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WAVELET TIGHT FRAMES IN WALSH ANALYSIS

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Abstract. We describe two type of wavelet tight frames associated with the generalized Walsh functions: (1) Parseval frames for L^2-spaces on Vilenkin groups, (2) finite tight frames for the space $\ell^2(\mathbb{Z}_N)$. In particular cases these tight frames coincide with orthogonal wavelet bases associated with the classical Walsh functions.

1. Inroduction

Wavelet tight frames, from the works of Ron and Shen [35, 36], have been a productive research area, both in theory and applications, particularly because the applications to areas as diverse as signal processing, quantum information theory, multivariate orthogonal polynomials and splines, and compressed sensing (see [4, 5, 25, 41] and the references therein). Let us recall that a family $\{g_m : m \in M\}$ is a *frame* for a Hilbert space $\mathcal{H}$ if there exist positive constants A and B such that, for every $f \in \mathcal{H}$,

$$A\|f\|^2 \le \sum_{m\in M} |\langle f, g_m\rangle|^2 \le B\|f\|^2.$$

The constants A and B are known respectively as lower and upper frame bounds. A frame is called a *tight frame* if the lower and upper frame bounds are

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equal; $A = B$. A frame is a *Parseval frame* if $A = B = 1$. It is well-known that a sequence $\{g_m\}$ is a Parseval frame for $\mathcal{H}$ if and only if $f = \sum_{m\in M}\langle f, g_m\rangle g_m$ for every $f \in \mathcal{H}$. Therefore, the concept of a Parseval frame generalizes the concept of an orthonormal basis to systems that do not have the minimal property.

The Walsh functions can be identified with characters of the Cantor group (the dyadic or 2-series local field). In the introduction to the book [1] is noted that this fact was first recognized by Gelfand in 1940s, who offered to Vilenkin [40] study series with respect to characters of a large class of abelian groups which includes the Cantor group as a special case. Fine [20] observed independently that Walsh functions are the characters of the compact Cantor (dyadic) group. Orthogonal wavelets on the Cantor group have been initiated in [26]; recent results in this directions including some orthogonal wavelets on Vilenkin groups can be found in [14]-[17]. Wavelet frames for L^2-space on the Cantor group are studied in [11, 18], where the Parseval frame related to the Walsh-Dirichlet kernel is given, as well as analogs of Cohen's condition and Daubechies' admissible condition. In a recent paper [2], the connection between discrepancy theory and Parseval frames defined by Walsh matrices is discussed.

Let us introduce notation and recall some basic definitions. The set of integers, non-negative integers, positive integers, and non-negative real numbers will be denoted by $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{N}$, and $\mathbb{R}_+$, respectively. Let $p \in \mathbb{N}$, $p \geq 2$. The Vilenkin group G_p consists of sequences $x = (x_j)$, where $x_j \in \{0, 1, \dots, p-1\}$ for $j \in \mathbb{Z}$ and with at most finite number of negative j such that $x_j \neq 0$. The zero sequence is denoted by θ. If $x \neq \theta$, then there exists a unique $k = k(x)$ such that $x_k \neq 0$ and $x_j = 0$ for all $j < k$. The group operation $\oplus$ on G_p is defined as the coordinatewise addition modulo p,

$$(z_j) = (x_j) \oplus (y_j) \iff z_j = x_j + y_j \pmod p \quad \text{for all} \quad j \in \mathbb{Z};$$

the topology on G_p is introduced via the complete system of neighbourhoods of zero

$$U_l = \{(x_j) \in G_p \,:\, x_j = 0 \text{ for all } j \leq l\}, \quad l \in \mathbb{Z}.$$

The equality $z = x \ominus y$ means that $z \oplus y = x$. For $p = 2$ we have $x \oplus y = x \ominus y$ and the group G_2 coincides with the locally compact Cantor group $\mathcal{C}$.

Notice that for $p = 2$ the subgroup $U := U_0$ is isomorphic to the compact Cantor group $\mathcal{C}_0$; i.e., the topological Cartesian product of a countable set of cyclic groups with discrete topology. It is well known that $\mathcal{C}_0$ is a perfect nowhere dense totally disconnected metrizable space, and therefore U_0 is homeomorphic to the Cantor ternary set.

One can show that $G := G_p$ is self-dual. The duality pairing on G takes $x = (x_j)$ and $\omega = (\omega_j)$ to

$$\chi(x, \omega) = \exp\left(\frac{2\pi i}{p} \sum_{j \in \mathbb{Z}} x_j \omega_{1-j}\right).$$

There exists a Haar measure on G normalized so that the measure of U is 1. For simplicity, we shall denote this measure by dx. As usual, the Lebesgue space $L^2(G)$ consists of all square integrable functions on G. For each function $f \in L^1(G) \cap L^2(G)$, its Fourier transform $\widehat{f}$,

$$\widehat{f}(\omega) = \int_G f(x)\overline{\chi(x, \omega)}\, dx, \quad \omega \in G,$$

belongs to $L^2(G)$. The Fourier operator

$$\mathcal{F} : L^1(G) \cap L^2(G) \to L^2(G), \quad \mathcal{F}f = \widehat{f},$$

extends uniquely to the whole space $L^2(G)$. See [23] and [37] for further details about harmonic analysis on the group G.

