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UNITARY EQUIVALENCE OF LOWEST DIMENSIONAL REPRODUCING FORMULAE OF TYPE $\mathcal{E}_2 \subset Sp(2,\mathbb{R})$

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Abstract. All two-dimensional reproducing formulae, i.e. of $L^2(\mathbb{R}^2)$, resulting out of restrictions of the projective metaplectic representation to connected Lie subgroups of $Sp(2,\mathbb{R})$ and of type $\mathcal{E}_2$, were listed and classified up to conjugation within $Sp(2,\mathbb{R})$ in [2], [3]. A full classification, up to conjugation within $\mathbb{R}^2 \rtimes Sp(1,\mathbb{R})$, of one-dimensional reproducing formulae, i.e. of $L^2(\mathbb{R})$, resulting out of restrictions of the extended projective metaplectic representation to connected Lie subgroups of $\mathbb{R}^2 \rtimes Sp(1,\mathbb{R})$ was obtained in [13], [14]. In dimension one, there are no reproducing formulae with one-dimensional parametrizations, yet in dimension two, there are reproducing formulae with two-dimensional parametrizations. Two-dimensional reproducing subgroups of $Sp(2,\mathbb{R})$ of type $\mathcal{E}_2$ are a novelty. They exhibit completely new phase space phenomena. We show, that they are all unitarily equivalent via natural choices of coordinate systems, and we derive the consequences of this equivalence.

1. Introduction

Let $(\mathfrak{P},\nu)$ be a measure space, and $\{\phi_\mathfrak{p}\}_{\mathfrak{p}\in\mathfrak{P}}$ a measurable field with values in a Hilbert space $\mathcal{H}$ (see e.g. Section 5.3 of [1]). We say that $\{\phi_\mathfrak{p}\}_{\mathfrak{p}\in\mathfrak{P}}$ is a

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reproducing system in $\mathcal{H}$, with the parameter measure ν, if for every $f \in \mathcal{H}$

$$f = \int_{\mathfrak{P}} \langle f, \phi_{\mathfrak{p}} \rangle \phi_{\mathfrak{p}} \, d\nu(\mathfrak{p}), \tag{1.1}$$

where the convergence in (1.1) is understood in the weak sense. Via polarization formula (1.1) is equivalent to

$$||f||^2 = \int_{\mathfrak{P}} |\langle f, \phi_{\mathfrak{p}} \rangle|^2 \, d\nu(\mathfrak{p}), \tag{1.2}$$

valid for all $f \in \mathcal{H}$. Form (1.2) of (1.1) is more convenient than (1.1) in formal arguments and we will use it frequently. In case ν is the counting measure the system $\{\phi_{\mathfrak{p}}\}_{\mathfrak{p} \in \mathfrak{P}}$ is called a Parseval frame. Formulae of the form (1.1) are called reproducing formulae.

The group $Sp(d, \mathbb{R})$ consists of $2d \times 2d$ invertible matrices, with real coefficients, preserving the symplectic form. The extended projective metaplectic representation μ_e of $\mathbb{R}^{2d} \rtimes Sp(d, \mathbb{R})$ assigns to an affine transformation $g \in \mathbb{R}^{2d} \rtimes Sp(d, \mathbb{R})$ of the phase space $\mathbb{R}^{2d} = \left\{(x, \xi) | \, x, \xi \in \mathbb{R}^d\right\}$ the corresponding unitary operator $\mu_e(g)$ acting on $L^2(\mathbb{R}^d)$. The definition of μ_e, we provide next, is based on the Wigner distribution. There are many alternative models for defining the extended projective metaplectic representation. The choice of the model depends on specific targets. In the current context we want to stress the phase space geometry phenomena captured by the Wigner distribution. The operator $\mu_e(g)$ translates the affine action of g performed on the Wigner distribution W_ϕ, $\phi \in L^2(\mathbb{R}^d)$, to the level of ϕ, i.e. to any $\phi \in L^2(\mathbb{R}^d)$ it assigns $\mu_e(g)\,\phi$ via the formula

$$W_{\mu_e(g)\,\phi}(x, \xi) = W_\phi\left(g^{-1} \cdot (x, \xi)\right), \tag{1.3}$$

where $W_\phi(x, \xi) = \int_{\mathbb{R}^d} e^{-2\pi i \langle \xi, y \rangle} \phi(x + y/2) \overline{\phi(x - y/2)} \, dy$. The function $\mu_e(g)\,\phi$ of formula (1.3) is defined up to a phase factor, since the Wigner distribution identifies square integrable functions up to phase factors. As the outcome of (1.3), we obtain the extended projective metaplectic representation μ_e. The name extended comes from the fact that we add phase space translations represented as $\mathbb{R}^{2d}$ to the linear action of $Sp(d, \mathbb{R})$. The name metaplectic is usually used for the (non-projective, i.e. exact, as far as the phase factors are concerned) representation of the double cover of $Sp(d, \mathbb{R})$, satisfying (1.3). For a comprehensive treatment of the metaplectic representation, from the point of view of analysis in phase space, we refer the reader to books by Folland [15], Gröchenig [18], De Gosson [10], and the survey article by De Mari-De Vito [12].

The classical interpretation of the Wigner distribution identifies it as the best possible surrogate of the non-existent joint probability distribution of position and momentum. It is a well established expectation of the mathematical physics community that reproducing formulae should be in one to one correspondence with phase space coverings obtained via the Wigner distribution. It is therefore of primary importance to identify and investigate all reproducing formulae for $L^2(\mathbb{R}^d)$ constructed out of restrictions of the extended metaplectic representation to connected Lie subgroups of $\mathbb{R}^{2d} \rtimes Sp(d,\mathbb{R})$. A subgroup of $\mathbb{R}^{2d} \rtimes Sp(d,\mathbb{R})$ is called reproducing, if it is possible to construct a reproducing formula out of its action on $L^2(\mathbb{R}^d)$, just by properly choosing the generating function. All one-dimensional, i.e. with $d = 1$, reproducing formulae of this type were classified up to a conjugation by an affine transformation of the time-frequency plane in [13], [14]. As a particular consequence, the classification demonstrated, that in one dimension none of the one-dimensional connected Lie subgroups of $\mathbb{R}^2 \rtimes Sp(1,\mathbb{R})$ is reproducing. The situation is different in two dimensions, i.e. for $d = 2$. It is possible to construct reproducing formulae for $L^2(\mathbb{R}^2)$ out of restrictions of the extended projective metaplectic representation to two-dimensional connected Lie subgroups of $\mathbb{R}^4 \rtimes Sp(2,\mathbb{R})$. All reproducing formulae, constructed out of restrictions of the projective metaplectic representation of $Sp(2,\mathbb{R})$ to connected Lie subgroups of type $\mathcal{E}_2$, were classified up to a conjugation by a linear transformation of the phase space $\mathbb{R}^4$ in [2] and [3]. It came as a surprise that in two dimensions also two-dimensional reproducing subgroups are possible. The other possible dimensions of reproducing subgroups are three and four. All one-dimensional reproducing formulae classified in [13], [14] may be interpreted geometrically as corresponding to coverings of the time-frequency plane constructed via the action of the reproducing subgroup applied to a compact set. The same phase space geometric interpretation is valid for all other standard reproducing formulae, but not for the two-dimensional reproducing Lie subgroups of $Sp(2,\mathbb{R})$ of type $\mathcal{E}_2$. In these special cases the set providing the phase space covering via the action of the reproducing subgroup must be non-compact. We refer the reader to [20] for detailed descriptions of phase space coverings corresponding to the two-dimensional lifts of Shannon wavelets. All these lifts are adaptable via unitary maps, introduced below in Table 2, to all representations treated in the current paper.

Let $\mathcal{Q}$ be the standard maximal parabolic subgroup of $Sp(d,\mathbb{R})$ consisting of matrices of the form

$$\begin{bmatrix} h & 0 \\ \sigma h & {}^t h^{-1} \end{bmatrix}, \tag{1.4}$$

where $h \in GL(d,\mathbb{R})$, $\sigma \in \mathrm{Sym}(d,\mathbb{R})$. Let us recall that $GL(d,\mathbb{R})$ consists of all $d \times d$ invertible matrices, and $\mathrm{Sym}(d,\mathbb{R})$ of all $d \times d$ symmetric matrices. In both cases the coefficients are real. Any $g \in \mathcal{Q}$ may be factored out as

$$g = \begin{bmatrix} 1 & 0 \\ \sigma & 1 \end{bmatrix} \begin{bmatrix} m & 0 \\ 0 & {}^t m^{-1} \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}, \tag{1.5}$$

where $a \in \mathbb{R}$, $a \neq 0$, and $m \in SL(d,\mathbb{R})$, i.e. $m \in GL(d,\mathbb{R})$, $\det m = 1$. Formulas (1.4), (1.5) show that

$$\mathcal{Q} = \mathrm{Sym}(d,\mathbb{R}) \rtimes GL(d,\mathbb{R}),$$

with the group law

$$(\sigma, h) \cdot (\sigma', h') = \left(\sigma + {}^t h^{-1}\sigma' h^{-1}, hh'\right).$$

A subgroup of $\mathcal{Q}$ is called of type $\mathcal{E}_d$ if it is of the form $\Sigma \rtimes H$, where $0 \neq \Sigma \subset$ $\subset \mathrm{Sym}(d,\mathbb{R})$ is a vector space, and $1 \neq H \subset GL(d,\mathbb{R})$ a connected Lie subgroup. For a group of type $\mathcal{E}_d$ represented as $\Sigma \rtimes H$, we have a very explicit form of the projective metaplectic representation, namely

$$\mu_e(\sigma, h)f(x) = |\det h|^{-\frac{1}{2}}\, e^{-2\pi i \Phi(x)\sigma} f\left(h^{-1}x\right), \tag{1.6}$$

where for $x \in \mathbb{R}^d$, functional $\Phi(x) \in \Sigma^*$ is defined as $\Phi(x)\sigma = -\frac{1}{2}\sigma x \cdot x$. Function $\Phi : \mathbb{R}^d \to \Sigma^*$ is called the symbol associated to Σ.

