ON SOME CONSEQUENCES OF RECENTLY PROVED CONJECTURES

Jean-Marie De Koninck[1] (Québec, Canada)
Imre Kátai and Bui Minh Phong (Budapest, Hungary)

Communicated by Karl-Heinz Indlekofer

(Received March 6, 2019; accepted April 12, 2019)

Abstract. We provide some consequences of recently proved conjectures of Kátai regarding the values taken by arithmetic functions at consecutive integers.

1. Introduction

We provide an update on some consequences of some old conjectures formulated by Kátai, many of which have recently been proved by O. Klurman [2] and others by O. Klurman and A.P. Mangerel [3], [4].

2. Notation

Let $T := \{z \in \mathbb{C} : |z| = 1\}$ stand for the set of the points on the unit circle and let $\mathcal{M}_1$ stand for the set of multiplicative functions f such that $|f(n)| = 1$ for all positive integers n. Given $f \in \mathcal{M}_1$, consider the arithmetic function $\delta(n) = \delta_f(n) := f(n+1)\overline{f(n)}$. Given $x \in \mathbb{R}$, we set $\|x\| = \min_{n\in\mathbb{Z}} |x-n|$. As is common, we let $\mathcal{A}$ stand for the set of real-valued additive functions. Finally, given $h \in \mathcal{A}$, we set $\Delta h(n) := h(n+1) - h(n)$.

Key words and phrases: Characterization of arithmetic functions.
2010 Mathematics Subject Classification: 11N37, 11N64.

[1] Research supported in part by a grant from NSERC.

https://doi.org/10.71352/ac.49.123

3. Some old conjectures of Kátai and their recent proofs

We first state some conjectures.

Conjecture 1. (Kátai [1]) *Let $f \in \mathcal{M}_1$ and consider its corresponding function $\delta = \delta_f$. If $\lim_{x\to\infty} \frac{1}{x}\sum_{n\le x} |\delta(n) - 1| = 0$, then $f(n) = n^{it}$ for some $t \in \mathbb{R}$.*

Conjecture 2. (Kátai [1]) *Let $f \in \mathcal{M}_1$ and consider its corresponding function $\delta = \delta_f$. If $\lim_{x\to\infty} \frac{1}{\log x}\sum_{n\le x} \frac{1}{n}|\delta(n) - 1| = 0$, then $f(n) = n^{it}$ for some $t \in \mathbb{R}$.*

Conjecture 1 was proved by Klurman [2], whereas Conjecture 2 can be proved in a similar manner.

Conjecture 3. *Let $f \in \mathcal{M}_1$ and consider its corresponding function $\delta = \delta_f$. Assume that there exists some $w \in T$ and some $\varepsilon > 0$ for which $|\delta(n)w - 1| \ge \varepsilon$ for all $n \in \mathbb{N}$. Then $f(n) = g(n)n^{it}$ for some $t \in \mathbb{R}$, where $g(n)^k = 1$ for all $n \in \mathbb{N}$ and some $k \in \mathbb{N}$.*

Conjecture 3 was proved by Klurman and Mangerel [3].

Conjecture 4. *Let $f \in \mathcal{M}_1$ and consider its corresponding function $\delta = \delta_f$. Assume that there exist some $w \in T$ and some $\varepsilon > 0$ for which*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{\substack{n\le x \\ |\delta(n)w-1|<\varepsilon}} 1 = 0.$$

Then $f(n) = g(n)n^{it}$ for some $t \in \mathbb{R}$, where $g(n)^k = 1$ for all $n \in \mathbb{N}$ and some $k \in \mathbb{N}$.

Klurman and Mangerel claim (private communication) that they can prove Conjecture 4.

The above statements can be reformulated for additive functions through the following theorem.

Theorem A. *Let $h \in \mathcal{A}$ and assume that either*

$$\lim_{x\to\infty} \frac{1}{x}\sum_{n\le x} \|\Delta h(n)\| = 0 \tag{3.1}$$

or

$$\lim_{x\to\infty} \frac{1}{\log x}\sum_{n\le x} \frac{1}{n}\|\Delta h(n)\| = 0 \tag{3.2}$$

holds. Then there exists some $c \in \mathbb{R}$ such that $h(n) \equiv c\log n \pmod 1$ for all $n \in \mathbb{N}$.