Consider the mapping $\lambda : G \to \mathbb{R}_+$ defined by

$$\lambda(x) = \sum_{j \in \mathbb{Z}} x_j p^{-j}, \qquad x = (x_j) \in G.$$

Take in G a discrete subgroup $H = \{(x_j) \in G : x_j = 0 \text{ for } j > 0\}$. The image of the subgroup H under λ is the set of non-negative integers: $\lambda(H) = \mathbb{Z}_+$. For each $k \in \mathbb{Z}_+$, let $h_{[k]}$ denote the element of H such that $\lambda(h_{[k]}) = k$ (clearly, $h_{[0]} = \theta$). The *generalized Walsh functions* on G can be defined by

$$W_k(x) = \chi(x, h_{[k]}), \quad x \in G, \quad k \in \mathbb{Z}_+.$$

So, these functions are characters for G. Also, it is well-known that $\{W_k : : k \in \mathbb{Z}_+\}$ is an orthonormal basis for $L^2(U)$ (when $p = 2$, we have the classical Walsh system).

The first results on orthogonal wavelets for the Vilenkin group G were obtained in [6]. In this paper, for any integer $p, n \geq 2$, the compactly supported scaling functions are defined on G, each of which satisfies a scaling equation with p^n coefficients and generates an MRA for $L^2(G)$. Moreover, a method to estimate the moduli of continuity of scaling functions is given in [6] which leads to the sharp estimates for small p and n (cf. [33]). In addition, it is shown that the conception of adapted multiresolution analysis suggested by Sendov [39] is applicable for orthogonal wavelets on G. Further results for wavelets defined on the Vilenkin/Cantor groups are reflected in [14, 17].

The aim of this paper is to present a review of wavelet tight frames related to the generalized Walsh functions. In Section 2, we define an MRA-based tight frame on G and consider the corresponding algorithm for constructing Parseval frames. This section reflects recent results from [13], [14], and [18]. Then, in Section 3, we use the generalized Walsh functions to define Parseval frames in the space $\ell^2(\mathbb{Z}_N)$ with $N = p^n$ (cf. [8, 10, 19, 22, 41]).

2. Wavelet tight frames for $L^2(G)$

As above, we let $N = p^n$. We define an automorphism $A \in \operatorname{Aut} G$ by the formula $(Ax)_j = x_{j+1}$. It is easy to see that the quotient group $H/A(H)$ contains p elements. The sets

$$U_{n,\,s} := A^{-n}(h_{[s]}) \oplus A^{-n}(U), \quad 0 \le s \le N-1,$$

are cosets of the subgroup $A^{-n}(U)$ in the group U. For every $0 \le \alpha \le N-1$ the Walsh function W_α is constant on each $U_{n,\,s}$.

The *Vilenkin-Chrestenson transform* translates an arbitrary vector $\boldsymbol{b} = (b_0, b_1, \dots, b_{N-1})$ of the space $\mathbb{C}^N$ into a vector $\boldsymbol{a} = (a_0, a_1, \dots, a_{N-1})$ with components

$$a_\alpha = \frac{1}{N}\sum_{s=0}^{N-1} b_s W_s(A^{-n}h_{[\alpha]}), \quad 0 \le \alpha \le N-1. \tag{2.1}$$

The inverse transform acts by the formula

$$b_s = \sum_{\alpha=0}^{N-1} a_\alpha \overline{W_\alpha(A^{-n}h_{[s]})}, \qquad 0 \le s \le N-1. \tag{2.2}$$

These transforms can be realized by the fast algorithms (see, for instance, [3], [23, Sect. 11.2], [37, p. 463]).

A mask $m_{\boldsymbol{b}}$ associated with a vector $\boldsymbol{b}$ has the form

$$m_{\boldsymbol{b}}(\omega) = \sum_{k=0}^{N-1} a_k \overline{W_k(\omega)}, \quad \omega \in G,$$

where the coefficients a_k are defined by (2.1). A compactly supported function $\varphi \in L^2(G)$ is a *refinable function* with the mask $m_{\boldsymbol{b}}$ if it satisfies the equation

$$\varphi(x) = p\sum_{k=0}^{N-1} a_k \varphi(A\,x \ominus h_{[k]}), \tag{2.3}$$

or, in the Fourier domain, $\widehat{\varphi}(\omega) = m_{\boldsymbol{b}}(A^{-1}\omega)\widehat{\varphi}(A^{-1}\omega)$.

For any $f \in L^2(G)$ we let

$$f_{j,k}(x) := p^{j/2} f(A^j x \ominus h_{[k]}), \quad j \in \mathbb{Z},\ k \in \mathbb{Z}_+.$$

Given $\Psi := \{\psi^{(1)}, \dots, \psi^{(r)}\} \subset L^2(G)$ with $r \geq p$, we define the *wavelet system* as

$$X(\Psi) := \{\psi_{j,k}^{(\nu)} \ :\ 1 \leq \nu \leq r,\ j \in \mathbb{Z},\ k \in \mathbb{Z}_+\}.$$

The system $X(\Psi)$ is a Parseval frame (or a *wavelet tight frame*) for $L^2(G)$ if

$$\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}_+} \sum_{\nu=1}^{r} |\langle f\,, \psi_{j,k}^{(\nu)}\rangle|^2 = \|f\|^2$$

for all $f \in L^2(G)$. This is equivalent to $f = \sum_{g\in X(\Psi)} \langle f\,, g\rangle g$ for all $f \in L^2(G)$.