Table 1 presents a complete list of non-conjugate, two-dimensional reproducing groups of type $\mathcal{E}_2$, obtained in [2], [3]. Conjugacy is defined via inner automorphisms of $Sp(2,\mathbb{R})$.

Subgroup Type	Σ $u \in \mathbb{R}$	H $t \in \mathbb{R}$	Φ
$\mathrm{I}, \alpha \in [-1, 0)$	$\begin{bmatrix} u & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} e^{\alpha t} & 0 \\ 0 & e^{(\alpha+1)t} \end{bmatrix}$	$-\frac{1}{2}x_1^2$
II	$\begin{bmatrix} u & 0 \\ 0 & 0 \end{bmatrix}$	$e^t \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$	$-\frac{1}{2}x_1^2$
$\mathrm{III}, \alpha \in [0, \infty)$	$\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$	$e^t \begin{bmatrix} \cos \alpha t & \sin \alpha t \\ -\sin \alpha t & \cos \alpha t \end{bmatrix}$	$-\frac{1}{2}(x_1^2 + x_2^2)$
$\mathrm{IV}, \alpha \in [0, \infty)$	$\begin{bmatrix} u & 0 \\ 0 & -u \end{bmatrix}$	$e^t \begin{bmatrix} \cosh \alpha t & \sinh \alpha t \\ \sinh \alpha t & \cosh \alpha t \end{bmatrix}$	$-\frac{1}{2}(x_1^2 - x_2^2)$

Table 1

In Table 1 rows describe the choices of representatives of non-conjugate conjugacy classes of subgroups. Columns identify parameters of direct products and the projective metaplectic representation. The authors of [2], [3] identified explicitly admissibility conditions in all of the cases of reproducing subgroups listed in Table 1. They developed general tools in Theorems 2–5 of [3], and then applied them directly. An introduction of orbit equivalence in Section 3 of [3] allowed them to treat each of the cases I, III, IV, parametrized by α, in a uniform fashion.

The main result of the current paper is summarized in Table 2. For each of the cases listed in Table 1 we identify a coordinate system providing unitary equivalence with case I , $\alpha = -1$.

Subgroup Type	U	New Coordinates	Original $\mathcal{H}$	Resulting $\mathcal{H}$
I , $\alpha \in [-1, 0)$	$y_1^{\frac{\alpha+1}{2\alpha}} f_c(y_1, y_2)$	$\begin{cases} y_1 = x_1 \\ y_2 = x_1^{-\frac{\alpha+1}{\alpha}} x_2 \end{cases}$	$L^2(\mathbb{R}_+ \times \mathbb{R})$	$L^2(\mathbb{R}_+ \times \mathbb{R})$
II	$y_1^{\frac{1}{2}} f_c(y_1, y_2)$	$\begin{cases} y_1 = x_1 \\ y_2 = \frac{x_2 - x_1 \log x_1}{x_1} \end{cases}$	$L^2(\mathbb{R}_+ \times \mathbb{R})$	$L^2(\mathbb{R}_+ \times \mathbb{R})$
III, $\alpha \in [0, \infty)$	$(r')^{\frac{1}{2}} f_c(r', \theta')$	$\begin{cases} r' = r \\ \theta' = \theta - \alpha \log r \end{cases}$	$L^2(\mathbb{R}^2)$	$L^2(\mathbb{R}_+ \times \mathbb{T})$
IV, $\alpha \in [0, \infty)$	$(r')^{\frac{1}{2}} f_c(r', \theta')$	$\begin{cases} r' = r \\ \theta' = \theta - \alpha \log r \end{cases}$	$L^2(\mathbb{R}_+ \times \mathbb{R})$	$L^2(\mathbb{R}_+ \times \mathbb{R})$

Table 2

In Table 2 column U describes unitary maps, and f_c expresses f in new coordinates. In cases III, IV r, θ represent standard polar and hyperbolic polar coordinates respectively. There is a clear intuitive explanation of the choices of coordinate systems of Table 2, which are described in more detail in formulas (3.7), (3.8), (3.9), (3.10) of Section 3. The general guideline for the choices is: remove the effect of dilations from the second coordinate. We present relevant calculations for all of the cases I-IV. Left hand sides refer to the values of the second coordinate occurring in the proof of (i) of Theorem 3.1, right before the main substitution,

I. $$\left(s^{-\alpha} x_1\right)^{-\frac{\alpha+1}{\alpha}} s^{-(\alpha+1)} x_2 = x_1^{-\frac{\alpha+1}{\alpha}} x_2,$$

II. $$\frac{s x_2 + s x_1 \log s - s x_1 \log(s x_1)}{s x_1} = \frac{s x_2 + s x_1 \log x_1}{s x_1} = \frac{x_2 + x_1 \log x_1}{x_1},$$

III., IV. $$\theta + \alpha \log s - \alpha \log(sr) = \theta - \alpha \log r.$$

Our proofs are independent of the work done in [2], [3] and our results go one step further by identifying explicitly the relationships between cases I–IV.

Our approach is direct and avoids the usage of advanced integration theory tools. The guidelines are clear, perform an appropriate change of coordinates, so that the action of the representation is removed from the second coordinate. The authors of the current paper expect that, with an appropriate adaptation, the orbit equivalence method of Section 3 of [3] is also applicable to the current context.

In Section 2 we introduce representations $\mu^{(l)}$, $\mu^{(q)}$, with $\mu^{(l)}$ allowing direct adaptations of Shannon lifts results and constructions of [20] to the current setup, where we make a transition to the Fourier transform domain, and we restrict to positive frequencies, and with $\mu^{(q)}$ allowing their further transfer to the context of $Sp(2,\mathbb{R})$, in the case $\mathrm{I}, \alpha = -1$ of Table 1. The case $\mathrm{I}, \alpha = -1$ allows still further transfers to all the $\mu^{\mathcal{J}}$ cases, described in Table 1. In Section 3 we introduce and study the unitary maps of Table 2 allowing reductions of all currently known reproducing formulae of $L^2(\mathbb{R}^2)$ with two dimensional parameterizations of the cases of $\mu^{\mathcal{J}}$ to the case $\mathrm{I}, \alpha = -1$.

The book by Führ [17] approaches constructions of wavelet type expansions via powerful tools of representation theory. The existence of a generating function for $\mu^{(l)}$ is guaranteed by the general theory of wavelet transforms developed there. It is enough to observe that $\mu^{(l)}$ is unitarily equivalent to a countably infinite multiple of the standard, square-integrable representation σ of the $ax+b$ group acting on $L^2(\mathbb{R}_+)$, i.e. $\mu^{(l)} = \oplus_{n\in\mathbb{N}}\sigma_n$. The existence of a generating function or an admissible vector (as it is called in [17]) for $\mu^{(l)}$ follows from Corollary 4.27 of [17], since each σ_n has an admissible vector and the $ax+b$ group is type I and nonunimodular.

The paper by Aronszajn [4] discusses the origins of the theory of reproducing kernels. The book by Ali-Antoine-Gazeau [1] presents both the current stage of development of the theory of reproducing formulae, as well as the background results. Our current results follow the approaches of De Mari-Nowak [13], [14], Cordero-De Mari-Nowak-Tabacco [5], [6], [7], De Mari-De Vito and collaborators [11], [2], [3], [12], Cordero-Tabacco [8]. The books by Daubechies [9], Gröchenig [18], Folland [15], Wojtaszczyk [22] are comprehensive references on phase-space analysis and wavelets. We refer the reader to books by Łojasiewicz [19], Rudin [21], and Folland [16] for the background results we use in our proofs.