Proof. This result is an obvious consequence of Conjectures 1 and 2. Indeed, setting $f(n) := e^{2\pi i h(n)}$, we have that $f \in \mathcal{M}_1$ and $\delta_f(n) - 1 \asymp \|\Delta h(n)\|$, implying that (3.1) is equivalent to the condition of Conjecture 1 whereas (3.2) is equivalent to the condition of Conjecture 2. ∎

We state our last conjecture.

Conjecture 5. *Let $h \in \mathcal{A}$, $\xi \in [0,1)$ and $\varepsilon > 0$. Let $n_1 < n_2 < \cdots$ be a sequence of positive integers of positive density. Assume that*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{\substack{n_j \le x \\ \|\Delta h(n_j) - \xi\| < \varepsilon}} 1 = 0.$$

Then, there exists $k \in \mathbb{N}$ such that $k\xi \in \mathbb{Z}$.

One can easily see that Conjecture 5 is actually a reformulation of Conjecture 4.

4. Main result

Theorem 1. *Let $h \in \mathcal{A}$ and $\tau \in \mathbb{R} \setminus \mathbb{Q}$. Assume that*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} \|\Delta h(n)\| = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} \|\tau \Delta h(n)\| = 0 \tag{4.1}$$

or

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{n\le x} \frac{1}{n} \|\Delta h(n)\| = 0 \text{ and } \lim_{x\to\infty} \frac{1}{\log x} \sum_{n\le x} \frac{1}{n} \|\tau \Delta h(n)\| = 0. \tag{4.2}$$

Then, there exists $c \in \mathbb{R}$ such that $h(n) = c \log n$ for all $n \in \mathbb{N}$.

5. Proof of Theorem 1

It follows from Theorem A that there exist $c_1, c_2 \in \mathbb{R}$ and integer valued additive functions $u(n)$ and $v(n)$ such that

$$h(n) = c_1 \log n + u(n) \quad \text{and} \quad \tau h(n) = c_2 \log n + v(n) \qquad \text{for all } n \in \mathbb{N}.$$

Since $\tau h(n) = c_1 \tau \log n + \tau u(n)$, we have that, for all $n \in \mathbb{N}$,

$$D \log n = v(n) - \tau u(n), \quad \text{where } D = c_1 \tau - c_2. \tag{5.1}$$

If $D = 0$, then $v(n) = \tau u(n)$ for every $n \in \mathbb{N}$, implying that $u(n) = v(n) = 0$ for each integer $n \geq 1$, thus completing the proof of Theorem 1 in the case $D = 0$.

From here on, we can therefore assume that $D \neq 0$. From (5.1), we have that

$$\log n = \frac{v(n)}{D} - \frac{\tau u(n)}{D},$$

so that, for arbitrary positive integers p and q, we have

$$\begin{aligned} Du(q) \log p &= u(q)v(p) - \tau u(p)u(q), \\ Du(p) \log q &= u(p)v(q) - \tau u(p)u(q), \end{aligned}$$

from which we obtain that

$$D \log \left(\frac{p^{u(q)}}{q^{u(p)}} \right) = u(q)v(p) - u(p)u(q) =: L(p,q). \tag{5.2}$$

So, let us first assume that there exist distinct primes p, q and co-prime prime powers P, Q for which $L(p,q) \neq 0$ and $L(P,Q) \neq 0$. Let A, B be such that

$$\frac{A}{B} = \frac{L(p,q)}{L(P,Q)}.$$

It follows that

$$\log \left(\frac{p^{u(q)}}{q^{u(p)}} \right)^B = \log \left(\frac{P^{u(Q)}}{Q^{u(P)}} \right)^A.$$

But, in light of the uniqueness of prime factorisation, this can hold only if $u(P) = u(Q) = 0$ and $u(p) = u(q) = 0$, which contradicts our condition $D \neq 0$.