Now, we let $N_1 := p^{n-1}$ and denote by $\boldsymbol{F}(p,n)$ the set of all vectors $\boldsymbol{b} = (b_0, b_1, \dots, b_{N-1})$ of the space $\mathbb{C}^N$ such that

$$b_0 = 1, \quad |b_l|^2 + |b_{l+N_1}|^2 + \cdots + |b_{l+(p-1)N_1}|^2 \leq 1, \tag{2.4}$$

for all $0 \leq l \leq N_1 - 1$. The following algorithm allows us to construct a Parseval frame for $L^2(G)$ from any vector $\boldsymbol{b} \in \boldsymbol{F}(p,n)$.

Algorithm A

- **Step 1**. Choose an arbitrary vector $\boldsymbol{b} = (b_0, b_1, \dots, b_{N-1})$ in $\boldsymbol{F}(p,n)$.
- **Step 2**. Compute a_α, $0 \leq \alpha \leq N-1$, by (2.1) and define

$$m_0(\omega) = \sum_{\alpha=0}^{N-1} a_\alpha \overline{W_\alpha(\omega)}.$$

- **Step 3**. Define $\varphi \in L^2(G)$ such that

$$\widehat{\varphi}(\omega) = \prod_{j=1}^{\infty} m_0(A^{-j}\omega), \quad \omega \in G. \tag{2.5}$$

- **Step 4**. Given $r \geq p$ find the Walsh polynomials

$$m_\nu(\omega) = \sum_{\alpha=0}^{N-1} a_\alpha^{(\nu)} \overline{W_\alpha(\omega)}, \quad 1 \leq \nu \leq r,$$

such that, for each $\omega \in G$, the rows of the matrix $M(\omega)$ form an orthonormal system, where

$$M(\omega) := \begin{bmatrix} m_0(\omega) & m_1(\omega) & \dots & m_r(\omega) \\ m_0(\omega \oplus \delta_1) & m_1(\omega \oplus \delta_1) & \dots & m_r(\omega \oplus \delta_1) \\ \vdots & \vdots & \vdots & \vdots \\ m_0(\omega \oplus \delta_{p-1}) & m_1(\omega \oplus \delta_{p-1}) & \dots & m_r(\omega \oplus \delta_{p-1}) \end{bmatrix}$$

with $\delta_l \in U$, $\lambda(\delta_l) = l/p$, $l = 0, \dots, p-1$.

- **Step 5**. Define $\psi^{(1)}, \dots, \psi^{(r)}$ as follows:

$$\psi^{(\nu)}(x) = p \sum_{\alpha=0}^{N-1} a_\alpha^{(\nu)} \varphi(Ax \ominus h_{[\alpha]}), \quad 1 \le \nu \le r.$$

Note that, according to Step 1 and Step 2,

$$\sum_{l=0}^{p-1} |m_0\,(\omega \oplus \delta_l)|^2 \le 1, \qquad \omega \in G.$$

Hence, by [13, Theorem 11], the function φ on Step 3 belongs to $L^2(G)$. Let us show how to calculate the values $\widehat{\varphi}(\omega)$ for $\omega \in G$. For each $0 \le s \le N-1$ we set

$$\gamma(i_1, i_2, ..., i_n) = b_s,$$

if

$$s = i_1 p^0 + i_2 p^1 + \dots + i_n p^{n-1}, \quad i_j \in \{0, 1, \dots, p-1\}.$$

Then for an integer l with the p-ary expansion

$$l = \sum_{j=0}^{k} \mu_j p^j, \quad \mu_j \in \{0, 1, \dots, p-1\}, \quad \mu_k \ne 0, \quad k = k(l) \in \mathbb{Z}_+, \tag{2.6}$$

we define d_l as follows

$$d_l = \gamma(\mu_0, 0, 0, \dots, 0, 0), \quad \text{if} \quad k(l) = 0;$$

$$d_l = \gamma(\mu_1, 0, 0, \dots, 0, 0)\gamma(\mu_0, \mu_1, 0, \dots, 0, 0), \quad \text{if} \quad k(l) = 1;$$

. . . .

$$d_l = \gamma(\mu_k, 0, 0, \dots, 0, 0)\gamma(\mu_{k-1}, \mu_k, 0, \dots, 0, 0) \dots \gamma(\mu_0, \mu_1, \mu_2, \dots, \mu_{n-2}, \mu_{n-1}),$$

if $k = k(l) \geq n-1$. Further, denote by $\mathbb{M}_0$ the set of all positive integers $l \geq N_1$ whose p-ary expansion (2.6) contains no n-tuple $(\mu_j, \mu_{j+1}, \dots, \mu_{j+n-1})$ coinciding with any of the n-tuples

$$(0,0,\dots,0,1), (0,0,\dots,0,2), \dots, (0,0,\dots,0,p-1).$$

and set $\mathbb{M} = \{1,2,\dots,N_1-1\} \cup \mathbb{M}_0$. In particular, if $n=2$ then

$$\mathbb{M} = \left\{ \sum_{j=0}^{k} \mu_j p^j \;:\; \mu_j \in \{1,2,\dots,p-1\},\ k \in \mathbb{Z}_+ \right\}.$$