2. Expansions via one-dimensional affine and wavelet lattice actions

We define two reference representations $\mu^{(l)}$, $\mu^{(q)}$. Letters l, q stand for linear and quadratic oscillations respectively. Let (Y,κ) be a measure space equipped with a non-negative, complete, σ-finite measure κ. Let $\mathcal{H}$ denote the space $L^2(\mathbb{R}_+ \times Y, dx \times d\kappa(y))$, where dx is the Lebesgue measure on $\mathbb{R}_+$, and $dx \times d\kappa(y)$ is the completion of the product measure defined on $\mathbb{R}_+ \times Y$. For $f \in \mathcal{H}$ we define

$$\mu^{(l)}_{(u,s)} f(\xi, y) = s^{1/2} f(s\xi, y) e^{2\pi i u \xi}, s > 0, u \in \mathbb{R}, \tag{2.1}$$

$$\mu^{(q)}_{(v,t)} f(r, y) = t^{1/2} f(tr, y) e^{\pi i v r^2}, t > 0, v \in \mathbb{R}. \tag{2.2}$$

The maps $(u, s) \mapsto \mu^{(l)}_{(u,s)}$, $(v, t) \mapsto \mu^{(q)}_{(v,t)}$ are unitary representations on $\mathcal{H}$ of groups $G^{(l)} = \{(u, s) \,|\, u \in \mathbb{R}, s > 0\}$, with the composition rule $(u', s') \circ (u, s) = = (s'u + u', s's)$ and the left Haar measure $\frac{du\,ds}{s^2}$, and $G^{(q)} = \{(v, t) \,|\, v \in \mathbb{R}, t > 0\}$, with the composition rule $(v', t') \circ (v, t) = \big((t')^2 v + v', t't\big)$ and the left Haar measure $\frac{dv\,dt}{t^3}$. Representation (2.1) is the standard one-dimensional wavelet action applied to the first coordinate, represented in the frequency domain, and restricted to positive frequencies. Representation (2.2) is an adaptation of (2.1) to the context of the projective metaplectic representation of $Sp(2, \mathbb{R})$, where only quadratic oscillations occur. The lattice $\Lambda^{(l)} = = \big\{(2^k m, 2^k)\big\}_{k,m\in\mathbb{Z}}$ properly discretizes $\mu^{(l)}$, $G^{(l)}$, and $\Lambda^{(q)} = \big\{(2^{k+1} m, 2^{k/2})\big\}_{k,m\in\mathbb{Z}}$ is its appropriate adaptation to $\mu^{(q)}$, $G^{(q)}$.

Proposition 2.1. *Let $U : \mathcal{H} \to \mathcal{H}$ be defined as $Uf(r, y) = (2r)^{1/2} f(r^2, y)$.*

(i) *The operator U is unitary, and it intertwines representations $\mu^{(l)}$ and $\mu^{(q)}$,*

$$\mu^{(l)}_{(u,s)} = U^* \mu^{(q)}_{\left(2u, s^{1/2}\right)} U. \tag{2.3}$$

(ii) *Let $\psi \in \mathcal{H}$. The system $\left\{\mu^{(l)}_{(u,s)} \psi\right\}_{\substack{u\in\mathbb{R}\\ s>0}} \subset \mathcal{H}$, with the parameter measure $\frac{du\,ds}{s^2}$, is reproducing if and only if the system $\left\{\mu^{(q)}_{(v,t)} U\psi\right\}_{\substack{v\in\mathbb{R}\\ t>0}} \subset \mathcal{H}$, with the parameter measure $\frac{dv\,dt}{t^3}$, is reproducing.*

(iii) *Let $\psi \in \mathcal{H}$. The system $\left\{\mu^{(l)}_{\lambda} \psi\right\}_{\lambda\in\Lambda^{(l)}} \subset \mathcal{H}$ is reproducing if and only if the system $\left\{\mu^{(q)}_{\lambda} U\psi\right\}_{\lambda\in\Lambda^{(q)}} \subset \mathcal{H}$ is reproducing. In both cases the parameter measure is the counting measure.*

Proof. We start with the proof of (i). A direct calculation shows that $U^{-1} f(r, y) = \left(2r^{1/2}\right)^{-1/2} f(r^{1/2}, y)$. Simple changes of variables verify that both U and U^{-1} preserve the norm of $\mathcal{H}$. For $f, g \in \mathcal{H}$ we have

$$\begin{aligned}\left\langle f, \mu^{(l)}_{(u,s)} g\right\rangle &= \int_Y \int_{\mathbb{R}_+} f(\xi, y) s^{1/2} \overline{g}(s\xi, y) e^{-2\pi i u \xi} d\xi \, d\kappa(y) = \\ &= \int_Y \int_{\mathbb{R}_+} f(r^2, y) s^{1/2} \overline{g}(sr^2, y) e^{-2\pi i u r^2} 2r dr \, d\kappa(y) = \\ &= \int_Y \int_{\mathbb{R}_+} (2r)^{1/2} f(r^2, y) t^{1/2} (2tr)^{1/2} \overline{g}((tr)^2, y) e^{-\pi i v r^2} dr \, d\kappa(y) = \\ &= \left\langle Uf, \mu^{(q)}_{(v,t)} Ug\right\rangle = \left\langle f, U^* \mu^{(q)}_{(v,t)} Ug\right\rangle,\end{aligned}$$

where we have substituted ξ by r^2, s by t^2, and $2u$ by v. From the formula above we obtain

$$\mu^{(l)}_{(u,s)} = U^* \mu^{(q)}_{(v,t)} U,$$

and this finishes the proof of (i). We apply (i) in order to prove (ii). Substitutions $\frac{v}{2}$ for u, and t^2 for s give

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} \left| \left\langle f, \mu^{(l)}_{(u,s)} \psi \right\rangle \right|^2 \frac{du\, ds}{s^2} = \int_{\mathbb{R}_+} \int_{\mathbb{R}} \left| \left\langle Uf, \mu^{(q)}_{(v,t)} U\psi \right\rangle \right|^2 \frac{dv\, dt}{t^3}.$$

The polarization formula and the fact that U is unitary allow us to conclude (ii). The proof of (iii) follows in the same way as (ii), with integrals substituted by sums. ■

Theorem 2.2. *Let us consider $\psi \in \mathcal{H}$. The system*

$$\left\{\mu^{(l)}_{(u,s)} \psi\right\}_{\substack{u \in \mathbb{R} \\ s > 0}} \subset \mathcal{H},$$

with the parameter measure $\frac{du\, ds}{s^2}$, is reproducing if and only if the map

$$g \mapsto \int_Y \overline{\psi\,(s, y)} g(y) \, d\kappa(y),$$

from $L^2(Y, d\kappa(y))$ into $L^2(\mathbb{R}_+, \frac{ds}{s})$, preserves inner products.

Proof. Let $f \in \mathcal{H}$ have the form $f(\xi, y) = f_1(\xi) f_2(y)$, with $f_1 \in L^1 \cap L^\infty$ defined on $\mathbb{R}_+$, with measure $d\xi$, and $f_2 \in L^1 \cap L^\infty$ defined on Y, with measure $d\kappa(y)$. In the first step, we express the inner product as an iterated integral.

We obtain

(2.4)
$$\int\limits_{\mathbb{R}^2_+} \left|\langle f, \mu^{(l)}_{(u,s)}\psi\rangle\right|^2 \frac{du\,ds}{s^2} = \int\limits_{\mathbb{R}^2_+} \left|\int\limits_{\mathbb{R}_+}\int\limits_{Y} f(\xi,y)\overline{\psi(s\xi,y)}\,d\kappa(y)\,e^{-2\pi i u\xi}d\xi\right|^2 \frac{du\,ds}{s}.$$

Representation of the inner product as an iterated integral is justified by the fact, that for u, s fixed, function $f(\xi,y)\,\mu^{(l)}_{(u,s)}\psi(\xi,y)$ is integrable with respect to $d\xi \times d\kappa(y)$. In the second step we change the outer integral over $\mathbb{R}^2_+$ into an iterative form, with the integration with respect to u performed internally and with respect to s externally. The change into an iterated form is justified by the non-negativity of the expression under the integral sign. We apply Plancherel's formula with respect to u and formula (2.4) becomes

$$\int\limits_{\mathbb{R}^2_+} \left|\langle f, \mu^{(l)}_{(u,s)}\psi\rangle\right|^2 \frac{du\,ds}{s^2} = \int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+} \left|\int\limits_{Y} f_1(\xi)f_2(y))\overline{\psi(s\xi,y)}\,d\kappa(y)\right|^2 d\xi\frac{ds}{s}. \tag{2.5}$$

In the third step we change the order of integration with respect to ξ and s and we apply multiplicative invariance of measure $\frac{ds}{s}$. Formula (2.5) becomes

$$\int\limits_{\mathbb{R}^2_+} \left|\langle f, \mu^{(l)}_{(u,s)}\psi\rangle\right|^2 \frac{du\,ds}{s^2} = \int\limits_{\mathbb{R}_+} \left|f_1(\xi)\right|^2 d\xi \int\limits_{\mathbb{R}_+} \left|\int\limits_{Y} f_2(y)\overline{\psi(s,y)}\,d\kappa(y)\right|^2 \frac{ds}{s}. \tag{2.6}$$

Change of the order of integration is justified by the non-negativity of the expression under the integral sign.

