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RECOVERING BANACH-VALUED COEFFICIENTS OF SERIES WITH RESPECT TO CHARACTERS OF ZERO-DIMENSIONAL GROUPS

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Abstract. Henstock type integrals for Banach-space-valued functions on zero-dimensional compact abelian group are introduced and used to recover, by generalized Fourier formulae, the vector-valued coefficients of series with respect to characters of such a group. The problem of convergence of Fourier-Henstock series is also investigated.

1. Introduction

One of the aim of the present paper is to investigate whether some results on series with respect to system of characters of zero-dimensional compact abelian groups known in the scalar-valued case can be extended to the case of Banach-space-valued coefficients. First of all we are interested in the problem of recovering, by generalized Fourier formulae, the vector-valued coefficients of such a series. We consider also the problem of convergence of Fourier series of Banach-space-valued functions. The scalar-valued case was considered in [13].

During the last decades, the study in vector-valued Fourier analysis demonstrates that in some cases classical results about scalar-valued functions remain

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true for any Banach space, some other results can be carried over only to the finite-dimensional case and there are also cases in which the validity of a result depends on the structure and geometry of the Banach spaces involved. A prominent example is the vector-valued extension of Carleson's celebrated theorem on point-wise convergence of Fourier series which is possible only in the case of UMD (unconditionality of martingale differences) spaces and was obtained a few years ago for a wide class of these spaces in the case of Fourier series with respect to Walsh and trigonometric systems (see [5, 6]) and also for Vilenkin system of bounded type (see [16]). Another example is the theory involving type and cotype of Banach spaces.

All the situations mentioned above occurs in the case of problems considered here for series with respect to system of characters. The result of [13] related to the problem of recovering the coefficients in the scalar case is extended here for any Banach space. But convergence of Fourier series in the sense of some considered here integrals remains valid only in the finite-dimension case. Moreover if we consider the rate of divergence of Fourier series for functions with value in an infinite-dimensional space then it turns out to depend on the structure of the space.

As it was in the scalar case, the problem of recovering the coefficients and the problem of convergence can be reduced to the correspondent problems in the theory of differentiation or integration of a certain functions associated with the series. In particular a solution of the problem of recovering the coefficients is obtained by reducing it to the one of recovering a primitive. In turn to solve this problem some Henstock type generalizations of Bochner and Pettis integrals are introduced and investigated.

Some of the results presented here are a generalization of those ones obtained in [10] for the case of Walsh and Haar series.

In Section 2 we recall some definitions and facts from harmonic analysis on zero-dimensional compact abelian group and from Banach space theory. In Section 3 several Henstock type integrals for Banach-valued functions on the group, needed to solve the problem of recovering coefficients, are introduced. A differential properties of these integrals are considered in Section 4 and a related problem of recovering a primitive from its generalized derivative is solved. One of the essential result of this section is Theorem 4.6. It states that for any infinite-dimensional Banach space there exists a function integrable in the sense of a Henstock type integral, introduced in the previous section, and the indefinite integral of this function is nowhere differentiable in a respective sense. In Section 5 a relation between convergence properties of a series with respect to the system of characters of zero-dimensional compact abelian groups and differential properties of a certain function associated with the series is established and in this way the problem of recovering, by generalized Fourier formulae, the vector-valued coefficients of a convergent series is solved by reducing it to the problem of recovering primitives considered in the previous section. Some results related to convergence of Fourier–Henstock series with respect to characters are also obtained as corollaries of results of the previous section. In particular it is proved for any infinite-dimensional Banach space that there exists a function with values in this space such that its Fourier–Henstock series diverges everywhere. A rate of growth of the partial sums of such a divergent series is also discussed.

2. Preliminaries

Let G be a zero-dimensional compact abelian group with second countability axiom. It is known (see [1]) that a topology in such a group can be given by a chain of subgroups

$$G = G_0 \supset G_1 \supset G_2 \cdots \supset G_n \supset \cdots \tag{2.1}$$

with $G = \bigcup_{n=0}^{+\infty} G_n$ and $\{0\} = \bigcap_{n=0}^{+\infty} G_n$. The subgroups G_n are clopen sets with respect to this topology. As G is compact, the factor group G_0/G_n and also the factor groups G_n/G_{n+1} for each n are finite. Let the order of the group G_n/G_{n+1} be p_n. Then the order of the group G_0/G_n is $m_n := p_0 \cdot p_1 \cdot \ldots \cdot p_{n-1}$, with $p_i \geq 2$ for all i (we agree that $m_0 := 1$). We denote by K_n any coset of the subgroup G_n and we numerate them so that $G_n = K_n^1$ and the rest of them are K_n^i, $i = 2, \ldots m_n$. For any $g \in G$ we denote by $K_n(g)$ a coset of the subgroup G_n which contains the element g, i.e.,

$$K_n(g) = g + G_n. \tag{2.2}$$

For each $g \in G$ the sequence $\{K_n(g)\}$ is decreasing and $\{g\} = \bigcap_n K_n(g)$.

We denote by μ the normalized Haar measure on the group G. We can make this measure to be complete by including all the subsets of the sets of measure zero into the class of measurable sets.

Since $\mu(G_0) = 1$ and μ is translation invariant then

$$\mu(G_n) = \mu(K_n) = \frac{1}{m_n} \tag{2.3}$$

for all cosets K_n, $n \geq 0$. It is easy to see that μ is outer regular.