Proposition 2.1. *Suppose that m_0 and d_l are defined as above. If the function $\widehat{\varphi}$ is given by (2.5), then*

$$\widehat{\varphi}(\omega) = \begin{cases} 1, & \omega \in U_{n-1,0}, \\ d_l, & \omega \in U_{n-1,l},\ l \in \mathbb{M}, \\ 0, & \omega \in U_{n-1,l},\ l \notin \mathbb{M}. \end{cases} \tag{2.7}$$

Proof. By definition, for each $\omega \in U_{n-1,0}$ we have $A^{-j}\omega \in U_{n,0}$ for all $j \in \mathbb{N}$. But $m_0(\omega) \equiv 1$ on $U_{n,0}$. Using (2.5), we obtain the first equality in (2.7). Further, since $\boldsymbol{b} \in \boldsymbol{F}(p,n)$, we have

$$b_{N_1} = \dots = b_{(p-1)N_1} = 0. \tag{2.8}$$

Now, take $l \in \mathbb{N}$ with expansion (2.6) and find

$$j_0 = \min\{j \;:\; j \in \mathbb{N},\ p^{j-1} > l+1\}.$$

Then for any $\omega \in U_{n-1,l}$ and $j \geq j_0$ we have $A^{-j}\omega \in U_{n,0}$ and hence $m_0(A^{-j}\omega) = 1$. Therefore, by (2.5) we obtain

$$\widehat{\varphi}(\omega) = \prod_{j=1}^{j_0-1} m_0(A^{-j}\omega), \quad \omega \in U_{n-1,l}. \tag{2.9}$$

In the case $l = \mu_0$ (i.e. when $k(l)=0$) in (2.6) we have $j_0 = 2$, $A^{-1}\omega \in U_{n,\mu_0}$ and $A^{-j}\omega \in U_{n,0}$. Thus, if $l \in \{1,2,\dots,p-1\}$ and $\omega \in U_{n-1,l}$, then

$$\widehat{\varphi}(\omega) = b_1 = \gamma(\mu_0,0,0,\dots,0,0) = d_l.$$

Further, if $k(l) = 1$ then

$$m_0(A^{-1}\omega) = \gamma(\mu_1,0,0,\dots,0,0) \quad m_0(A^{-2}\omega) = \gamma(\mu_0,\mu_1,0,\dots,0,0), \dots,$$

and, hence, $\widehat{\varphi}(\omega) = d_l$. In general, from (2.5) it follows, that

$$\frac{l}{p^{n+j-1}} = \frac{1}{p^n}\left(\frac{\mu_0}{p^{j-1}} + \dots + \frac{\mu_{j-2}}{p} + \mu_{j-1} + \mu_j p + \dots + \mu_k p^{k-n}\right)$$

for all $j \in \mathbb{N}$. Beside, if $\omega \in U_{n-1,l}$ then $A^{-j}\omega \in U_{n+j-1,l}$. For $l \notin \mathbb{M}$ in view of (2.8) among the factors in (2.9) there is zero. Moreover, if $l \in \mathbb{M}$, then in (2.9) we have

$$A^{-1}\omega \in U_{n,\mu_k} \quad A^{-2}\omega \in U_{n,\mu_{k-1}+p\mu_k}, \ \ldots, \ A^{-j_0}\omega \in U_{n,\nu_0},$$

where

$$\nu_0 = \mu_{j_0-1} + \mu_{j_0}p + \cdots + \mu_k p^{k-n}.$$

Therefore, since m_0 is H-periodic, we conclude that (2.7) is true. ■

As a consequence of Proposition 2.1, we have

$$\widehat{\varphi}(\omega) = \mathbf{1}_{U_{n-1,0}}(\omega) + \sum_{l \in \mathbb{M}} d_l \mathbf{1}_{U_{n-1,0}}(\omega \ominus A^{1-n}h_{[l]}), \quad \omega \in G, \tag{2.10}$$

where $\mathbf{1}_E$ is the characteristic function of a set E. A formal application of the inverse Fourier transform to (2.10) gives the expansion

$$\varphi(x) = (1/p^{n-1})\mathbf{1}_U(A^{1-n}x)(1 + \sum_{l \in \mathbb{M}} d_l W_l(A^{1-n}x)), \quad x \in G. \tag{2.11}$$

A few examples of expansion (2.11) are given in [6, 14, 17, 33]. Example 4.3 in [17] shows that, in general, this expansion does not converge absolutely. The necessary and sufficient conditions on the mask of equation (2.3) ensuring that the refinable function φ will not be a step function are contained in [14, Theorem 3.9].