If the system $\left\{\mu^{(l)}_{(u,s)}\psi\right\}_{u\in\mathbb{R},s>0}$ is reproducing, then the integrals considered above are finite, and via formula (2.6) we conclude that the map $f \mapsto \int_Y \overline{\psi\,(s,y)} f(y)\,d\kappa(y)$ restricted to $f \in L^1 \cap L^\infty$ preserves norms. Standard density, polarization arguments allow us to conclude that it extends to an isometry from $L^2(Y, d\kappa(y))$ into $L^2(\mathbb{R}_+, \frac{ds}{s})$, and that it preserves inner products.

Conversely, if the map $f \mapsto \int_Y \overline{\psi\,(s,y)} f(y)\,d\kappa(y)$ preserves inner products, then it preserves norms, the integrals considered above are finite, and (2.6) allows us to conclude via polarization that for $f, g \in \mathcal{H}$ being finite sums of tensor products of the form $f_1(x)\,f_2(y)$, $g_1(x)\,g_2(y)$, with $f_1, f_2, g_1, g_2 \in L^1 \cap L^\infty$ we have

$$\int\limits_{\mathbb{R}^2_+} \langle f, \mu^{(l)}_{(u,s)}\psi\rangle\langle\mu^{(l)}_{(u,s)}\psi, g\rangle\frac{du\,ds}{s^2} = \langle f, g\rangle.$$

A standard density argument allows us to extend the equality to all $f, g \in \mathcal{H}$.

■

Corollary 2.3. *Let us consider* $\psi \in \mathcal{H}$*. The system*

$$\left\{\mu^{(q)}_{(v,t)}\psi\right\}_{\substack{v\in\mathbb{R}\\ t>0}} \subset \mathcal{H},$$

with the parameter measure $\frac{dv\,dt}{t^3}$*, is reproducing if and only if the map*

$$g \mapsto \int\limits_Y \overline{\psi\,(r,y)}g(y)\,d\kappa(y),$$

from $L^2(Y, d\kappa(y))$ *into* $L^2(\mathbb{R}_+, \frac{dr}{r^2})$*, preserves inner products.*

Proof. The result is a direct consequence of Proposition 2.1 (ii) and Theorem 2.2. ■

For a measurable function f, defined on a topological space X, equipped with a Borel measure ν, we define its essential support ess-supp f as the intersection of all closed sets F, satisfying $f(x) = 0$ for ν-almost every x in the complement of F. We will need the following standard representation of the inner product on $L^2(\mathbb{R})$, valid for band limited functions (see e.g. Lemma 2.1 in [20])

Lemma 2.4. *Let us suppose that for* $f, g \in L^2(\mathbb{R})$ *we have ess-supp* $f, g \subset \subset [0, 2^{-k}]$*. Then*

$$\int\limits_0^{2^{-k}} f(\xi)\overline{g(\xi)}\,d\xi = 2^k \sum_{m\in\mathbb{Z}} \hat{f}\left(2^k m\right) \overline{\hat{g}\,(2^k m)}.$$

Theorem 2.5. *Let us consider* $\psi \in \mathcal{H}$*. Suppose that for almost every* $y \in Y$ *ess-supp* $\psi(\cdot, y) \subset [0, 1]$*. The system*

$$\left\{\mu^{(l)}_\lambda\psi\right\}_{\lambda\in\Lambda^{(l)}} \subset \mathcal{H},$$

with the parameter measure being the counting measure on $\Lambda^{(l)}$*, is reproducing if and only if for every pair* $f, g \in L^2(Y, d\kappa(y))$ *the equality*

$$\langle f, g\rangle = \sum_k \int\limits_Y \overline{\psi\,(2^k\xi, y)}f(y)\,d\kappa(y) \int\limits_Y \psi\left(2^k\xi, y\right)\overline{g(y)}\,d\kappa(y) \tag{2.7}$$

holds for almost every $\xi \in \mathbb{R}_+$*.*

Proof. We assume that $f(\xi,y) = f_1(\xi)\, f_2(y)$, with $f_1 \in L^1 \cap L^\infty$ on $\mathbb{R}_+$, and $f_2 \in L^1 \cap L^\infty$ on Y. In the first step we express the inner product as an iterated integral. We obtain

(2.8)
$$\sum_{\lambda\in\Lambda^{(l)}} \left|\langle f, \mu_\lambda^{(l)}\psi\rangle\right|^2 = \sum_{k,m\in\mathbb{Z}} \left| \int_{\mathbb{R}_+}\int_Y f(\xi,y) 2^{k/2}\overline{\psi(2^k\xi,y)}\, d\kappa(y)\, e^{-2\pi i 2^k m\xi} d\xi \right|^2.$$

Representation of the inner products as iterated integrals is justified by the fact, that for λ fixed, functions $f(\xi,y)\,\mu_\lambda^{(l)}\psi(\xi,y)$, $g(\xi,y)\,\mu_\lambda^{(l)}\psi(\xi,y)$are integrable with respect to $d\xi \times d\kappa(y)$. In the second step we change the summation over $k,m \in \mathbb{Z}$ into an iterative form, with the summation with respect to m performed internally and with respect to k externally. Non-negativity of the summation terms justifies the transition. We are allowed to apply Lemma 2.4 and formula (2.8) becomes

$$\sum_{\lambda\in\Lambda^{(l)}} \left|\langle f, \mu_\lambda^{(l)}\psi\rangle\right|^2 = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}_+} \left| \int_Y f_1(\xi) f_2(y)) \overline{\psi(2^k\xi,y)}\, d\kappa(y) \right|^2 d\xi. \tag{2.9}$$

The usage of Lemma 2.4 is justified by the fact that for almost every $y \in Y$ we have $\text{ess-supp}\,\psi(2^k\cdot,y) \subset \left[0,2^{-k}\right]$. We represent the square integrable kernel $\overline{\psi(2^k\cdot,\cdot)}$, defined on $\mathbb{R}_+ \times Y$, k is fixed, as an infinite sum of orthogonal tensor products of functions, with compact support, with respect to the first coordinate, and square integrable, with respect to the second coordinate. Then, we apply Lemma 2.4 to finite sums, and next we pass to norm limits in both expressions, the original one, and the one obtained by an application of Lemma 2.4. In the third step we change the order of integration with respect to ξ and summation with respect to k. Formula (2.9) becomes

$$\sum_{\lambda\in\Lambda^{(l)}} \left|\langle f, \mu_\lambda^{(l)}\psi\rangle\right|^2 = \int_{\mathbb{R}_+} |f_1(\xi)|^2 \sum_{k\in\mathbb{Z}} \left| \int_Y f_2(y) \overline{\psi(2^k\xi,y)}\, d\kappa(y) \right|^2 d\xi. \tag{2.10}$$

Change of the order of integration and summation is justified by non-negativity of the terms.

If the system $\left\{\mu_\lambda^{(l)}\psi\right\}_{\lambda\in\Lambda^{(l)}}$ is reproducing, then, via formula (2.10), we have

$$\int_{\mathbb{R}_+} |f_1(\xi)|^2 \sum_k \left| \int_Y f_2(y) \overline{\psi(2^k\xi,y)}\, d\kappa(y) \right|^2 d\xi = ||f_1||^2 ||f_2||^2,$$

for all $f_1 \in L^1 \cap L^\infty$ on $\mathbb{R}_+$, and $f_2 \in L^1 \cap L^\infty$ on Y. Therefore for every

$f \in L^1 \cap L^\infty$ on Y

$$\sum_k \left| \int_Y \overline{\psi(2^k \xi, y)} f(y)\, d\kappa(y) \right|^2 = ||f||^2, \tag{2.11}$$

for almost every $\xi \in \mathbb{R}_+$. A standard density argument, making use of the convergence in the mixed norm space $L^\infty(l^2)$, allows us to conclude that for every $f \in L^2(Y, d\kappa(y))$ (2.11) holds for almost every $\xi \in \mathbb{R}_+$. The fact that for every pair $f, g \in L^2(Y, d\kappa(y))$ the equality (2.7) holds for almost every $\xi \in \mathbb{R}_+$ follows by polarization. Conversely, if for every pair $f, g \in L^2(Y, d\kappa(y))$ the equality (2.7) holds for almost every $\xi \in \mathbb{R}_+$, then (2.10) allows us to conclude, via polarization, that for $f, g \in \mathcal{H}$ being finite sums of tensor products of the form $f_1(\xi)\, f_2(y)$, $g_1(\xi)\, g_2(y)$, with $f_1, g_1 \in L^1 \cap L^\infty$ on $\mathbb{R}_+$, and $f_2, g_2 \in L^1 \cap L^\infty$ on Y. we have

$$\sum_{\lambda \in \Lambda^{(l)}} \langle f, \mu_\lambda^{(l)} \psi \rangle \langle \mu_\lambda^{(l)} \psi, g \rangle = \langle f, g \rangle.$$

Again, a standard density argument allows us to extend the equality to all $f, g \in \mathcal{H}$. ■

Corollary 2.6. *Let us consider $\psi \in \mathcal{H}$. Suppose that for almost every $y \in Y$ ess-supp $\psi(\cdot, y) \subset [0,1]$. The system*

$$\left\{ \mu_\lambda^{(q)} \psi \right\}_{\lambda \in \Lambda^{(q)}} \subset \mathcal{H},$$

with the parameter measure being the counting measure on $\Lambda^{(q)}$, is reproducing if and only if for every pair $f, g \in L^2(Y, d\kappa(y))$ the equality

$$\langle f, g \rangle = \frac{1}{2r} \sum_k 2^{-k/2} \int_Y \overline{\psi\left(2^{k/2} r, y\right)} f(y)\, d\kappa(y) \int_Y \psi\left(2^{k/2} r, y\right) \overline{g(y)}\, d\kappa(y)$$

holds for almost every $r \in \mathbb{R}_+$.