Let Γ denotes the dual group of G, i.e., the group of characters of the group G. It is known (see [1]) that under assumption imposed on G the group Γ is discrete abelian group (with respect to the pointwise multiplication of characters) and it can be represented as a sum of increasing chain of finite subgroups

$$\Gamma_0 \subset \Gamma_1 \subset \Gamma_2 \subset \cdots \subset \Gamma_n \subset \cdots, \tag{2.4}$$

where $\Gamma_0 = \{\gamma_0\}$ with $\gamma_0(g) = 1$ for all $g \in G$. For each $n \in \mathbb{N}$ the group Γ_n is the annulator of G_n, i.e.,

$$\Gamma_n := \{\gamma \in \Gamma : \gamma(g) = 1 \ \text{ for all } \ g \in G_n\}.$$

The factor groups Γ_{n+1}/Γ_n and G_n/G_{n+1} are isomorphic (see [1]) and so they are of finite order p_n for each $n \in \mathbb{N}$.

It is easy to check that if $\gamma \in \Gamma_n$ then γ is constant on each coset K_n of G_n, and if $\gamma \in \Gamma \setminus \Gamma_n$ then $\int_{K_n} \gamma d\mu = 0$ for each coset K_n (see [13]).

This implies that if γ_1 and γ_2 are not equal identically on K_n, then they are orthogonal on K_n, i.e.,

$$\int\limits_{K_n} \gamma_1 \overline{\gamma_2} d\mu = 0.$$

So the characters γ constitute a countable orthogonal system on G with respect to normalized measure μ and we can consider a series

$$\sum_{\gamma \in \Gamma} a_\gamma \gamma \tag{2.5}$$

with respect to this system. We define the convergence of this series at a point g as the convergence of its partial sums

$$S_n(g) := \sum_{\gamma \in \Gamma_n} a_\gamma \gamma(g) \tag{2.6}$$

when n tends to infinity.

If coefficients a_γ are Banach-valued we can consider strong and weak convergence of this series.

Now we recall some definitions and facts from Banach space theory (see, for example, [2]).

A sequence $\{e_k\}_{k=1}^{\infty}$ in a Banach space X is called a *basic sequence* if it is a basis for the closed linear span of $\{e_k\}_{k=1}^{\infty}$.

If $\{e_k\}_{k=1}^{\infty}$ is a basis for a Banach space X then the number $K = \sup_n \|S_n\|$, where $\{S_n\}$ are the natural projections associated with the bases, i.e.,

$$S_n\Big(\sum_{i=1}^{\infty} a_k e_k\Big) = \sum_{i=1}^{n} a_k e_k$$

is called the *basis constant*.

The following statement is attributed to Mazur (see [2, Corollary 1.5.3]).

Proposition 2.1. *Every infinite-dimensional Banach space contains for any $K > 1$ a basic sequence with basis constant less than K.*

If X and Y are two isomorphic Banach spaces, the *Banach-Mazur distance* between X and Y is defined as

$$\inf\left\{\|T\|\,\|T^{-1}\| : T : X \to Y \text{ is an isomorphism}\right\}.$$

Let X and Y be infinite-dimensional Banach spaces. We say that X is *finitely representable* in Y if given any finite-dimensional subspace E of X and $C > 1$ there is a finite-dimensional subspace F of Y with $\dim F = \dim E$, and a linear isomorphism $T : E \to F$, satisfying $\inf \|T\|\,\|T^{-1}\| < C$, that is, the Banach–Mazur distance between E and F is less than C.

In these terms famous Dvoretzky's Theorem can be formulated as follows:

Proposition 2.2. [2, Theorem 11.3.13]. *The space l_2 is finitely representable in every infinite-dimensional Banach space.*

We denote by X^* the dual space of X, i. e., the space of bounded linear functionals on X.

3. Henstock type generalization of Bochner and Pettis integrals on the group G

We extend here to the Banach-space-valued case some definitions of generalized Henstock types (Kurzweil–Henstock types, to be exact) integrals on the group which were considered in [13] for the scalar case. For definition of the classical Kurzweil–Henstock integral on the real line for real-valued and for Banach-space-valued functions see [7] and [9]. First we recall the construction of the correspondent derivation basis.

For any function $\nu : G \to \mathbb{N}$, called a *gage*, we define the set

$$\beta_\nu := \{(I, g) : g \in G, I = K_n(g), n \geq \nu(g)\}.$$

Then our *derivation basis* $\mathcal{B}$ is the family $\{\beta_\nu\}_\nu$ where ν runs over the set of all natural-valued functions on G. In the terminology of the derivation basis theory each coset K_n, $n \geq 0$, can be called *$\mathcal{B}$-interval of rank n*. We denote by $\mathcal{I}$ the family of all $\mathcal{B}$-intervals.

This basis has all the usual properties of a general derivation basis (see [8]).

First of all it has the *filter base property*: $\emptyset \notin \mathcal{B}$ and for every $\beta_{\nu_1}, \beta_{\nu_2} \in \in \mathcal{B}$ there exists $\beta_\nu \in \mathcal{B}$ such that $\beta_\nu \subset \beta_{\nu_1} \cap \beta_{\nu_2}$ (it is enough to take $\nu = = \max\{\nu_1, \nu_2\}$).

For a fixed gage ν a *β_ν-partition* is a finite collection π of elements of β_ν, where the distinct elements (I', g') and (I'', g'') in π have I' and I'' disjoint. If L is a $\mathcal{B}$-interval and $\bigcup_{(I,g)\in\pi} I = L$ then π is called β_ν-partition *of* L and if $\bigcup_{(I,g)\in\pi} I \subset L$ then π is called β_ν-partition *in* L.