Example 2.2. For the case $p = 2, n = 3$, we obtain from (2.11) the following step functions:

1) $\varphi(x) = (1/4)\mathbf{1}_{[0,1)}(A^{-2}x) \quad (b_1 = 0)$,

2) $\varphi(x) = (1/4)\mathbf{1}_{[0,1)}(A^{-2}x)\left(1 + b_1W_1(A^{-2}x)\right) \quad (b_1 \neq 0,\ b_2 = b_3 = 0)$,

3) $\varphi(x) = (1/4)\mathbf{1}_{[0,1)}(A^{-2}x)\left(1 + b_1W_1(A^{-2}x) + b_1b_2W_2(A^{-2}x)\right)$
$(b_1b_2 \neq 0,\ b_3 = b_4 = b_5 = 0)$,

4) $\varphi(x) = (1/4)\mathbf{1}_{[0,1)}(A^{-2}x)\left(1 + b_1W_1(A^{-2}x) + b_1b_3W_3(A^{-2}x)\right)$
$(b_1b_3 \neq 0,\ b_2 = b_6 = b_7 = 0)$,

5) $\varphi(x) = (1/4)\mathbf{1}_{[0,1)}(A^{-2}x)(1 + b_1W_1(A^{-2}x) + b_1b_2W_2(A^{-2}x) +$
$+ b_1b_3w_3(A^{-2}x) + b_1b_3b_6W_6(A^{-2}x))$
$(b_1b_2b_3b_6 \neq 0,\ b_4 = b_5 = b_7 = 0)$.

In connection with Example 2.2, see [31, Example 3] and examples of refinable step functions in [28, 33].

Let us set
$$b_s^{(\nu)} = m_\nu(A^{-n}h_{[s]}), \quad 0 \le s \le N-1.$$
Then, by Step 1, we have
$$b_0^{(0)} = 1, \quad |b_l^{(0)}|^2 + |b_{l+N_1}^{(0)}|^2 + \cdots + |b_{l+(p-1)N_1}^{(0)}|^2 \le 1, \quad 0 \le l \le N_1 - 1.$$
According to Step 4, for each l, we must find
$$b_l^{(1)}, b_{l+N_1}^{(1)}, \ldots, b_{l+(p-1)N_1}^{(1)}, \quad \ldots, \quad b_l^{(r)}, b_{l+N_1}^{(r)}, \ldots, b_{l+(p-1)N_1}^{(r)},$$
such that the matrices
$$\mathcal{M}_l = \begin{bmatrix} b_l^{(0)} & b_{l+N_1}^{(0)} & \ldots & b_{l+(p-1)N_1}^{(0)} \\ b_l^{(1)} & b_{l+N_1}^{(1)} & \ldots & b_{l+(p-1)N_1}^{(1)} \\ \vdots & \vdots & \vdots & \vdots \\ b_l^{(r)} & b_{l+N_1}^{(r)} & \ldots & b_{l+(p-1)N_1}^{(r)} \end{bmatrix}$$
satisfy the condition $\mathcal{M}_l\mathcal{M}_l^* = I$, where $\mathcal{M}_l^*$ is the Hermitian conjugate matrix of $\mathcal{M}_l$ and I denotes the identity matrix. To find such matrices $\mathcal{M}_l$ can be used the methods given in [13].

Application of the unitary extension principle (cf. [13], [30, Sect. 1.8], [38]) gives the following

Theorem 2.3. *Let $\Psi = \{\psi^{(1)}, \ldots, \psi^{(r)}\}$ be the wavelet system determined in Algorithm A. Then $X(\Psi)$ is a Parseval frame for $L^2(G)$.*

Remark 2.1. If a refinable function $\varphi_{\boldsymbol{b}}$ associated with the mask $m_{\boldsymbol{b}}$ satisfies equation (2.3) and generates a Parseval frame for $L^2(G)$, then $\boldsymbol{b} \in \boldsymbol{F}(p,n)$ (see [13, Theorem 10]).

We write $\boldsymbol{b} \in \boldsymbol{G}(p,n)$, if for a vector $\boldsymbol{b} \in \boldsymbol{F}(p,n)$ all inequalities in (2.4) become equalities. Further, denote by $\boldsymbol{W}(p,n)$ the set of all vectors $\boldsymbol{b} \in \boldsymbol{G}(p,n)$ for which
$$V(\varphi_{\boldsymbol{b}}) := \{\varphi_{\boldsymbol{b}}(\cdot \ominus h) \ : \ h \in H\}$$
is an orthonormal system in $L^2(G)$.

Remark 2.2. Algorithm A with $\boldsymbol{b} \in \boldsymbol{G}(p,n)$ in Step 1 can be applied for $r = p-1$ and Theorem 2.2 is still valid in this case (see [13]). Moreover, it is known that Algorithm A with $\boldsymbol{b} \in \boldsymbol{W}(p,n)$ leads to orthogonal MRA-based wavelets

$\psi^{(1)}, \dots, \psi^{(p-1)}$ in $L^2(G)$. There are three ways to verify the orthogonality of $V(\varphi_{\boldsymbol{b}})$: (a) the modified Cohen criterion [7, 27], (b) the blocking sets criterion [7, 31, 33], and (c) the N-valid trees method [28]. For instance, if a vector $\boldsymbol{b} = (b_0, b_1, \dots, b_{N-1})$ lies in $\boldsymbol{G}(p, n)$ and $b_l \neq 0$ for all $1 \leq l \leq N_1 - 1$, then $\boldsymbol{b} \in \boldsymbol{W}(p, n)$ (see [7, Example 5]).