Proof. The result is a direct consequence of Proposition 2.1 (iii) and Theorem 2.5. ■

Let $\psi^S(\xi) = \chi_{(1,2]}(\xi)$. The system $\{\psi_\lambda^S\}_{\lambda \in \Lambda^{(l)}} \subset L^2(\mathbb{R}_+)$, with

$$\psi_\lambda^S(\xi) = 2^{k/2} \chi_{(1,2]}\left(2^k \xi\right) e^{2\pi i 2^k m \xi},$$

is the Shannon wavelet system adapted to $\mathbb{R}_+$, with $L^2(\mathbb{R}_+)$ representing the Fourier transform domain. It is a family of standard trigonometric systems adapted to the dyadic partition of $\mathbb{R}_+$. We describe its lifts to $L^2(\mathbb{R}_+ \times Y)$, with Y being $\mathbb{R}$ and $\mathbb{T}$, lifts adapting the constructions done in [20] to the current context. First, we do it for $\mu^{(l)}$, and then we transfer the resulting systems to $\mu^{(q)}$ via the unitary map of Proposition 2.1.

We move now to $L^2(\mathbb{R}_+ \times Y)$. We introduce $e_{k,l}(y) = \chi_{(k,k+1]}(y)\, e^{2\pi i l y}$, with $k, l \in \mathbb{Z}$, and $f_m(s) = \chi_{(2^{-m}, 2^{-m+1}]}(s)$, with $m \geq 1, m \in \mathbb{Z}$, i.e. $m \in \mathbb{N}$. The system $\{e_{k,l}\}_{k,l\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R})$, and $\left\{c_f^{-1} f_m\right\}_{m\geq 1}$, where $c_f = (\log 2)^{1/2}$, is an orthonormal system of $L^2(\mathbb{R}_+, \frac{ds}{s})$. Let $D_{\mathbb{R}} : \mathbb{Z} \times \mathbb{Z} \to \to \mathbb{N}$, $D_{\mathbb{T}} : \mathbb{Z} \to \mathbb{N}$ be two bijections. We define the corresponding generating functions $\psi^{D_{\mathbb{R}}} \in L^2(\mathbb{R}_+ \times \mathbb{R})$, $\psi^{D_{\mathbb{T}}} \in L^2(\mathbb{R}_+ \times \mathbb{T})$ as

$$\psi^{D_{\mathbb{R}}}(\xi, y) = \sum_{k,l\in\mathbb{Z}} f_{D_{\mathbb{R}}(k,l)}(\xi) e_{k,l}(y), \tag{2.12}$$

$$\psi^{D_{\mathbb{T}}}(\xi, y) = \sum_{l\in\mathbb{Z}} f_{D_{\mathbb{T}}(l)}(\xi) e_{0,l}(y). \tag{2.13}$$

The following two lemmas summarize the basic properties of the generating functions $\psi^{D_{\mathbb{R}}}$, $\psi^{D_{\mathbb{T}}}$. Their proofs follow exactly the steps of the proof of Lemma 2.2 of [20], and are omitted from the current presentation.

Lemma 2.7. *Let $\psi^{D_{\mathbb{R}}}$ be the generating function defined in (2.12). Then*

(i) *the sum (2.12) representing $\psi^{D_{\mathbb{R}}}(\xi, y)$ consists of a single term $f_{D_{\mathbb{R}}(k,l)}(\xi) e_{k,l}(y)$, for $\xi \in (0, 1]$, with the unique k, l satisfying $\xi \in (2^{-D_{\mathbb{R}}(k,l)}, 2^{-D_{\mathbb{R}}(k,l)+1}]$, and it contains no non-zero terms for $\xi \notin (0, 1]$,*

(ii) *ess-supp $\psi^{D_{\mathbb{R}}}(\cdot, y) \subset [0, 1]$ for every $y \in \mathbb{R}$,*

(iii) $\displaystyle\int_{\mathbb{R}_+\times\mathbb{R}} \left|\psi^{D_{\mathbb{R}}}(\xi, y)\right|^2 dy\, d\xi = 1,$

(iv) $\displaystyle S_N^{D_{\mathbb{R}}}(\xi, y) = \sum_{|k|\leq N, |l|\leq N} f_{D_{\mathbb{R}}(k,l)}(\xi) e_{k,l}(y)$ *converges to* $\psi^{D_{\mathbb{R}}}(\xi, y)$ *in $L^2(\mathbb{R}_+ \times \mathbb{R})$, as $N \to \infty$.*

Lemma 2.8. *Let $\psi^{D_{\mathbb{T}}}$ be the generating function defined in (2.13). Then*

(i) *the sum (2.13) representing $\psi^{D_{\mathbb{T}}}(\xi, y)$ consists of a single term $f_{D_{\mathbb{T}}(l)}(\xi)e_{0,l}(y)$, for $\xi \in (0,1]$, with the unique l satisfying $\xi \in (2^{-D_{\mathbb{T}}(l)}, 2^{-D_{\mathbb{T}}(l)+1}]$, and it contains no non-zero terms for $\xi \notin (0,1]$,*

(ii) *ess-supp $\psi^{D_{\mathbb{T}}}(\cdot, y) \subset [0,1]$ for every $y \in \mathbb{R}$,*

(iii) $\displaystyle\int_{\mathbb{R}_+ \times \mathbb{T}} \left|\psi^{D_{\mathbb{T}}}(\xi, y)\right|^2 dy\, d\xi = 1,$

(iv) $S_N^{D_{\mathbb{T}}}(\xi, y) = \displaystyle\sum_{|l| \le N} f_{D_{\mathbb{T}}(l)}(\xi)e_{0,l}(y)$ *converges to* $\psi^{D_{\mathbb{T}}}(\xi, y)$ *in $L^2(\mathbb{R}_+ \times \mathbb{T})$, as $N \to \infty$.*

Theorem 2.9. *The systems $\{c_f^{-1}\mu_{(u,s)}^{(l)}\psi^{D_{\mathbb{R}}}\}_{u\in\mathbb{R},s>0}$, $\{c_f^{-1}\mu_{(u,s)}^{(l)}\psi^{D_{\mathbb{T}}}\}_{u\in\mathbb{R},s>0}$, with generating functions $\psi^{D_{\mathbb{R}}}$, $\psi^{D_{\mathbb{T}}}$ defined in (2.12), (2.13), both with the same parameter measure $\frac{du\,ds}{s^2}$, are reproducing in $L^2(\mathbb{R}_+ \times \mathbb{R})$, $L^2(\mathbb{R}_+ \times \mathbb{T})$, respectively.*

Proof. Both proofs, based on Theorem 2.2, follow the steps of the proof of Corollary 1.2 of [20], with the adjustments indicated in Lemmas 2.7, 2.8, respectively. ∎

Theorem 2.10. *The systems $\{\mu_\lambda^{(l)}\psi^{D_{\mathbb{R}}}\}_{\lambda\in\Lambda^{(l)}}$, $\{\mu_\lambda^{(l)}\psi^{D_{\mathbb{T}}}\}_{\lambda\in\Lambda^{(l)}}$, with $\psi^{D_{\mathbb{R}}}$, $\psi^{D_{\mathbb{T}}}$ defined in (2.12), (2.13), are orthonormal bases of $L^2(\mathbb{R}_+ \times \mathbb{R})$, $L^2(\mathbb{R}_+ \times \mathbb{T})$, respectively.*

Proof. Both proofs, based on Theorem 2.5, follow the steps of the proof of Corollary 1.4 of [20], with the adjustments indicated in Lemmas 2.7, 2.8, respectively. ∎

Corollary 2.11. *The systems $\{c_f^{-1}\mu_{(v,t)}^{(q)}U\psi^{D_{\mathbb{R}}}\}_{v\in\mathbb{R},t>0}$, $\{c_f^{-1}\mu_{(v,t)}^{(q)}U\psi^{D_{\mathbb{T}}}\}_{v\in\mathbb{R},t>0}$, with generating functions $U\psi^{D_{\mathbb{R}}}$, $U\psi^{D_{\mathbb{T}}}$, defined via an application of the unitary map U of Proposition 2.1 to functions (2.12), (2.13), both systems with the same parameter measure $\frac{dv\,dt}{t^3}$, are reproducing in $L^2(\mathbb{R}_+ \times \mathbb{R})$, $L^2(\mathbb{R}_+ \times \mathbb{T})$, respectively.*