Basis $\mathcal{B}$ has the *partitioning property.* It means that the following conditions hold: for each finite collection $I_0, I_1, \ldots, I_n$ of $\mathcal{B}$-intervals with $I_1, \ldots, I_n \subset I_0$ and I_i, $i = 1, 2, \ldots$, being disjoint, the difference $I_0 \setminus \bigcup_{i=1}^n I_i$ can be expressed as a finite union of pairwise disjoint $\mathcal{B}$-intervals; for each $\mathcal{B}$-interval L and for any $\beta_\nu \in \mathcal{B}$ there exists a β_ν-partition of L.

We write

$$\beta_\nu(E) := \{(I, g) \in \beta_\nu : I \subset E\} \quad \text{and} \quad \beta_\nu[E] := \{(I, g) \in \beta_\nu : g \in E\}.$$

Note that any two $\mathcal{B}$-intervals I' and I'' are either disjoint or one of them is contained in the other one and that, given a point $g \in G$, any β_ν-partition contains only one pair (I, g) with this point g.

We say that a $\mathcal{B}$-interval function F is *$\mathcal{B}$-continuous* at a point g, with respect to the basis $\mathcal{B}$, if $\lim_{n\to\infty} F(K_n(g)) = 0$.

Now we define $H_{\mathcal{B}}$-integral with respect to the basis $\mathcal{B}$.

Definition 3.1. Let X be a Banach space and $L \in \mathcal{I}$. A function $f : L \to X$ on L is said to be *$H_{\mathcal{B}}$-integrable on L*, with *$H_{\mathcal{B}}$-integral* $A \in X$, if for every $\varepsilon > 0$, there exists a gage $\nu : L \mapsto \mathbb{N}$ such that for any β_ν-partition π of L we have:

$$\left\| \sum_{(I,g)\in\pi} f(g)\mu(I) - A \right\|_X < \varepsilon. \tag{3.1}$$

We denote the integral value A by $(H_{\mathcal{B}}) \int_L f$.

It is easy to check that if a function f is $H_{\mathcal{B}}$-integrable on a $\mathcal{B}$-interval L then it is also $H_{\mathcal{B}}$-integrable on any $\mathcal{B}$-interval $K \subset L$. So the indefinite $H_{\mathcal{B}}$-integral on L is defined and it can be easily proved that it is an additive $\mathcal{B}$-interval function on the set of all $\mathcal{B}$-subintervals of L and is $\mathcal{B}$-continuous at each point of L. We can check also that $H_{\mathcal{B}}$-integral is invariant under translation given by some element $g \in G$.

The fact that $H_{\mathcal{B}}$-integral is a generalization of the Bochner integral on G can be checked similar to the case of integrals on an interval of the real line (see [7] or [9]).

An essential part of the theory of the Kurzweil-Henstock integral on an interval of the real line is based on the so called Saks-Henstock Lemma (see [9, Lemma 3.4.1]). The following generalization of this Lemma for the case of our basis can be proved by similar arguments.

Lemma 3.1. *If a function $f : L \to \mathbb{R}$ is $H_{\mathcal{B}}$-integrable on $L \in \mathcal{I}$ and the inequality (3.1) holds for any β_ν-partition π of L, where ν is chosen by $\varepsilon > 0$ according to Definition 3.1, then for any β_ν-partition π_1 in L we have*

$$\left\| \sum_{(I,g)\in\pi_1} (f(g)\mu(I) - \int_I f) \right\| \leq \varepsilon.$$

In the scalar-valued case a stronger statement known as Kolmogorov–Henstock Lemma (see a version of it in [7, Lemma 3.9]) holds.

Lemma 3.2. *If a function $f : L \to \mathbb{R}$ on $L \in \mathcal{I}$ is $H_{\mathcal{B}}$-integrable on L with F being its indefinite $H_{\mathcal{B}}$-integral, then for every $\varepsilon > 0$, there exists a gage $\nu : L \to \mathbb{N}$ such that for any β_ν-partition π of L we have*

$$\sum_{(I,g)\in\pi} \left| f(g)\mu(I) - \int_I f \right| < \varepsilon.$$

The property, described in this lemma, gives in the scalar-valued case an equivalent definition of the $H_{\mathcal{B}}$-integral. It is not so in the Banach-space-valued case (see [12] where this problem is considered for integrals on the real line).

So if we accept the property from the lemma to hold, we get another version of a Henstock type integral with respect to $\mathcal{B}$, the so called *variational* Henstock type integral. For the full interval basis on the real line such an integral was defined in [12].

Definition 3.2. Let X be a Banach space and $L \in \mathcal{I}$. A function $f : L \to X$ on L is said to be *$VH_{\mathcal{B}}$-integrable on L, with a given $\mathcal{B}$-interval additive function $F : \mathcal{I} \to X$ as the indefinite $VH_{\mathcal{B}}$-integral*, if for every $\varepsilon > 0$, there exists a gage $\nu : L \to \mathbb{N}$ such that for any β_ν-partition π of L we have:

$$\sum_{(I,g)\in\pi} \|f(g)\mu(I) - F(I)\|_X < \varepsilon.$$

It is clear that each $VH_{\mathcal{B}}$-integrable function is also $H_{\mathcal{B}}$-integrable and the integral values coincide.

It is easy to check that a function which is equal to zero almost everywhere on $L \in \mathcal{I}$, is $H_{\mathcal{B}}$-integrable (and also $H_{\mathcal{B}}$-integrable) on L with integral value zero. This implies that $H_{\mathcal{B}}$-integrability of a function and the value of the

$H_{\mathcal{B}}$-integral does not depend on values of the function on a set of measure zero.

We consider also a Pettis type definition.

