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# AN ELEMENTARY METHOD OF SOLVING FUNCTIONAL EQUATIONS

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**Abstract.** In the present paper we prove in an elementary way that if for a fixed $\lambda \in (0,1)$ the functions $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ satisfy the equation

$$\frac{f(b)-f(a)}{b-a} = g\left(\lambda a + (1-\lambda)b\right)$$

for all $b > a$ then $f$ is a quadratic polynomial, and $g = f'$. Moreover, if $\lambda \neq \frac{1}{2}$, then $f$ is a linear polynomial and $g = f'$. This result is obtained with no regularity assumptions on $f$ or $g$ and generalizes a theorem from [4].

## 1. Introduction

Roman Ger (cf. [5]) has drawn our attention to the paper [4] where the authors ask the following questions:

**Question 1.** Which diferentiable functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfy

$$\frac{f(b)-f(a)}{b-a} = f'\left(\frac{a+b}{2}\right) \tag{1.1}$$

for all $b > a$?

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**Question 2.** Let us fix a $\lambda \in (0,1)$. Which differentiable functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfy

$$\frac{f(b) - f(a)}{b - a} = f'(\lambda a + (1 - \lambda)b) \tag{1.2}$$

for all $b > a$?

The following theorem is proved in [4].

**Theorem.** *Let us fix a $\lambda \in (0,1)$, and suppose that a (differentiable) function $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the condition*

$$\frac{f(b) - f(a)}{b - a} = f'(\lambda a + (1 - \lambda)b)$$

*for all $a < b$. Then $f$ is a quadratic polynomial. If $\lambda \neq \frac{1}{2}$, then $f$ is a linear polynomial.*

In the proof the authors use **triple** differentiability of $f$. It has been known, at least since J. Aczél's result (cf. [1]) that **no regularity** is needed to solve even a more general equation than (1.1). Indeed, Aczél has shown that

$$\frac{f(b) - f(a)}{b - a} = g(a + b), \tag{1.3}$$

has a unique solution and this is of the form

- $f(x) = \alpha x^2 + \beta x + \gamma$
- $g(x) = \alpha x + \beta$

for some constants $\alpha$, $\beta$ and $\gamma$. Later J. Aczél and M. Kuczma published a paper [2] where they proved, among others, the above result for functions $f$ and $g$ defined in an interval $I \subseteq \mathbb{R}$.

Functional equations stemming from the Mean Value Theorems have been studied by several mathematicians, let me quote J. Aczél [1], J. Aczél and M. Kuczma [2], S. Haruki [6], B. Koclęga-Kulpa and T. Szostok [7], [8], [9] and [10], B. Koclęga-Kulpa, T. Szostok and Sz. Wąsowicz [11] and [12], A. Lisak and M. S. [13], I. Pawlikowska [15], T. Riedel and M. S. [16], M. S. [17] and [18], T. Szostok [23], and quite recent papers by Z. M. Balogh, O. O. Ibrogimov and B. S. Mityagin [3], also M. Schwarzenberger [22], and above all by P. K. Sahoo and T. Riedel in the book [20]. It is astonishing that the authors were unaware of these items, especially the last one.

Let us observe that (1.2) may be written in more general form

$$\frac{f(b) - f(a)}{b - a} = g(\lambda a + (1 - \lambda)b) \tag{1.4}$$

for all $a < b$. Here $f$ and $g$ are arbitrary functions mapping $\mathbb{R}$ into itself. Since the left-hand side is symmetric in $a$, $b$ we infer that

$$\frac{f(b)-f(a)}{b-a} = g\left((1-\lambda)a + \lambda b\right) \tag{1.5}$$

for all $b < a$. Denoting

$$m_\lambda(a,b) = \begin{cases} \lambda a + (1-\lambda)b & \text{if } a < b \\ (1-\lambda)a + \lambda b & \text{if } a \geq b, \end{cases} \tag{1.6}$$

we can write (1.4) and (1.5) as

$$\frac{f(b)-f(a)}{b-a} = g\left(m_\lambda(a,b)\right)$$

for all $a$, $b$, $a \neq b$, or, after multiplying both sides by $b-a$ as

$$f(b) - f(a) = g\left(m_\lambda(a,b)\right)(b-a) \tag{1.7}$$

for all $a$, $b \in \mathbb{R}$.