Proof. Both proofs follow directly out of Theorem 2.9 and Proposition 2.1 (ii). ∎

Corollary 2.12. *The systems $\{\mu_\lambda^{(q)}U\psi^{D_{\mathbb{R}}}\}_{\lambda\in\Lambda^{(q)}}$, $\{\mu_\lambda^{(q)}U\psi^{D_{\mathbb{T}}}\}_{\lambda\in\Lambda^{(q)}}$, with $U\psi^{D_{\mathbb{R}}}$, $U\psi^{D_{\mathbb{T}}}$ defined via an application of the unitary map U of Proposition 2.1* *to functions* (2.12), (2.13), *are orthonormal bases of* $L^2(\mathbb{R}_+ \times \mathbb{R})$, $L^2(\mathbb{R}_+ \times \mathbb{T})$, *respectively.*

Proof. Both proofs follow directly out of Theorem 2.10 and Proposition 2.1 (iii). ■

3. Unitary equivalence of restrictions to reproducing subgroups of type $\mathcal{E}_2$

We list representatives, up to conjugation within $Sp(2,\mathbb{R})$, of all reproducing formulae obtained out of restrictions of the projective metaplectic representation of $Sp(2,\mathbb{R})$ to two-dimensional, connected Lie subgroups of $\mathcal{E}_2$. Each such reproducing formula is conjugate to exactly one reproducing formula of the list. All reproducing formulae of the list are non-conjugate. We refer the reader to [2], [3] for details and a comprehensive presentation of the topic. For the sake of simplicity, we restrict attention to the cases of single connected components of multiplicative actions on the first coordinate, and we choose $\mathbb{R}_+$ for them. The choice of $\mathbb{R}_-$ can be treated in a similar manner. The transition to two components $\mathbb{R}_+ \cup \mathbb{R}_-$ follows in a standard way, see e.g. [13], [3]. In the parametrizations of two-dimensional subgroups of $\mathcal{E}_2$ listed below we use $u, t \in \mathbb{R}$.

I. Additional parameter $\alpha \in [-1, 0)$, Hilbert space $\mathcal{H} = L^2(\mathbb{R}_+ \times \mathbb{R})$, action on $f \in \mathcal{H}$

$$\mu^{I_\alpha}_{(u,t)} f(x_1, x_2) = e^{-(2\alpha+1)t/2} e^{\pi i u x_1^2} f\left(e^{-\alpha t} x_1, e^{-(\alpha+1)t} x_2\right), \tag{3.1}$$

with the corresponding composition rule

$$(u', t') \circ (u, t) = \left(u' + e^{-2\alpha t'} u, t' + t\right)$$

and the left Haar measure $-\alpha\, du\, e^{2\alpha t} dt$.

II. No additional parameters, Hilbert space $\mathcal{H} = L^2(\mathbb{R}_+ \times \mathbb{R})$, action on $f \in \mathcal{H}$

$$\mu^{II}_{(u,t)} f(x_1, x_2) = e^{-t} e^{\pi i u x_1^2} f\left(e^{-t}(x_1, x_2 - t x_1)\right), \tag{3.2}$$

with the corresponding composition rule

$$(u', t') \circ (u, t) = \left(u' + e^{-2t'} u, t' + t\right)$$

and the left Haar measure $du\, e^{2t} dt$.

In order to describe case III, we introduce standard polar coordinates

$$\begin{cases} x_1 = r\cos 2\pi\theta \\ x_2 = r\sin 2\pi\theta \end{cases}, \tag{3.3}$$

$r > 0$, $\theta \in [0,1)$, and we interpret the interval $[0,1)$ as the unit circle $\mathbb{T}$. We define rotations

$$R_\theta = \begin{bmatrix} \cos 2\pi\theta & \sin 2\pi\theta \\ -\sin 2\pi\theta & \cos 2\pi\theta \end{bmatrix}.$$

For a function $f \in L^2(\mathbb{R}^2)$, we write f_p for its representation in polar coordinates, i.e. $f_p(r,\theta) = f(x_1,x_2)$.

III. In this case the additional parameter is $\alpha \in [0,\infty)$. The Hilbert space $\mathcal{H}$ is $L^2(\mathbb{R}^2)$, and the action of the representation on $f \in \mathcal{H}$, is

$$\mu^{III_\alpha}_{(u,t)} f(x_1,x_2) = e^{-t} e^{\pi i u\left(x_1^2+x_2^2\right)} f\left(e^{-t} R_{-\alpha t}(x_1,x_2)\right), \tag{3.4}$$

with the corresponding composition rule

$$(u',t') \circ (u,t) = \left(u' + e^{-2t'}u, t'+t\right)$$

and the left Haar measure $du\, e^{2t} dt$.

In order to describe case IV, we introduce hyperbolic polar coordinates

$$\begin{cases} x_1 = r\cosh \theta \\ x_2 = r\sinh \theta \end{cases}, \tag{3.5}$$

$r, \theta \in \mathbb{R}$, and hyperbolic rotations

$$A_\theta = \begin{bmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{bmatrix}.$$

For a function $f \in L^2(\mathbb{R}^2)$, we write f_h for its representation in hyperbolic polar coordinates, i.e. $f_h(r,\theta) = f(x_1,x_2)$.

IV. In this case the additional parameter is $\alpha \in [0,\infty)$. The Hilbert space $\mathcal{H}$ is $L^2(\mathbb{R}_+ \times \mathbb{R})$, and the action of the representation on $f \in \mathcal{H}$, is

$$\mu^{IV_\alpha}_{(u,t)} f(x_1,x_2) = e^{-t} e^{\pi i u\left(x_1^2-x_2^2\right)} f\left(e^{-t} A_{-\alpha t}(x_1,x_2)\right), \tag{3.6}$$

with the corresponding composition rule

$$(u',t') \circ (u,t) = \left(u' + e^{-2t'}u, t'+t\right)$$

and the left Haar measure $du\, e^{2t} dt$.

In what follows we introduce coordinate systems needed for the reductions of cases I–IV to $\mu^{(q)}$.

I, $-1 \le \alpha < 0$.

$$(3.7)\qquad \begin{cases} y_1 = x_1 \\ y_2 = x_1^{-\frac{\alpha+1}{\alpha}} x_2 \end{cases}, \begin{cases} x_1 = y_1 \\ x_2 = y_1^{\frac{\alpha+1}{\alpha}} y_2 \end{cases}, \text{ Jacobian} = \frac{\partial x_2}{\partial y_2} = y_1^{\frac{\alpha+1}{\alpha}}.$$

II.

$$(3.8)\qquad \begin{cases} y_1 = x_1 \\ y_2 = \frac{x_2 - x_1 \log x_1}{x_1} \end{cases}, \begin{cases} x_1 = y_1 \\ x_2 = y_1 y_2 + y_1 \log y_1 \end{cases}, \text{ Jacobian} = \frac{\partial x_2}{\partial y_2} = y_1.$$

III, $\alpha \ge 0$.

$$(3.9)\qquad \begin{cases} r' = r \\ \theta' = \theta - \alpha \log r \end{cases}, \begin{cases} r = r' \\ \theta = \theta' + \alpha \log r' \end{cases}, \text{ Jacobian} = \frac{\partial \theta}{\partial \theta'} = 1,$$

where (r, θ) are the standard polar coordinates of (3.3).

IV, $\alpha \ge 0$.

$$(3.10)\qquad \begin{cases} r' = r \\ \theta' = \theta - \alpha \log r \end{cases}, \begin{cases} r = r' \\ \theta = \theta' + \alpha \log r' \end{cases}, \text{ Jacobian} = \frac{\partial \theta}{\partial \theta'} = 1,$$

where (r, θ) are the hyperbolic polar coordinates of (3.5).

For $\mathcal{J} = \mathrm{I}_\alpha, \mathrm{II}, \mathrm{III}_\alpha, \mathrm{IV}_\alpha$ we define the corresponding lattice $\Lambda^{\mathcal{J}}$ as the image of $\Lambda^{(q)}$ via the inverse of $(u,t) \mapsto (u, e^{-\alpha t})$ for $\mathcal{J} = \mathrm{I}_\alpha$, i.e. it is $\left\{\left(2^{k+1}m, -\frac{\log 2}{2\alpha}k\right)\right\}_{m,k\in\mathbb{Z}}$, and via the inverse of $(u,t) \mapsto (u, e^{-t})$ for $\mathcal{J} = \mathrm{II}, \mathrm{III}_\alpha, \mathrm{IV}_\alpha$, i.e. it is $\left\{\left(2^{k+1}m, -\frac{\log 2}{2}k\right)\right\}_{m,k\in\mathbb{Z}}$.