3. Finite tight frames for the space $\ell^2(\mathbb{Z}_N)$

The notation used in this section is consistent with some previous publications (e.g., [8, 21]) on related topics. Let $\mathbb{Z}_N$ denote the set $\{0, 1, \dots, N-1\}$ with $N = p^n$. For $a, b \in \mathbb{Z}_N$ we define

$$a \oplus_p b := \sum_{\nu=0}^{n-1} |a_\nu - b_\nu| \, p^\nu,$$

where

$$a = \sum_{\nu=0}^{n-1} a_\nu p^\nu, \quad b = \sum_{\nu=0}^{n-1} b_\nu p^\nu, \quad a_\nu, b_\nu \in \{0, 1, \dots, p-1\}.$$

As usual, $c = a \ominus_p b$ means that $a = c \oplus_p b$.

Let us denote by $\ell^2(\mathbb{Z}_N)$ the space of complex N-periodic sequences

$$x = (\dots, x(-1), x(0), x(1), x(2), \dots), \quad x(j+N) = x(j), \quad j \in \mathbb{Z}.$$

An arbitrary x from $\ell^2(\mathbb{Z}_N)$ is given if the values of $x(j)$ are known for $j \in \mathbb{Z}_N$; therefore, the sequence x can be identified with the vector

$$(x(0), x(1), \dots, x(N-1)).$$

Hence, the space $\ell^2(\mathbb{Z}_N)$ has an inner product defined by

$$\langle x, y \rangle := \sum_{j=0}^{N-1} x(j)\overline{y(j)}$$

and the resulting norm $\|x\| = \sqrt{\langle x, x \rangle}$.

Let $\varepsilon_p = \exp(2\pi i/p)$. The *generalized Walsh functions** $w_0^{(N)}, w_1^{(N)}, \dots, w_{N-1}^{(N)}$ for the space $\ell^2(\mathbb{Z}_N)$ can be defined by

*These functions are sometimes called the *Chrestenson functions* or the *Chrestenson–Levy functions* (see e.g. [3], [23]).

$$w_k^{(N)}(l) = \varepsilon_p^{\sigma(k,l)}, \quad w_k^{(N)}(j) = w_k^{(N)}(j+N), \quad j \in \mathbb{Z},$$

where $\sigma(k,l) = \sum_{\nu=0}^{n-1} k_\nu l_{n-\nu-1}$ and

$$k = \sum_{\nu=0}^{n-1} k_\nu p^\nu, \quad l = \sum_{\nu=0}^{n-1} l_\nu p^\nu, \quad k_\nu, l_\nu \in \{0,1,\dots,p-1\}.$$

For example, if $p=3$ and $\varepsilon = \varepsilon_3$, then, for $n=1$,

$$(w_k^{(3)}(l)) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \varepsilon & \varepsilon^2 \\ 1 & \varepsilon^2 & \varepsilon \end{bmatrix}, \qquad k,l \in \{0,1,2\}.$$

and, for $n=2$,

$$(w_k^{(9)}(l)) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & \varepsilon & \varepsilon & \varepsilon & \varepsilon^2 & \varepsilon^2 & \varepsilon^2 \\ 1 & 1 & 1 & \varepsilon^2 & \varepsilon^2 & \varepsilon^2 & \varepsilon & \varepsilon & \varepsilon \\ 1 & \varepsilon & \varepsilon^2 & 1 & \varepsilon & \varepsilon^2 & 1 & \varepsilon & \varepsilon^2 \\ 1 & \varepsilon & \varepsilon^2 & \varepsilon & \varepsilon^2 & 1 & \varepsilon^2 & \varepsilon & 1 \\ 1 & \varepsilon & \varepsilon^2 & \varepsilon^2 & 1 & \varepsilon & \varepsilon & \varepsilon^2 & 1 \\ 1 & \varepsilon^2 & \varepsilon & 1 & \varepsilon^2 & \varepsilon & 1 & \varepsilon^2 & \varepsilon \\ 1 & \varepsilon^2 & \varepsilon & \varepsilon & 1 & \varepsilon^2 & \varepsilon^2 & \varepsilon & 1 \\ 1 & \varepsilon^2 & \varepsilon & \varepsilon^2 & \varepsilon & 1 & \varepsilon & 1 & \varepsilon^2 \end{bmatrix}, \qquad k,l \in \{0,1,\dots,8\}.$$

It is known also, that

$$\sum_{k=0}^{N-1} w_k^{(N)}(j) = \begin{cases} N, & j = 0 \,(\mathrm{mod}\, N), \\ 0, & j \neq 0 \,(\mathrm{mod}\, N). \end{cases}$$