Let us substitute $x := m_\lambda(a,b)$ and $y := b-a$. This substitution is an automorphism of $\mathbb{R}^2$ onto itself. Moreover, we have

$$\begin{cases} a = x - (1-\lambda)y \\ b = x + \lambda y \end{cases} \tag{1.8}$$

if $y > 0$, and

$$\begin{cases} a = x - \lambda y \\ b = x + (1-\lambda)y \end{cases} \tag{1.9}$$

if $y \leq 0$. Now, substituting (1.8) or (1.9) into (1.7) we obtain

$$g(x)y = \begin{cases} f(x+\lambda y) - f(x-(1-\lambda)y) & \text{if } y > 0 \\ f(x+(1-\lambda)y) - f(x-\lambda y) & \text{if } y \leq 0. \end{cases} \tag{1.10}$$

It is the above equation that we are going to solve.

### 1.1. Solution of (1.10)

Let us introduce the following operator acting in the space of functions of two variables. For every function $\varphi : (x,y) \longmapsto \varphi(x,y)$, and every pair $(u,v)$ we put

$$\Gamma_{(u,v)}\varphi(x,y) := \varphi(x+u, y+v) - \varphi(x,y),$$

for $(x,y) \in \mathrm{Dom}\varphi$ fulfilling $(x+u, y+v) \in \mathrm{Dom}\varphi$ as well. Observe that in particular we have for $\varphi(x,y) = g(x)y$ the following

$$\begin{aligned}\Gamma_{(u,v)} g(x)y &= g(x+u)(y+v) - g(x)y = \\ &= [g(x+u) - g(x)]\, y + g(x+u)v = \\ &= [\Delta_u g(x)]\, y + g(x+u)v,\end{aligned} \tag{1.11}$$

where $\Delta$ is the Fréchet difference operator. For arbitrary $(x,y) \in \mathbb{R}^2$ let us choose arbitrarily $u_1 \in \mathbb{R}$, and select a $v_1 \in \mathbb{R}$ so that $u_1 + \lambda v_1 = 0$ (resp. $u_1 + (1-\lambda)v_1 = 0$) depending on whether $y > 0$ or $y \le 0$. Applying $\Gamma_{(u_1,v_1)}$ to both sides of (1.10) we obtain

$$[\Delta_{u_1} g(x)]\, y + g(x+u_1)v_1 = \begin{cases} -\Gamma_{(u_1,v_1)} f(x-(1-\lambda)y) & \text{if } y > 0 \\ -\Gamma_{(u_1,v_1)} f(x - \lambda y) & \text{if } y \le 0. \end{cases} \tag{1.12}$$

Now, let choose arbitrarily $u_2 \in \mathbb{R}$ and select a $v_2 \in \mathbb{R}$ so that $u_2 - (1-\lambda)v_2 = 0$ (resp. $u_2 - \lambda v_2 = 0$) depending on whether $y > 0$ or $y \le 0$. Applying $\Gamma_{(u_2,v_2)}$ to both sides of (1.12) we obtain

$$[\Delta_{u_2}\Delta_{u_1} g(x)]\, y + \psi(x, u_1, u_2) = 0, \tag{1.13}$$

where $\psi(x,u_1,u_2) = \Delta_{u_1} g(x+u_2)v_2 + \Delta_{u_2} g(x+u_1)v_1$ does not depend on $y$. The left-hand side of (1.13) is a polynomial in $y$, so we have

$$\Delta_{u_2}\Delta_{u_1} g(x) = 0,$$

for every $x, u_1, u_2 \in \mathbb{R}$. But this means (cf. eg. [21]) that there exist a constant $\beta$ and an additive function $A_1$ such that

$$g(x) = \beta + A_1(x) \tag{1.14}$$

for every $x \in \mathbb{R}$. From (1.7), putting $a = 0$ and using equalities (1.6) we obtain

$$f(b) = f(0) + \begin{cases} g((1-\lambda)b)b & \text{if } 0 < b \\ g(\lambda b)b & \text{if } 0 \ge b. \end{cases} \tag{1.15}$$