Definition 3.3. A function $f : G \to X$ is *Henstock–Pettis integrable with respect to basis* $\mathcal{B}$ (or $HP_{\mathcal{B}}$*-integrable*) on G if $x^*(f)$ is $H_{\mathcal{B}}$-integrable on each $I \in \mathcal{I}$, for each $x^* \in X^*$ and there exists $A_I \in X$ such that

$$x^*(A_I) = (H_{\mathcal{B}}) \int_I x^*(f)$$

for each x^*. A_I is the value of the *indefinite* $HP_{\mathcal{B}}$ *integral* on I and we write

$$A_I = (HP_{\mathcal{B}}) \int_I f.$$

If we define a variational version of Henstock–Pettis integral then due to Lemma 3.2 it would be equivalent to Henstock–Pettis integral.

4. Recovering the primitive and problem of differentiation

Let X be a Banach space and F be an X-valued $\mathcal{B}$-interval function, i.e., a function defined on $\mathcal{I}$. The $\mathcal{B}$*-derivative* of F at a point g is

$$D_{\mathcal{B}}F(g) := \lim_{n\to\infty} \frac{F(K_n(g))}{\mu(K_n(g))}$$

if this limit exists.

We define also a weak derivative of F. Namely, we say that $wD_{\mathcal{B}}F(t) \in X$ is the *weak* $\mathcal{B}$*-derivative* of F at g, with respect to the basis $\mathcal{B}$, if for any $x^* \in X^*$

$$\lim_{n\to\infty} \frac{x^*(F(K_n(g))}{\mu(K_n(g))} = x^*(wD_{\mathcal{B}}F(g)).$$

In this case we say that F is *weakly* $\mathcal{B}$*-differentiable* at g.

Let F be an additive $\mathcal{B}$-interval function and E an arbitrary subset of G. For a fixed gage ν, we set

$$Var(E, F, \nu) := \sup_{\pi \subset \beta_\nu[E]} \sum \|F(I)\|.$$

We put also

$$V_F(E) = V(E, F, \mathcal{B}) := \inf_{\nu} Var(E, F, \nu).$$

The extended real-valued set function $V_F(\cdot)$ is called *variational measure* generated by F, with respect to the basis $\mathcal{B}$. Following the proof given in [15] for the interval bases in $\mathbb{R}$ it is possible to show that $V_F(\cdot)$ is an outer measure.

First we prove the following proposition.

Proposition 4.1. *Let an additive function $F : \mathcal{I} \to X$ be $\mathcal{B}$-differentiable on G outside a set E such that $V_F(E) = 0$. Then the function*

$$f(x) := \begin{cases} D_{\mathcal{B}}F(x) & \text{if it exists,} \\ 0 & \text{if } x \in E \end{cases}$$

is $VH_{\mathcal{B}}$-integrable on G and F is its indefinite $VH_{\mathcal{B}}$-integral.

Proof. Fix $\varepsilon > 0$ and according to definition of variational measure find $\nu : E \to X$ such that for any $\beta_\nu[E]$-partition π_1 we have $\sum_{\pi_1} \|F(I)\| < \frac{\varepsilon}{2}$. For each point g where F is $\mathcal{B}$-differentiable find $\nu(g)$ such that

$$\big\|F(K_n(g)) - f(g)\mu(K_n(g))\big\| < \frac{\varepsilon}{2}\mu(K_n(g))$$

if $n \geq \nu(g)$. In this way a gage ν is defined at each point of G. Then for any β_ν-partition π of G we have we get

$$\sum_{(I,g)\in\pi} \big\|F(K_n(g)) - f(g)\mu(K_n(g))\big\| \leq \sum_{(I,g)\in\pi,\, g\notin E} \|f(g)\mu(I) - F(I)\| +$$
$$+ \sum_{(I,g)\in\pi,\, g\in E} \|f(t)\mu(I) - F(I)\| \leq \frac{\varepsilon}{2} \sum_{(I,g)\in\pi,\, g\notin E} \mu(I) + \frac{\varepsilon}{2} \leq \varepsilon.$$

Thus F is indefinite $VH_{\mathcal{B}}$-integral of f. In particular

$$F(G) = (VH_{\mathcal{B}}) \int_G f. \qquad \blacksquare$$

We formulate also a weak version of the above proposition:

Proposition 4.2. *Let an additive function $F : \mathcal{I} \to X$ be $w\mathcal{B}$-differentiable on G outside a set E such that $V_{x^*F}(E) = 0$ for any $x^* \in X^*$. Then the function*

$$f(x) := \begin{cases} wD_{\mathcal{B}}F(x) & \text{if it exists,} \\ 0 & \text{if } x \in E \end{cases}$$

is $HP_{\mathcal{B}}$-integrable on G and F is its indefinite $HP_{\mathcal{B}}$-integral.

We get the next two theorems as corollaries of the above propositions.

Theorem 4.3. *Let an additive function $F : \mathcal{I} \to X$ be $\mathcal{B}$-differentiable everywhere on G outside of a set E with $\mu(E) = 0$, and*

$$\overline{\lim}_{n\to\infty} \frac{\|F(K_n(g))\|}{\mu(K_n(g))} < \infty \tag{4.1}$$

everywhere on E except on a countable set M where F is $\mathcal{B}$-continuous. Then the function

$$f(x) := \begin{cases} D_{\mathcal{B}}F(x) & \text{if it exists,} \\ 0 & \text{if } x \in E \end{cases}$$

is $VH_{\mathcal{B}}$-integrable on G and F is its indefinite $VH_{\mathcal{B}}$-integral.

Proof. To apply Proposition 4.1 we need to prove only that $V_F(E) = 0$.