Moreover, the functions $w_0^{(N)}, w_1^{(N)}, \dots, w_{N-1}^{(N)}$ constitute an orthogonal basis in $\ell^2(\mathbb{Z}_N)$ and

$$\|w_k^{(N)}\|^2 = N \quad \text{for all} \quad k \in \mathbb{Z}_N.$$

The Vilenkin–Chrestenson transform $\widehat{x}$ of each $x \in \ell^2(\mathbb{Z}_N)$ coincides with the sequence of the Fourier coefficients of x with respect to the basis $\{w_k^{(N)}\}_{k=0}^{N-1}$:

$$\widehat{x}(k) := \frac{1}{N} \sum_{j=0}^{N-1} x(j) \overline{w_k^{(N)}(j)}, \quad k \in \mathbb{Z}_N.$$

Therefore, for any $x \in \ell^2(\mathbb{Z}_N)$,

$$x(j) = \sum_{k=0}^{N-1} \widehat{x}(k) w_k^{(N)}(j), \quad j \in \mathbb{Z}_N.$$

For each $k \in \mathbb{Z}_N$, the *p-adic shift operator* $T_k : \ell^2(\mathbb{Z}_N) \to \ell^2(\mathbb{Z}_N)$ is defined by

$$(T_k x)(j) := x(j \ominus_p k), \quad x = x(j) \in \ell^2(\mathbb{Z}_N).$$

It follows from the definitions that for $x, y \in \ell^2(\mathbb{Z}_N)$ and $k, l \in \mathbb{Z}_N$ we have

$$\widehat{(T_k x)}(l) = \overline{w_k^{(N)}(l)}\,\widehat{x}(l), \quad \langle x, y\rangle = N\langle\,\widehat{x}, \widehat{y}\,\rangle.$$

We recall that I_p denotes the identity matrix of order p. The following theorem is proved in [19] for the case $p = 2$ (see also [8] for the orthogonal case).

Theorem 3.1. *Let $u_0, u_1, \dots, u_r \in \ell^2(\mathbb{Z}_N)$, where $r \ge p-1$. Suppose that*

$$M(l) := \frac{N}{\sqrt{p}} \begin{bmatrix} \widehat{u}_0(l) & \dots & \widehat{u}_r(l) \\ \widehat{u}_0(l+N_1) & \dots & \widehat{u}_r(l+N_1) \\ \widehat{u}_0(l+2N_1) & \dots & \widehat{u}_r(l+2N_1) \\ \vdots & \vdots & \vdots \\ \widehat{u}_0(l+(p-1)N_1) & \dots & \widehat{u}_r(l+(p-1)N_1) \end{bmatrix}$$

and, for each $l \in \{0, 1, \dots, N_1 - 1\}$,

$$M(l)M^*(l) = I_p, \tag{3.1}$$

where $M^*(l)$ is the Hermitian conjugate matrix of $M(l)$. Then

$$B(u_0, u_1, \dots, u_r) := \{T_{p\,k} u_0\}_{k=0}^{N_1-1} \cup \{T_{p\,k} u_1\}_{k=0}^{N_1-1} \cup \dots \cup \{T_{p\,k} u_r\}_{k=0}^{N_1-1}$$

is a Parseval frame for $\ell^2(\mathbb{Z}_N)$.

To illustrate this theorem, we give several examples.

Example 3.2. Let $p = r = 2$, $n = 1$. Then

$$M(l) = \sqrt{2} \begin{bmatrix} \widehat{u}_0(l) & \widehat{u}_1(l) & \widehat{u}_2(l) \\ \widehat{u}_0(l+1) & \widehat{u}_1(l+1) & \widehat{u}_2(l+1) \end{bmatrix}, \quad l = 0.$$

The condition (3.1) will be satisfied if we set

$$\widehat{u}_i(0) = \frac{\sqrt{2}}{2}\,x_i, \quad \widehat{u}_i(1) = \frac{\sqrt{2}}{2}\,y_i, \quad i = 0, 1, 2,$$

where (x_0, x_1, x_2) and (y_0, y_1, y_2) are orthogonal vectors with unit lengths:

$$x_0\bar{y}_0 + x_1\bar{y}_1 + x_2\bar{y}_2 = 0,$$

$$|x_0|^2+|x_1|^2+|x_2|^2=1, \quad |y_0|^2+|y_1|^2+|y_2|^2=1.$$

In particular, if $x_0=a$, $y_0=b$, $|a|^2+|b|^2\le 1$, then we can take

$$x_1=0, \quad x_2=\sqrt{1-|a|^2},$$

$$y_2=-\frac{a\bar{b}}{\sqrt{1-|a|^2}}, \quad y_1=\sqrt{1-|b|^2-|y_2|^2}.$$

As a result, for each pair of complex numbers (a,b), satisfying the condition $0<|a|^2+|b|^2\le 1$, we get the Parseval frame $\{u_0,u_1,u_2\}$ for $\ell^2(\mathbb{Z}_2)$ (cf. [8, example 2]).