Theorem 3.1. *Let $\mu^{(q)}$ be defined in (2.2), with $Y = \mathbb{R}$ in cases* I, II, IV, *and $Y = \mathbb{T}$ in case* III. *In all cases κ is the Lebesgue measure.*

(i) *In each case f_c expresses f in the adequate coordinate system.*

I. *Let us define $U^{I_\alpha} f(y_1, y_2) = y_1^{\frac{\alpha+1}{2\alpha}} f_c(y_1, y_2)$, where f_c is the expression of f in coordinates (3.7). Then U^{I_α} is unitary and we have the following intertwining property*

$$U^{I_\alpha} \mu^{I_\alpha}_{(u,t)} = \mu^{(q)}_{(u,e^{-\alpha t})} U^{I_\alpha}.$$

II. *Let us define* $U^{II}f(y_1,y_2)=y_1^{\frac{1}{2}}f_c(y_1,y_2)$, *where* f_c *is the expression of* f *in coordinates* (3.8). *Then* U^{II} *is unitary and we have the following intertwining property*

$$U^{II}\mu^{II}_{(u,t)}=\mu^{(q)}_{(u,e^{-t})}U^{II}.$$

III. *Let us define* $U^{III_\alpha}f(r',\theta')=(r')^{\frac{1}{2}}f_c(r',\theta')$, *where* f_c *is the expression of* f_p *in coordinates* (3.9), *and* f_p *expresses* f *in standard polar coordinates* (3.3). *Then* U^{III_α} *is unitary and we have the following intertwining property*

$$U^{III_\alpha}\mu^{III_\alpha}_{(u,t)}=\mu^{(q)}_{(u,e^{-t})}U^{III_\alpha}.$$

IV. *Let us define* $U^{IV_\alpha}f(r',\theta')=(r')^{\frac{1}{2}}f_c(r',\theta')$, *where* f_c *is the expression of* f_h *in coordinates* (3.10), *and* f_h *expresses* f *in hyperbolic polar coordinates* (3.5). *Then* U^{IV_α} *is unitary and we have the following intertwining property*

$$U^{IV_\alpha}\mu^{IV_\alpha}_{(u,t)}=\mu^{(q)}_{(u,e^{-t})}U^{IV_\alpha}.$$

(ii) *In each of the cases* $\mathcal{J}=\mathrm{I}_\alpha,\ \mathrm{II},\ \mathrm{III}_\alpha,\ \mathrm{IV}_\alpha$ *the system* $\{\mu^{\mathcal{J}}_{(u,t)}\psi\}_{u,t\in\mathbb{R}}$, *with the Hilbert space and the left Haar measure described in* (3.1), (3.2), (3.4), (3.6), *respectively, is reproducing, if and only if, the system* $\{\mu^{(q)}_{(v,s)}U^{\mathcal{J}}\psi\}_{\substack{v\in\mathbb{R},\\ s>0}}$, *with the parameter measure* $\frac{dv\,ds}{s^3}$, *is reproducing in* $L^2(\mathbb{R}_+\times Y)$, $Y=\mathbb{R}$ *in cases* I, II, IV, *and* $Y=\mathbb{T}$ *in case* III.

(iii) *In each of the cases* $\mathcal{J}=\mathrm{I}_\alpha,\ \mathrm{II},\ \mathrm{III}_\alpha,\ \mathrm{IV}_\alpha$ *the system* $\{\mu^{\mathcal{J}}_\lambda\psi\}_{\lambda\in\Lambda^{\mathcal{J}}}$, *with the Hilbert space described in* (3.1), (3.2), (3.4), (3.6), *respectively, is reproducing, if and only if, the system* $\{\mu^{(q)}_\lambda U^{\mathcal{J}}\psi\}_{\lambda\in\Lambda^{(q)}}$ *is reproducing in* $L^2(\mathbb{R}_+\times Y)$, $Y=\mathbb{R}$ *in cases* I, II, IV, *and* $Y=\mathbb{T}$ *in case* III. *In all cases the parameter measure is the counting measure.*

Proof. We begin with the proof of (i) and handle all four subgroup types simultaneously. We will freely identify $h\in H$ with the real parameter t and $\sigma\in\Sigma$ with u as given in Table 1. Let $P=\mathbb{R}_+\times\mathbb{R}$ for $\mathcal{J}\neq\mathrm{III}_\alpha$ and $=\mathbb{R}^2$ if $\mathcal{J}=\mathrm{III}_\alpha$, and endow P with Lebesgue measure. For $f\in L^2(P)$, we write in coordinate free form

$$\mu^{(\mathcal{J})}_{(\sigma,h)}f(p)=|\det h|^{-1/2}f(h^{-1}p)e^{\pi i\sigma p\cdot p}.$$

We need two smooth change of coordinates in general. Let (x_1,x_2) be the usual euclidean coordinates for P. For the real parameter t given in Table 1, let $s=e^{-t}$ for subgroup type $\mathcal{J}\neq I_\alpha$ and $s=e^{-\alpha t}$ for $\mathcal{J}=\mathrm{I}_\alpha$. Let $Y=\mathbb{T}$ if $\mathcal{J}=\mathrm{II}_\alpha$ and $Y=\mathbb{R}$ otherwise. The first change of coordinates is required for subgroup types III_α and IV_α. In particular, $(x_1, x_2) \to (\xi_1, \xi_2)$ with domain P and codomain $\mathbb{R}_+ \times Y$ and must have the two properties

$$e^{\pi i \sigma p \cdot p} = e^{\pi i u \xi_1^2},\ \sigma \in \Sigma,$$
$$f(h^{-1}p) = f(s\xi_1, a(s, \xi_1, \xi_2)),\ h \in H,$$

where $a(s, \xi_1, \xi_2)$ is a smooth function. For the sake of notation, we take $\xi_1 = x_1$, $\xi_2 = x_2$ for $\mathcal{J} = \mathrm{I}_\alpha$, II. The second change of coordinates $(\xi_1, \xi_2) \to$ $\to (y_1, y_2)$ on $\mathbb{R}_+ \times Y$ must satisfy the following three properties:

$$\begin{aligned} y_1 &= \xi_1, \\ y_2 &= F(\xi_1, \xi_2),\ \text{where } F(\xi_1, \xi_2) \text{ is an } H\text{-invariant function}, \\ sJ(sy_1, y_2) &= |\det h|^{-1} J(y_1, y_2), \end{aligned}$$

where $J(y_1, y_2)$ is the Jacobian $|\partial(x_1, x_2)/\partial(y_1, y_2)|$. We note that the explicit form of the H-invariance of F is

$$F(s\xi_1, a(s, \xi_1, \xi_2)) = F(\xi_1, \xi_2),\ s > 0.$$

Finally, we define the unitary maps $U^{(\mathcal{J})} : L^2(P) \to L^2(\mathbb{R}_+ \times Y)$ by

$$U^{(\mathcal{J})} f(p) = J^{1/2}(y_1, y_2) f_c(y_1, y_2).$$

We now verify the equivalence between $\mu^{(q)}_{(u,s)}$ and $\mu^{(\mathcal{J})}_{(u,t)}$:

$$\begin{aligned} \mu^{(q)}_{(u,s)} U^{(\mathcal{J})} f(p) &= \mu^{(q)}_{(u,s)} \left[J^{1/2}(y_1, y_2) f_c(y_1, y_2) \right] = \\ &= s^{1/2} J^{1/2}(sy_1, y_2) e^{\pi i u y_1^2} f_c(sy_1, y_2) = \\ &= |\det h|^{-1/2} J^{1/2}(y_1, y_2) e^{\pi i u y_1^2} f_c(sy_1, y_2) = \\ &= U^{(\mathcal{J})} \left[|\det h|^{-1/2} e^{\pi i u y_1^2} f_c(sy_1, y_2) \right] = \\ &= U^{(\mathcal{J})} \mu^{(\mathcal{J})}_{(u,t)} f(p). \end{aligned}$$

To finish, we give the explicit form for $F(\xi_1, \xi_2)$ for each subgroup type and verify its H-invariance. For type I_α, $F(x_1, x_2) = x_1^{-\frac{\alpha+1}{\alpha}} x_2$ (see (3.7)) and

$$\left(e^{-\alpha t} x_1\right)^{-\frac{\alpha+1}{\alpha}} e^{-(\alpha+1)t} x_2 = x_1^{-\frac{\alpha+1}{\alpha}} x_2;$$

for type II, $F(x_1, x_2) = (x_2 - x_1 \log x_1)/x_1$ (see (3.8)) and

$$\frac{sx_2 + sx_1 \log s - sx_1 \log(sx_1)}{sx_1} = \frac{sx_2 + sx_1 \log x_1}{sx_1} = \frac{x_2 + x_1 \log x_1}{x_1}.$$

For the remaining two subgroup types, we must make the preliminary change of coordinates: to polar coordinates (type III_α) or to hyperbolic polar coordinates (type IV_α). For clarity, we use the appropriate notation of polar coordinates instead of the generic variables ξ_1, ξ_2. For both types III_α and IV_α, $F(r,\theta) = = \theta - \alpha \log r$ (see (3.9), (3.10)) and

$$\theta + \alpha \log s - \alpha \log(sr) = \theta - \alpha \log r.$$

The transformation property of the Jacobian $J(y_1, y_2)$ is straightforward to verify.