We insert the general forms (1.14) and (1.15) into (1.10) to obtain among others that

$$\begin{aligned}A_1(x)y + \beta(y) = {} & [\beta + A_1((1-\lambda)(x+\lambda y))]\,(x+\lambda y) - \\ & - [\beta + A_1((1-\lambda)(x-(1-\lambda)y))]\,(x-(1-\lambda)y)\end{aligned} \tag{1.16}$$

for all $x > 0$, $y \in \left(0, \frac{x}{1-\lambda}\right)$. Rearranging the terms in (1.16), and comparing those containing $x$ and $y$ we obtain in particular the equality

$$A_1(\lambda x)y = A_1((1-\lambda)y)x$$

for all $x > 0$ and $y \in \left(0, \frac{x}{1-\lambda}\right)$. Diving both sides by $xy$ we get

$$\frac{A_1(\lambda x)}{x} = \frac{A_1((1-\lambda)y)}{y}$$

or

$$\lambda \frac{A_1(\lambda x)}{\lambda x} = (1-\lambda)\frac{A_1((1-\lambda)y)}{(1-\lambda)y} \tag{1.17}$$

for all $x > 0$ and $y \in \left(0, \frac{x}{1-\lambda}\right)$. Fix $x > 0$ and define $\alpha := \frac{A_1(\lambda x)}{\lambda x}$. Putting $z := (1-\lambda)y$ we get from (1.17)

$$(1-\lambda)A_1(z) = \lambda \alpha z \tag{1.18}$$

for all $z \in (0, x)$. Thus $A_1$ is continuous on an interval $(0, x)$, and being additive it has to be so on the whole $\mathbb{R}$. From (1.17) we get $\lambda\alpha = (1-\lambda)\alpha$ or

$$(2\lambda - 1)\alpha = 0.$$

Thus either $\lambda = \frac{1}{2}$ or $\alpha = 0$. In the case $\lambda = \frac{1}{2}$ we get from (1.17) that $\frac{A_1(x)}{x} = \frac{A_1(y)}{y}$ for all $y \in (0, 2x)$ and the linearity of $A_1$ easily follows.

In other words we proved that if $(f, g)$ yields a solution to (1.10) then

$$g(x) = \begin{cases} \beta & \text{if } \lambda \neq \frac{1}{2} \\ \beta + \alpha x & \text{if } \lambda = \frac{1}{2}. \end{cases} \tag{1.19}$$

Now, going back to formula (1.15) we see that (with $\gamma = f(0)$)

$$f(x) = \begin{cases} \gamma + \beta x & \text{if } \lambda \neq \frac{1}{2} \\ \gamma + \beta x + \frac{1}{2}\alpha x^2 & \text{if } \lambda = \frac{1}{2}. \end{cases} \tag{1.20}$$

We summarize the results of the present section in

**Theorem 1.1.** *Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be functions and let $\lambda \in (0, 1)$. The pair $(f, g)$ satisfies the equation*

$$\frac{f(b) - f(a)}{b - a} = g\left(\lambda a + (1-\lambda)b\right)$$

*for all $a < b$ if, and only if, $f$ and $g$ are given by (1.20) and (1.19), respectively. In particular, $g = f'$.*

## 2. Remarks

**Remark 2.1.** We do not assume any regularity of $f$ or $g$. However, some regularity is "tacitly" assumed. Namely, if we considered (1.7) with $a$, $b$ belonging to a linear space $X$ over $\mathbb{R}$ then clearly $g(\lambda a + (1-\lambda)b)$ would belong to the space of additive mappings defined on $X$. Usually such a mapping needs not to be linear or continuous (if the space $X$ is endowed with a topology). In our present case ($X = \mathbb{R}$), following the authors of [4], we assume that acting on the difference $b - a$ is actually multiplying the coefficient by the argument. This means that the additive mapping $g(\lambda a + (1-\lambda)b)$ is highly regular, as highly as possible.

**Remark 2.2.** The main problem was to get (1.7) so that instead of investigating the problem for $a < b$, we have obtained an equation holding for **all** arguments from $\mathbb{R}$. This has an important meaning for the method of solving. However, in (1.7) we have now an operation $m_\lambda$ appearing in the argument of $g$. This prevents us from directly applying the general theory developed for some general equations for functions with arguments and values in Abelian groups (cf. [21], [18], [15], [13] and especially [19]). In all the mentioned papers the operations appearing in arguments of unknown functions were of the form $(x, y) \longrightarrow \mu(x) + \nu(y)$ where $\mu$ and $\nu$ are homomorphims.

**Remark 2.3.** We are going to develop further the general theory with operations not necessarily being sums of homomorphisms. The results will be published elsewhere.

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