Example 3.3. Let $p=r=n=2$. Choose u_0,u_1,u_2 in $\ell^2(\mathbb{Z}_4)$ such that

$$\sum_{s=0}^{2}\widehat{u}_s(l)\overline{\widehat{u}_s(l+2)}=0, \quad \sum_{s=0}^{2}|\widehat{u}_s(l)|^2=\sum_{s=0}^{2}|\widehat{u}_s(l+2)|^2=\frac{1}{8}, \quad l=0,1.$$

Then $\{u_0,u_1,u_2,T_2u_0,T_2u_1,T_2u_2\}$ is a Parseval frame for $\ell^2(\mathbb{Z}_4)$. Indeed, in this case

$$M(l)=\frac{4}{\sqrt{2}}\begin{pmatrix}\widehat{u}_0(l) & \widehat{u}_1(l) & \widehat{u}_2(l)\\ \widehat{u}_0(l+2) & \widehat{u}_1(l+2) & \widehat{u}_2(l+2)\end{pmatrix}, \quad l=0,1,$$

and (3.1) holds.

Example 3.4. Let $p=3$, $n=2$, $r=8$. Then $N=9$ and

$$M(l)=\frac{9}{\sqrt{3}}\begin{bmatrix}\widehat{u}_0(l) & \widehat{u}_1(l) & \dots & \widehat{u}_8(l)\\ \widehat{u}_0(l+3) & \widehat{u}_1(l+3) & \dots & \widehat{u}_8(l+3)\\ \widehat{u}_0(l+6) & \widehat{u}_1(l+6) & \dots & \widehat{u}_8(l+6)\end{bmatrix}, \qquad l=0,1,2.$$

Thus, we can choose the matrices $M(0)$, $M(1)$, $M(2)$ so that the matrix $[\widehat{u}_k(j)]_{k,j=0}^{j}$ will be proportional to $[w_k^{(9)}(j)]_{k,j=0}^{8}$ and (3.1) will be fulfilled. In a similar way, for any N, we can take $r=N-1$ and then use the matrix $[w_k^{(N)}(j)]_{k,j=0}^{N-1}$ to construct a Parseval frame for $\ell^2(\mathbb{Z}_N)$.

Suppose that the N-dimensional complex non-zero vector $(b_0,b_1,\dots,b_{N-1})$ satisfies the condition

$$(3.2) \qquad |b_l|^2+|b_{l+N_1}|^2+\dots+|b_{l+(p-1)N_1}|^2\le\frac{p}{N^2}, \quad l=0,1,\dots,N_1-1.$$

Then by Theorem 3.1 we have the following algorithm for constructing the Parseval frame $B(u_0,u_1,\dots,u_r)$ for $\ell^2(\mathbb{Z}_N)$.

Algorithm B

- **Step 1**. Find $u_0 \in \ell^2(\mathbb{Z}_N)$
  $\widehat{u}_0(l) = b_l, \widehat{u}_0(l+N_1) = b_{l+N_1}, \dots, \widehat{u}_0(l+(p-1)N_1) = b_{l+(p-1)N_1}$,
  $l = 0, 1, \dots, N_1 - 1$, where $b_0, b_1, \dots, b_{N-1}$ are taken from (3.2).
- **Step 2**. Find $u_1, \dots, u_r \in \ell^2(\mathbb{Z}_N)$ such that for the matrix

$$M(l) = \frac{N}{\sqrt{p}} \begin{bmatrix} \widehat{u}_0(l) & \widehat{u}_1(l) & \dots & \widehat{u}_r(l) \\ \widehat{u}_0(l+N_1) & \widehat{u}_1(l+N_1) & \dots & \widehat{u}_r(l+N_1) \\ \widehat{u}_0(l+2N_1) & \widehat{u}_1(l+2N_1) & \dots & \widehat{u}_r(l+2N_1) \\ \vdots & \vdots & \vdots & \\ \widehat{u}_0(l+(p-1)N_1) & \widehat{u}_1(l+(p-1)N_1) & \dots & \widehat{u}_r(l+(p-1)N_1) \end{bmatrix}$$

  the equality $M(l)M^*(l) = I_p$ holds for all $l = 0, 1, \dots, N_1 - 1$.
- **Step 3**. Define

$$B(u_0, u_1, \dots, u_r) = \{T_{p\,k} u_0\}_{k=0}^{N_1-1} \cup \{T_{p\,k} u_1\}_{k=0}^{N_1-1} \cup \dots \cup \{T_{p\,k} u_r\}_{k=0}^{N_1-1}.$$

Step 1 of this algorithm can be implemented by the inverse discrete Vilenkin–Chrestenson transform:

$$u_0(j) = \sum_{k=0}^{N-1} b_k w_k^{(N)}(j), \quad j \in \mathbb{Z}_N.$$

To find matrices in Step 2 can be used the methods given for Vilenkin groups in [13, 14].

Finally, we note that the initiating vector $(b_0, b_1, \dots, b_{N-1})$ in Algorithms A and B can be chosen in combination with adaptive methods used in signal processing (e.g. [5, 22, 29]). Examples of this approach for some applications of orthogonal and biorthogonal wavelets associated with the Walsh functions are given in [9, 10, 12, 34]. Properties of approximation, optimality and smoothness of dyadic wavelets and frames [13, 24, 32, 33] can also be useful in applications.

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