We apply (i) in order to prove (ii). In case I substitution of $e^{-\alpha t}$ by s gives

$$-\alpha \int_{\mathbb{R}} \int_{\mathbb{R}} \left|\left\langle f, \mu^{I_\alpha}_{(u,t)} g\right\rangle\right|^2 du\, e^{2\alpha t} dt = \int_{\mathbb{R}} \int_{\mathbb{R}} \left|\left\langle U^{I_\alpha} f, \mu^{(q)}_{(u,e^{-\alpha t})} U^{I_\alpha} g\right\rangle\right|^2 du\, e^{2\alpha t} dt =$$
$$= \int_{\mathbb{R}_+} \int_{\mathbb{R}} \left|\left\langle U^{I_\alpha} f, \mu^{(q)}_{(u,s)} U^{I_\alpha} g\right\rangle\right|^2 \frac{du\, ds}{s^3}.$$

In cases II, III, IV we substitute e^{-t} by s. For $\mathcal{J} = \mathrm{II}, \mathrm{III}_\alpha, \mathrm{IV}_\alpha$ we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left|\left\langle f, \mu^{\mathcal{J}}_{(u,t)} g\right\rangle\right|^2 du\, e^{2t} dt = = \int_{\mathbb{R}} \int_{\mathbb{R}} \left|\left\langle U^{\mathcal{J}} f, \mu^{(q)}_{(u,e^{-t})} U^{\mathcal{J}} g\right\rangle\right|^2 du\, e^{2t} dt =$$
$$= \int_{\mathbb{R}_+} \int_{\mathbb{R}} \left|\left\langle U^{\mathcal{J}} f, \mu^{(q)}_{(u,s)} U^{\mathcal{J}} g\right\rangle\right|^2 \frac{du\, ds}{s^3}.$$

Polarization formula and the fact that operators $U^{\mathcal{J}}$, $\mathcal{J} = \mathrm{I}_\alpha, \mathrm{II}, \mathrm{III}_\alpha, \mathrm{IV}_\alpha$, are unitary finish the proof in all cases. The proof of (iii) follows in the same way as (ii), with integrals substituted by sums. ∎

Corollary 3.2. *In each of the cases* $\mathcal{J} = \mathrm{I}_\alpha, \mathrm{II}, \mathrm{III}_\alpha, \mathrm{IV}_\alpha$, *the system* $\{\mu^{\mathcal{J}}_{(u,t)}\psi\}_{u,t\in\mathbb{R}}$, *with the Hilbert space and the parameter measure being the left Haar measure, both described in* (3.1), (3.2), (3.4), (3.6), *respectively, is reproducing if and only if the corresponding integral operator of the following table preserves inner products.*

Subgroup Type	*Integral Kernel*	*Domain*	*Codomain*
I, $\alpha \in [-1, 0)$	$\psi\left(r, r^{\frac{\alpha+1}{\alpha}} y\right)$	$L^2(\mathbb{R}, dy)$	$L^2\left(\mathbb{R}_+, r^{\frac{1-\alpha}{\alpha}} dr\right)$
II	$\psi\left(r, ry + r \log r\right)$	$L^2(\mathbb{R}, dy)$	$L^2\left(\mathbb{R}_+, \frac{dr}{r}\right)$
III, $\alpha \in [0, \infty)$	$\psi_p\left(r, y + \alpha \log r\right)$	$L^2(\mathbb{T}, dy)$	$L^2\left(\mathbb{R}_+, \frac{dr}{r}\right)$
IV, $\alpha \in [0, \infty)$	$\psi_h\left(r, y + \alpha \log r\right)$	$L^2(\mathbb{R}, dy)$	$L^2\left(\mathbb{R}_+, \frac{dr}{r}\right)$

By ψ_p, ψ_h we denote the representations of ψ in polar and hyperbolic polar coordinates, respectively.

Proof. The result is a direct consequence of Theorem 3.1 (ii) and Corollary 2.3. ∎

Corollary 3.3. *In each of the cases $\mathcal{J} = \mathrm{I}_\alpha, \mathrm{II}, \mathrm{III}_\alpha, \mathrm{IV}_\alpha$, the system $\{\mu_\lambda^{\mathcal{J}}\psi\}_{\lambda\in\Lambda^{\mathcal{J}}}$, with the Hilbert space described in (3.1), (3.2), (3.4), (3.6), respectively, is a Parseval frame, if and only if the corresponding integral operator of the following table preserves inner products for almost every $r \in \mathbb{R}_+$. Domains of the operators are the same as in Corollary 3.2. Codomains are weighted $l^2(\mathbb{Z})$ with the indicated weight.*

Subgroup Type	*Integral Kernel*	*Weight*
$I, \alpha \in [-1, 0)$	$2^{-1/2} r^{\frac{1}{2\alpha}} \psi\left(2^{k/2}r, \left(2^{k/2}r\right)^{\frac{\alpha+1}{\alpha}} y\right)$	$2^{\frac{k}{2\alpha}}$
II	$2^{-1/2}\psi\left(2^{k/2}r, 2^{k/2}ry + 2^{k/2}r\log\left(2^{k/2}r\right)\right)$	1
$III, \alpha \in [0, \infty)$	$2^{-1/2}\psi_p\left(2^{k/2}r, y + \alpha\log\left(2^{k/2}r\right)\right)$	1
$IV, \alpha \in [0, \infty)$	$2^{-1/2}\psi_h\left(2^{k/2}r, y + \alpha\log\left(2^{k/2}r\right)\right)$	1

By ψ_p, ψ_h we denote the representations of ψ in polar and hyperbolic polar coordinates. We assume that for almost every $y \in Y$, where $Y = \mathbb{R}$ for $\mathcal{J} = \mathrm{I}_\alpha, \mathrm{II}, \mathrm{IV}_\alpha$, and $Y = \mathbb{T}$ for $\mathcal{J} = \mathrm{III}_\alpha$, ess-supp $\psi(\cdot, y) \subset [0, 1]$ in cases $\mathcal{J} = \mathrm{I}_\alpha, \mathrm{II}$, and ess-supp $\psi_p(\cdot, y) \subset [0, 1]$, ess-supp $\psi_h(\cdot, y) \subset [0, 1]$ in cases $\mathcal{J} = \mathrm{III}_\alpha, \mathrm{IV}_\alpha$, respectively.

Proof. The result is a direct consequence of Theorem 3.1 (iii) and Corollary 2.6. ∎

We define $\psi^{\mathcal{J},D_\mathbb{R}} = \left(U^{\mathcal{J}}\right)^{-1} U\psi^{D_\mathbb{R}}$, for $\mathcal{J} = \mathrm{I}_\alpha, \mathrm{II}, \mathrm{IV}_\alpha$, and $\psi^{\mathcal{J},D_\mathbb{T}} = \left(U^{\mathcal{J}}\right)^{-1} U\psi^{D_\mathbb{T}}$, for $\mathcal{J} = \mathrm{III}_\alpha$, where $\psi^{D_\mathbb{R}}$, $\psi^{D_\mathbb{T}}$ are defined in (2.12), (2.13), respectively, and U is the unitary map of Proposition 2.1.

Corollary 3.4. *Systems $\{c_f^{-1}\mu^{\mathcal{J}}_{(u,t)}\psi^{\mathcal{J},D_\mathbb{R}}\}_{u,t\in\mathbb{R}}$, $\mathcal{J} = \mathrm{I}_\alpha, \mathrm{II}, \mathrm{IV}_\alpha$, $\{c_f^{-1}\mu^{\mathcal{J}}_{(u,t)}\psi^{\mathcal{J},D_\mathbb{T}}\}_{u,t\in\mathbb{R}}$, $\mathcal{J} = \mathrm{III}_\alpha$, where $c_f = (\log 2)^{1/2}$, with the Hilbert spaces and the parameter measures the same as in Corollary 3.2, are reproducing.*

Proof. The result is a direct consequence of Theorem 3.1 (ii) and Corollary 2.11. ∎

Corollary 3.5. *Systems* $\{\mu_\lambda^{\mathcal{J}}\psi^{\mathcal{J},D_{\mathbb{R}}}\}_{\lambda\in\Lambda^{\mathcal{J}}}$, $\mathcal{J}=\mathrm{I}_\alpha$, II, IV_α, $\{\mu_\lambda^{\mathcal{J}}\psi^{\mathcal{J},D_{\mathbb{T}}}\}_{\lambda\in\Lambda^{\mathcal{J}}}$, $\mathcal{J}=\mathrm{III}_\alpha$, *with the Hilbert spaces the same as in Corollary 3.3, are orthonormal bases.*

Proof. The result is a direct consequence of Theorem 3.1 (iii) and Corollary 2.12. ■

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