We note first that $\mathcal{B}$-continuity of F at each point of M and the fact that $V_F(\cdot)$ is an outer measure imply $V_F(M) = 0$. Now let

$$H := E \setminus M = \bigcup_{m\in\mathbb{N}} H_m$$

where

$$H_m := \left\{\xi \in E \setminus M : \overline{\lim} \frac{\|F(K_n(g))\|}{\mu(K_n(g))} < m\right\}.$$

As $\mu(H_m) = 0$ and the measure μ is outer regular, there exists, for any $\varepsilon > 0$, an open set $O_m \supset H_m$ such that $\mu(O_m) < \frac{\varepsilon}{m}$. Then for any $g \in H_m$ there exists $\nu(g) \in \mathbb{N}$ such that for any $n \geq \nu(g)$ we have

$$K_{n(g)} \subset O_m, \quad \text{and} \quad \|F(K_{n(g)})\| \leq m\mu(K_{n(g)}). \tag{4.2}$$

By this we have defined a gage ν on H_m for each m. Now taking any $\beta_\nu[H_m]$-partition π and using (4.2) we compute:

$$\sum_{(I,x)\in\pi} \|F(I)\| \leq m \sum_{(I,x)\in\pi} \mu(I) \leq m\mu(O_m) \leq m \cdot \frac{\varepsilon}{m} = \varepsilon.$$

Since ε is arbitrary we get $V_F(H_m) = 0$. Then, once again using the property of an outer measure we obtain

$$V_F(E) \leq V_F(M) + \sum_m V_F(H_m) = 0. \qquad \blacksquare$$

In the weak version of the previous theorem we need not use variational type integral by the reason mentioned in the previous section.

Theorem 4.4. *Let an additive function $F : \mathcal{I} \to X$ be weakly $\mathcal{B}$-differentiable everywhere on G outside of a set E with $\mu(E) = 0$, and for any $x^* \in X^*$*

$$\overline{\lim}_{n\to\infty} \frac{\|x^* F(K_n(g))\|}{\mu(K_n(g))} < \infty$$

everywhere on E except on a countable set M where F is $\mathcal{B}$-continuous. Then the function

$$f(x) := \begin{cases} wD_{\mathcal{B}}F(x) & \text{if it exists,} \\ 0 & \text{if } x \in E \end{cases}$$

is $HP_{\mathcal{B}}$-integrable on G and F is its indefinite $HP_{\mathcal{B}}$-integral.

It was proved in [13] that in the scalar-valued case the indefinite $H_{\mathcal{B}}$-integral of any $H_{\mathcal{B}}$-integrable function is $\mathcal{B}$-differentiable everywhere on G and $D_{\mathcal{B}}F(g) = f(g)$ almost everywhere. In a similar way this property can be proved in a case when a the range of a function is of finite dimension. Although, as we show below, this property of differentiability fails to be true for any infinite-dimensional Banach space, it is still true for $VH_{\mathcal{B}}$-integral in the case of any Banach space:

Theorem 4.5. *If a function $f : G \to X$ is $VH_{\mathcal{B}}$-integrable on G then the indefinite $VH_{\mathcal{B}}$-integral $F(K) = (VH)_{\mathcal{B}} \int_K f$ as an additive function on the set of all $\mathcal{B}$-intervals is $\mathcal{B}$-differentiable almost everywhere on G and*

$$D_{\mathcal{B}}F(g) = f(g) \quad \text{a.e.} \quad \text{on} \quad L. \tag{4.3}$$

Proof. The proof follows the lines of the proof in [13, Theorem 3.1] where a reference to Kolmogorov–Henstock lemma should be replaced with a reference to the inequality from Definition 3.2. ■

The next theorem shows that this result can not be extended to the case of $H_{\mathcal{B}}$-integral.

Theorem 4.6. *For any infinite-dimensional Banach space X, there exists a $H_{\mathcal{B}}$-integrable on G function $f : G \to X$ with the indefinite $H_{\mathcal{B}}$-integral which is $\mathcal{B}$-differentiable nowhere on G.*

Proof. We use some elements of construction in [4] and some ideas from [9]. First we define inductively a collection $\{A_k^n : n = 0, 1, \ldots ; k = 1, \ldots , m_n\}$ of disjoint nowhere dense sets of strictly positive measure such that $A_k^n \subset K_n^k$.

By Proposition 2.1 there is a basic sequence $\{x_n\}$ in X with basis constant $B \geq 1$. Take a blocking F_n of the basis with each subspace F_n of large enough dimension to find by Proposition 2.2 a m_n-dimensional subspace E_n of F_n such that the Banach–Mazur distance between E_n and $l_2^{m_n}$ is less than 2. Next find operators $T_n : l_2^{m_n} \to E_n$ such that $\|T_n\| \le 2$ and $\|T_n^{-1}\| = 1$. Let $u_k^n : k = 1, \dots, m_n$ be the standard unit vectors of $l_2^{m_n}$ and let $e_k^n = T_n u_k^n$.

