$

This immediately gives ($0 < \delta < 1/2$ is an arbitrarily fixed number) that

$$
\begin{aligned}
&\operatorname{mes}\Bigl(y \in Q : \sup_{k_\delta^2 \le N \in \mathbb{N}} |T_N f(y)| > \lambda\Bigr) \le \\
&\le \operatorname{mes}\Bigl(y \in Q : \sup_{k_\delta^2 \le N \in \mathbb{N}} |T_N f(y)|^2 > \lambda^2\Bigr) \le \\
&\le \operatorname{mes}\Bigl(y \in Q : \sup_{k_\delta^2 \le K \in \mathbb{N}} \bigl(A_K^1 + B_K^1\bigr) > \lambda^2\Bigr) \le \\
&\le \sum_{K=1}^{\infty} C \frac{1}{K^{2(1-\delta)}} \frac{\|f\|_1}{\lambda} \le \\
&\le C \frac{\|f\|_1}{\lambda}
\end{aligned}
$$

because $2(1-\delta) > 1$. Besides, because k_δ is a fixed natural number, which depends only on fixed parameter $0 < \delta < 1/2$, then it is trivial to have that

$$
\operatorname{mes}\Bigl(y \in Q : \sup_{k_\delta^2 > N \in \mathbb{N}} |T_N f(y)| > \lambda\Bigr) \le C \frac{\|f\|_1}{\lambda},
$$

where - of course C depends on k_δ, that is, on δ. These last two inequalities immediately complete the proof of Lemma 2.4. ■

Proof of Theorem 2.1. Recall that

$$
\sigma_N f = \sigma_N^0 f + \tilde{\sigma}_N f
$$

and the already proved fact that operator $\sup |\sigma_N^0|$ is of weak type (L^1, L^1). Also recall the definition of the stopping time ν_λ.

$$
\nu_\lambda(x) := \inf\Bigl\{n \in \mathbb{N} : E_n(|f|)(x) > \lambda\Bigr\} \quad (\inf \emptyset = +\infty).
$$

In [14] one can find the well-known inequality

$$
\operatorname{mes}\bigl\{\nu_\lambda < \infty\bigr\} = \operatorname{mes}\bigl\{f^* > \lambda\bigr\} \le \frac{C}{\lambda} \|f\|_1,
$$

where $f^* = \sup |E_n(f)|$. Therefore by Lemma 2.4 we have

$$\begin{aligned}
&\operatorname{mes}\Big\{x \in Q : \sup_N |\sigma_N f(x)| > 4\lambda\Big\} \le \\
&\quad \le \operatorname{mes}\Big\{x \in Q : \sup_N \big|\sigma_N^0 f(x)\big| > 2\lambda\Big\} + \\
&\qquad + \operatorname{mes}\Big\{x \in Q : \sup_N |\tilde{\sigma}_N f(x)| > 2\lambda\Big\} \le \\
&\quad \le C\frac{\|f\|_1}{\lambda} + \operatorname{mes}\{\nu_\lambda < \infty\} + \\
&\qquad + \operatorname{mes}\Big\{x \in Q : \nu_\lambda(x) = \infty, \sup_N |T_N f(x)| > \lambda\Big\} + \\
&\qquad + \operatorname{mes}\Big\{x \in Q : \nu_\lambda(x) = \infty, \sup_N |\tilde{\sigma}_N f(x) - T_N f(x)| > \lambda\Big\} \le \\
&\quad \le C\frac{\|f\|_1}{\lambda} + \sum_{N=1}^{\infty} \operatorname{mes}\Big\{x \in Q : \nu_\lambda(x) = \infty, |\tilde{\sigma}_N f(x) - T_N f(x)| > \lambda\Big\} \le \\
&\quad \le C\frac{\|f\|_1}{\lambda},
\end{aligned}$$

because we verify that $\tilde{\sigma}_N f(x) - T_N f(x) = 0$ for such an $x \in Q$ that $\nu_\lambda(x) = \infty$. Since in the case $\nu_\lambda(x) = \infty$ we have $1_{\{\nu_\lambda > i\}}(x) = 1$ for every i, then it follows

$$\begin{aligned}
&\tilde{\sigma}_N f(x) = \\
&\qquad = \frac{1}{N}\sum_{n=1}^{N}\sum_{i=0}^{A(n)-1} 1_{\{\nu_\lambda > i\}}(x)(f * d_{a(n),i})(x) = \\
&\qquad = T_N f(x).
\end{aligned}$$

In other words, we proved for the operator $\sup |\sigma_N|$

$$\operatorname{mes}\Big\{\sup_N |\sigma_N f| > \lambda\Big\} \le \frac{C}{\lambda}\|f\|_1,$$

where C depends only on parameter δ (which is a fixed one). Finally, since for each Walsh polynomial $P = \sum_{l=0}^{L} c_l \omega_l$ we have $\lim_N \sigma_N P = P$ everywhere (because $S_u P = P$ for $u > L$), then by the standard density argument (the set of Walsh polynomials is dense in $L^1(Q)$) and by the weak (L^1, L^1) typeness of the maximal operator $\sup |\sigma_N|$ we complete the proof of Theorem 2.1. ∎

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