Hence, it remains to consider the case where there exist at most three primes $\pi_1 < \pi_2 < \pi_3$ for which $u(\pi_j^{e_j}) \neq 0$ for some $e_j \in \mathbb{N}$ for $j = 1, 2, 3$. Consider the integers $n = \pi_1^{e_1} \nu$, where ν runs over those integers such that $(\nu, \pi_1 \pi_2 \pi_3) = 1$ and $(n+1, \pi_1 \pi_2 \pi_3) = 1$. In this case, we have

$$\Delta h(n) = h(n+1) - h(n) = c_1 \log \left(1 + \frac{1}{\pi_1^{e_1} \nu} \right) - u(\pi_1^{e_1}),$$

from which it follows that

$$\lim_{n = \pi_1^{e_1} \nu \to \infty} \Delta h(n) = -u(\pi_1^{e_1}),$$

which in turn implies that

$$\lim_{n=\pi_1^{e_1}\nu\to\infty} \Delta\tau h(n) = \lim_{n=\pi_1^{e_1}\nu\to\infty} (v(n+1)-v(n)) = -\tau u(\pi_1^{e_1}).$$

Now, since $v(n+1)-v(n)\in\mathbb{Z}$ and $u(\pi_1^{e_1})\neq 0$, we have established that, for a suitable $\delta>0$, there exists $n_0\in\mathbb{N}$ such that

$$\|\tau u(\pi_1^{e_1}) + (v(n+1)-v(n))\| > \delta > 0 \quad \text{for all } n\geq n_0,$$

again a contradiction. This completes the proof of Theorem 1 in this particular case.

It remains to consider the case where there exist only two primes $\pi_1<\pi_2$ for which for suitable $e_1,e_2\in\mathbb{N}$ we have $u(\pi_1^{e_1})\neq 0$ and $u(\pi_2^{e_2})\neq 0$. Similarly as above, let us consider those integers $n=\pi_1^{e_1}\nu$, where $(\nu,\pi_1\pi_2)=1$ and $(n+1,\pi_1\pi_2)=1$. We may then argue as above and conclude that this situation also leads to a contradiction.

Therefore, it only remains to consider the case where $u(P)=0$ for some prime power $P=p^\ell$ and $u(m)=0$ for every m coprime to P. Let us first assume that there exist positive integers Q_1 and Q_2 such that $(Q_1,Q_2)=1$ and $(p,Q_1Q_2)=1$ for which $v(Q_1)\neq 0$ and $v(Q_2)\neq 0$. We then have

$$\begin{aligned} D\log Q_j &= v(Q_j) \quad \text{for } j=1,2,\\ \frac{\log Q_1}{\log Q_2} &= \frac{v(Q_1)}{v(Q_2)}, \end{aligned}$$

which implies that $Q_1^{v(Q_2)}=Q_2^{v(Q_1)}$, which is clearly impossible. If $u(n)=0$ for all $n\in\mathbb{N}$ or if $v(n)=0$ for all $n\in\mathbb{N}$, we are done.

So, consider those integers $n=p^\ell\nu$, where ν runs over those positive integers satisfying $(\nu,p)=1$. In this case, we have $u(n+1)=0$ and $u(n)=u(p^\ell)$. Consequently,

$$\lim_{n=p^\ell\nu\to\infty} \Delta h(n) = -u(p^\ell) \quad \text{and}$$

$$\lim_{n=p^\ell\nu\to\infty} \Delta\tau h(n) = \lim_{n=p^\ell\nu\to\infty} (v(n+1)-v(n)) = -\tau u(p^\ell),$$

which is also impossible, thus completing the proof of Theorem 1. ■

References

[1] **Kátai, I.,** Some problems in number theory, *Studia Sci. Math. Hungar.*, **16(3–4)** (1983), 289–295.

[2] **Klurman, O.,** Correlations of multiplicative functions and applications, *Compos. Math.*, **153(8)** (2017), 1622–1657.

[3] **Klurman, O. and A.P. Mangerel,** Rigidity theorems of multiplicative functions, *Math. Ann.*, **372(1–2)** (2018), 651–697.

[4] **Klurman, O. and A.P. Mangerel,** On the orbits of multiplicative pairs, https://arxiv.org/abs/1810.08967

J.-M. De Koninck
Départment de mathématiques
Université Laval
Québec
Québec G1V 0A6
Canada
jmdk@mat.ulaval.ca

I. Kátai and B.M. Phong
Department of Computer Algebra
Faculty of Informatics
Eötvös Loránd University
Pázmány Péter sétány 1/C
H-1117 Budapest, Hungary
katai@inf.elte.hu
bui@inf.inf.elte.hu