Define a function $f : G \to X$ by

$$f(g) = \sum_{j=1}^{\infty} m_j^{-\frac{3}{4}} \sum_{k=1}^{m_j} \frac{\chi_{A_k^j}(g)}{\mu(A_k^j)} e_k^j.$$

The series here is obviously convergent as at each point g not more than one term is not equal to zero. We consider partial sums of this series:

$$f_n(g) = \sum_{j=1}^{n} m_j^{-\frac{3}{4}} \sum_{k=1}^{m_j} \frac{\chi_{A_k^j}(g)}{\mu(A_k^j)} e_k^j.$$

This simple functions are Bochner integrable and so are also $H_{\mathcal{B}}$-integrable with $H_{\mathcal{B}}$-integral

$$\int_G f_n(g) = \sum_{j=1}^{n} m_j^{-\frac{3}{4}} \sum_{k=1}^{m_j} e_k^j.$$

For a given $\varepsilon > 0$ and for any n we can find ν_n such that for any β_{ν_n}-partition π_n of G we have

$$\left\| \sum_{(I,g)\in\pi_n} f_n(g)\mu(I) - \int_G f_n \right\| < \varepsilon 2^{-(n+2)}. \tag{4.4}$$

The series

$$\sum_{j=1}^{\infty} m_j^{-\frac{3}{4}} \sum_{k=1}^{m_j} e_k^j$$

is convergent to some element $A \in X$ because

$$\left\| \sum_{k=1}^{m_n} e_k^n \right\|_X \le \|T_n\| \cdot \left\| \sum_{k=1}^{m_n} u_k^n \right\|_{l_2} \le 2 m_n^{\frac{1}{2}}.$$

So for the chosen ε we can find N such that for any $n \ge N$

$$\left\| \sum_{j=n}^{\infty} m_j^{-\frac{3}{4}} \sum_{k=1}^{m_j} e_k^j \right\| \le 2 \sum_{j=n}^{\infty} m_j^{-\frac{1}{4}} < \frac{\varepsilon}{4}. \tag{4.5}$$

Hence for $n \ge N$

$$\left\| \int_G f_n - A \right\| < \frac{\varepsilon}{4}. \tag{4.6}$$

We define the following sequence of sets. We put

$$E_N = \bigcup_{n=1}^{N}\bigcup_{k=1}^{m_n} A_k^n \cup (G \setminus \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{m_n} A_k^n)$$

and

$$E_n = \bigcup_{k=1}^{m_n} A_k^n \text{ for } n > N.$$

Note that $G = E_N \bigcup (\bigcup_{n>N} E_n)$.

To prove that f is $H_{mathcalB}$-integrable with the integral value A we define a gage ν corresponding to the given ε. We put

$$\nu(g) = \max\{\nu_1(g), \dots, \nu_N(g)\} \text{ if } g \in E_N,$$

$$\nu(g) = \max\{\nu_1(g), \dots, \nu_n(g)\} \text{ if } g \in E_n,\ n > N.$$

Let $\{(g_i, I_i)\}_{i=1}^{p}$ be a fixed β_ν-partition of G and let $n_0 = \max_i\{n : g_i \in E_n, n \geq$ $\geq N\}$. Note that if $g \in E_n$ then $f(g) = f_n(g)$. So $f(g_i) = f_{n_0}(g_i)$ for any i. We get

$$\sum_{i=1}^{p}\left\| f(g_i)\mu(I_i) - A\right\| \leq \sum_{i=1}^{p}\left\| f_{n_0}(g_i)\mu(I_i) - \int_G f_{n_0}\right\| + \left\|\int_G f_{n_0} - A\right\|.$$

By (4.6)

$$\left\|\int_G f_{n_0} - A\right\| < \frac{\varepsilon}{4}. \tag{4.7}$$

We have

$$\left\|\sum_{i=1}^{p} f_{n_0}(g_i)\mu(I_i) - \int_G f_{n_0}\right\| \leq \left\|\sum_{j=N}^{n_0}\sum_{i:g_i\in E_j}\left(f_{n_0}(g_i)\mu(I_i) - \int_{I_i} f_{n_0}\right)\right\| =$$

$$= \left\|\sum_{j=N}^{n_0}\sum_{i:g_i\in E_j}\left(f_j(g_i)\mu(I_i) - \int_{I_i} f_j\right)\right\| + \left\|\sum_{j=N}^{n_0-1}\sum_{i:g_i\in E_j}\left(\int_{I_i} f_j - \int_{I_i} f_{n_0}\right)\right\|.$$

Using (4.4) and Lemma 3.1 we estimate

$$\left\|\sum_{j=N}^{n_0}\sum_{i:g_i\in E_j}\left(f_j(g_i)\mu(I_i) - \int_{I_i} f_j\right)\right\| \leq \frac{\varepsilon}{2^{N+1}} < \frac{\varepsilon}{4}. \tag{4.8}$$

Note that if i is such that $g_i \in E_j$ then

$$\int_{I_i} (f_{n_0} - f_j) = \sum_{k=j+1}^{n_0} m_k^{-\frac{3}{4}} \sum_{s=1}^{m_k} \frac{\mu(A_s^k \cap I_i)}{\mu(A_s^k)} e_s^k.$$

Therefore,

$$\left\| \sum_{j=N}^{n_0-1} \sum_{i:g_i \in E_j} \left(\int_{I_i} f_{n_0} - \int_{I_i} f_j \right) \right\| = \left\| \sum_{j=N}^{n_0-1} \sum_{i:g_i \in E_j} \sum_{k=j+1}^{n_0} m_k^{-\frac{3}{4}} \sum_{s=1}^{m_k} \frac{\mu(A_s^k \cap I_i)}{\mu(A_s^k)} e_s^k \right\| \le$$

$$\le \sum_{k=N+1}^{n_0} m_k^{-\frac{3}{4}} \left\| \sum_{s=1}^{m_k} \sum_{j=N+1}^{k-1} \sum_{i:g_i \in E_j} \frac{\mu(A_s^k \cap I_i)}{\mu(A_s^k)} e_s^k \right\| \le$$

$$\le \sum_{k=N+1}^{n_0} 2 m_k^{-\frac{3}{4}} \left\| \sum_{s=1}^{m_k} \sum_{j=N+1}^{k-1} \sum_{i:g_i \in E_j} \frac{\mu(A_s^k \cap I_i)}{\mu(A_s^k)} u_s^k \right\|_{l_2}.$$

It is clear that

$$\left\| \sum_{s=1}^{m_k} \sum_{j=N+1}^{k-1} \sum_{i:g_i \in E_j} \frac{\mu(A_s^k \cap I_i)}{\mu(A_s^k)} u_s^k \right\| \le m_k^{\frac{1}{2}}.$$

So by (4.5)

$$\left\| \sum_{j=N}^{n_0-1} \sum_{i:g_i \in E_j} \left(\int_{I_i} f_{n_0} - \int_{I_i} f_j \right) \right\| \le \frac{\varepsilon}{2}. \tag{4.9}$$

Summing up the estimates (4.7), (4.8), and (4.9) we get

$$\sum_{i=1}^{p} \| f(g_i)\mu(I_i) - A \| < \varepsilon.$$

To show that the indefinite $H_{\mathcal{B}}$-integral of f is not $\mathcal{B}$-differentiable everywhere on G we fix a point g and a $\mathcal{B}$-interval $K_n(g)$ of rank n and estimate $\int_{K_n} f$. Let $K_n(g) = K_n^i$ for some i, $1 \le i \le m_n$.

Since the basis constant of the chosen basic sequence $\{x_n\}$ in X is B and since $||T_n^{-1}|| = 1$ we have

$$2B \left\| \int_{K_n(g)} f \right\| \ge \left\| \int_{K_n(g)} m_n^{-\frac{3}{4}} \sum_{p=1}^{m_n} \frac{\chi_{A_p^n}}{\mu(A_p^n)} e_p^n \right\|_X \ge$$

$$\ge \left\| m_n^{-\frac{3}{4}} \int_{K_n(g)} \frac{\chi_{A_i^n}}{\mu(A_i^n)} u_i^n \right\|_{l_2^{m_n}} = m_n^{-\frac{3}{4}}.$$

Then

$$\frac{\left\| \int\limits_{K_n(g)} f \right\|}{\mu(K_n(g))} \geq \frac{m_n^{\frac{1}{4}}}{2B}. \tag{4.10}$$

So the limit of this ratio tend to infinity when $n \to \infty$ for each $g \in G$. This proves the theorem. ■

Remark 4.1. Note that the inequality (4.10) implies a stronger result. In fact the sequence of the ratios

$$\frac{\int\limits_{K_n(g)} f}{\mu(K_n(g))}$$

is also weakly unbounded. So in the constructed example the indefinite $H_{\mathcal{B}}$-integral is also weakly $\mathcal{B}$-differentiable nowhere on G.

5. Application to the series with respect to the characters

We associate with the series (2.5) a function F defined on each coset K_n by

$$F(K_n) := \int\limits_{K_n} S_n(g) d\mu \tag{5.1}$$

where S_n are partial sums given by (2.6). Similar to the scalar case (see [13]) it is easy to check that F is an additive function on the family $\mathcal{I}$ of all $\mathcal{B}$-intervals. As it was in the case of Haar and Walsh series (see [10] and [11]) we call this function a *quasi-measure* associated with the series (2.5).

Properties of characters, described in Section 2, imply that the sum S_n, defined by (2.6), is constant on each K_n. Then by (5.1) we have

$$S_n(g) = \frac{F(K_n(g))}{\mu(K_n(g))}. \tag{5.2}$$

The following three lemmas are immediate consequences of the equality (5.2).

Lemma 5.1. *If the series (2.5) converges at some point $g \in G$ to a value $f(g)$ then the associated quasi-measure F is $\mathcal{B}$-differentiable at g and $D_{\mathcal{B}}F(g) = = f(g)$.*

Lemma 5.2. *If the series (2.5) converges weakly at some point $g \in G$ to a value $f(g)$ then the associated function F is $w\mathcal{B}$-differentiable at g and $wD_{\mathcal{B}}F(g) = = f(g)$.*

Lemma 5.3. *If the partial sums (2.6) satisfy at a point g the condition*

$$S_n(g) = o\left(\frac{1}{\mu_G(K_n(g))}\right) \tag{5.3}$$

then the associated function F is $\mathcal{B}_G$-continuous at the point g.

The next lemma gives a sufficient condition for the assumption (5.3) of the previous lemma to hold.

Lemma 5.4. *Suppose that the coefficients $\{a_\gamma\}$ of a series (2.5) satisfy the condition*

$$\max_{\gamma \in \Gamma_{(n+1)} \setminus \Gamma_n} ||a_\gamma||_X \to 0 \quad \text{if} \quad n \to \infty, \tag{5.4}$$

then (5.3) holds for partial sums $S_n(g)$ at each point $g \in G$.

Proof. The proof follows the lines of the proof in [13, Lemma 4.3] given for the scalar case. ■

The following statement is essential for establishing that a given series with respect to characters is the Fourier series in the sense of some general integral.

Theorem 5.1. *Let some integration process $\mathcal{A}$ be given which produces an integral additive on $\mathcal{I}$. Let a function F defined on $\mathcal{I}$ be the quasi-measure associated with the series (2.5). Then this series is the Fourier series of an $\mathcal{A}$-integrable function f iff $F(K) = (\mathcal{A})\int_K f$ for any $K \in \mathcal{I}$.*

Proof. Necessity follows from the following known formula for Dirichlet kernel of the considered system of characters (see [1]):

$$D_n(g) := \sum_{\gamma \in \Gamma_n} a_\gamma \gamma(g) = \begin{cases} m_n & \text{if } g \in G_n, \\ 0 & \text{if } g \notin G_n. \end{cases}$$

To prove the sufficiency, suppose that $F(K) = (\mathcal{A})\int_K f$ for any $\mathcal{B}$-interval K. Fix a character γ and choose n such that $\gamma \in \Gamma_n$. For this n the group G can be represented as a finite union $G = \cup_i K_n^i$. As we have mentioned in Section 2, the character γ and also the sum S_n are constant on each K_i^n. Let $S_n(g) = s_i$ and $\gamma(g) = \xi_i$ if $g \in K_n^i$. Then by (5.2) we get

$$a_\gamma = \int_G S_n \gamma d\mu = \sum_i s_i \xi_i \mu(K_n^i) = \sum_i \xi_i F(K_n^i) = \sum_i \xi_i \ (\mathcal{A})\int_{K_n^i} f d\mu =$$

$$= \sum_i (\mathcal{A})\int_{K_n^i} f\gamma d\mu = (\mathcal{A})\int_G f\gamma d\mu. \qquad ■$$

In view of (5.2), Lemmas 5.1–5.4 and Theorem 5.1, in order to solve the coefficient problem, it is enough to show that quasi-measure associated with the series (2.5) is the indefinite integral of its derivative (strong or weak, respectively).

By this we reduce the problem of recovering the coefficients to the one of recovering the primitive and we can use a corresponding theorem on primitives in Section 3.

Theorem 5.2. *Suppose that the partial sums (2.6) of the series (2.5) converge to a function f everywhere on G outside of a set E with $\mu(E) = 0$, and*

$$\overline{\lim}_{n\to\infty}||S_n(g)|| < \infty$$

everywhere on E except on a countable set M where (5.3) holds. Then f is $VH_{\mathcal{B}}$-integrable and (2.5) is the $VH_{\mathcal{B}}$-Fourier series of f.

Proof. By Lemma 5.1 the function F, defined by (5.1), is $\mathcal{B}$-differentiable at g with $D_G F(g) = f(g)$ at any point g at which the series (2.5) converges to $f(g)$. By (5.2) the inequality (4.1) is satisfied everywhere on E except on the set M, where by Lemma 5.3 F is $\mathcal{B}$-continuous. Therefore, by Theorem 4.3 f is $VH_{\mathcal{B}}$-integrable and F is its $VH_{\mathcal{B}}$-integral. Finally using Theorem 5.1, applied to $VH_{\mathcal{B}}$-integral, we complete the proof. ■

In the same way, using Theorem 5.1 for the case of $HP_{\mathcal{B}}$-integral we get

Theorem 5.3. *Suppose that the partial sums (2.6) of the series (2.5) converge weakly to a function f everywhere on G outside of a set E with $\mu(E) = 0$, and for any $x^* \in X^*$*

$$\overline{\lim}_{n\to\infty}|x^* S_n(g)|| < \infty$$

everywhere on E except on a countable set M where (5.3) holds. Then f is $HP_{\mathcal{B}}$-integrable and (2.5) is the $HP_{\mathcal{B}}$-Fourier series of f.

Remark 5.1. In view of Lemma 5.4 we can replace the condition (5.3) by the condition (5.4) in the assumption of the above theorems.

The following theorem is a particular case of Theorem 5.2 (or Theorem 5.3, respectively).

Theorem 5.4. *Suppose that the partial sums (2.6) of the series (2.5) converge (converge weakly) to a function f everywhere on G. Then f is $VH_{\mathcal{B}}$-integrable (resp. $HP_{\mathcal{B}}$-integrable) on G and the series (2.5) is the $VH_{\mathcal{B}}$-Fourier series (resp. $HP_{\mathcal{B}}$-Fourier series) of f.*

Now we consider the problem of convergence of Fourier series in the sense of $VH_{\mathcal{B}}$-integral and $H_{\mathcal{B}}$-integral. The partial sums $S_n(f, g)$ of Fourier series, with respect to the system Γ, of a function $f : G \to X$ integrable in the sense of these integrals can be represented, according to Theorem 5.1 and formula (5.2), as

$$S_n(f,g) = \frac{1}{\mu(K_n(g))} \int_{K_n(g)} f. \tag{5.5}$$

From this equality together with differentiability property of the indefinite $VH_{\mathcal{B}}$-integral (see Theorem 4.5) follows

Theorem 5.5. *The partial sums $S_n(f,g)$ of the $VH_{\mathcal{B}}$-Fourier series of a $VH_{\mathcal{B}}$-integrable on G function f are convergent to f almost everywhere on G.*

At the same time such a theorem fails to be true for $H_{\mathcal{B}}$-Fourier series.

Namely, Theorem 4.6 implies that for any infinite-dimensional Banach space there exists a $H_{\mathcal{B}}$-integrable function with values in this space such that its $H_{\mathcal{B}}$-Fourier series diverges everywhere. In fact the estimate (4.10) and the equality (5.5) gives us also a possible rate of growth of the partial sums of the divergent series.

Theorem 5.6. *For any infinite-dimensional Banach space X there exists $H_{\mathcal{B}}$-integrable function $f : G \to X$ such that partial sums of its $H_{\mathcal{B}}$-Fourier series with respect to the system Γ satisfy the estimate $\|S_n(f,g)\| \geq m_n^{\frac{1}{4}}$ for each $g \in G$.*

But such a rate of growth can not be made arbitrary large for the whole class of infinite-dimensional Banach spaces. For example it can be deduced from [10] that for a Pettis-integrable function f taking values in any infinite-dimensional Banach space with 2-Orlicz property (see definition in [3]) partial sums of its Fourier series with respect to the Walsh system, which is the systems of characters of a particular case of a zero-dimensional group, satisfy the relation $\|S_n(f,g)\| = o(2^{\frac{1}{2}n})$.

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