A UNIQUENESS THEOREM FOR ENTIRE FUNCTIONS HAVING A DIRICHLET SERIES REPRESENTATION

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Abstract. Dirichlet series and their analytic properties play a central role in analytic number theory. We prove a variant of George Pólya's four point theorem, resp. Rolf Nevanlinna's five point theorem for a large class of functions representable as Dirichlet series in some right half-plane.

1. Introduction and statement of the main result

One of the most spectacular theorems in complex analysis is Rolf Nevanlinna's five point theorem [16] which claims that *two meromorphic functions sharing five different values are identical.* Hermann Weyl called it "one of the few great mathematical events of [the twentieth] century" ([23], p.8). Recall that two meromorphic functions f and g are said to *share a value* $c \in \mathbb{C} \cup \{\infty\}$ if the sets of preimages of c under f and g are identical, i.e., $f^{-1}(c) :=$ $:= \{s \in \mathbb{C} : f(s) = c\} = g^{-1}(c)$; if in this case the roots of the equations $f(s) = c$ and $g(s) = c$ have the same multiplicity, then f and g are said to share the value c *counting multiplicity (CM)*, otherwise the value is shared *ignoring multiplicity (IM)* which is in this case often not explicitly mentioned. Moreover, *if four values are shared CM, then the functions are identical or can be transformed into one another by a Möbius transform.* This result is best

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possible since $\exp(\pm s)$ share $c = 0, \pm 1, \infty$ CM. It shall be noticed that George Pólya [18] proved a forerunner of Nevanlinna's uniqueness theorem already in 1921 showing that *two entire functions of finite order sharing four complex values CM are identical.* Since entire functions share the infinite value as well, Pólya's theorem is also about five points; however, they are shared counting multiplicities and the functions are assumed to be entire and of finite order. It seems that this result is almost forgotten. It is our aim to show how successfully Pólya's ansatz can be used in some instances of recent research without the heavy machinery of Nevanlinna theory!

In the last decade, the topic of sharing values has been discussed with respect to meromorphic functions appearing in number theory. In particular, it was shown by the second named author that *two distinct normalized elements from the Selberg class sharing a complex value CM are already identical* [20, 21]. Those functions are examples of generating functions to arithmetical data, hence restricting to this class one could expect that indeed a smaller number than five values can be shared. Indeed, the number of shared complex values is zero as follows from further analytic properties to be fulfilled by elements of the extended Selberg class (e.g., representation as a Dirichlet series as well as an Euler product, a functional equation and, finally, a certain normalization which rules out that with a function $\mathcal{L}$ also a complex multiple $\lambda\mathcal{L}$ with $\lambda \neq 0, 1$ is an element). In the frame of the Selberg class it would be desirable to show that different primitive functions cannot share zero IM, since this would imply unique factorization of these arithmetically relevant functions with important consequences as, for example, the Artin conjecture on the holomorphicity for L-functions to non-abelian algebraic extensions (see M.R. Murty & V.K. Murty [14], p. 184).

In this note we shall consider another more general class of functions and derive an optimal uniqueness theorem by using only basic results from complex analysis.

More precisely, we are interested in entire functions of finite order that can be represented as a Dirichlet series in some right half-plane, that is

$$L(s; f) := \sum_{n \geq 1} f(n) n^{-s}, \tag{1.1}$$

where the coefficients are given by an arithmetical function $f : \mathbb{N} \to \mathbb{C}$. The prototypical examples are Dirichlet L-functions $L(s; \chi)$ associated with a primitive residue class character χ (e.g., the Legendre symbol $\chi(n) = (\frac{a}{q})$ for a prime q), resp. their continuation on $\mathbb{N}$ by defining $\chi(n) = 0$ for all n not relatively prime with q. We denote our functions in a similar form as those Dirichlet L-functions or more general L-functions appearing in number theory although we do not assume the existence of an Euler product as is standard for L-functions in general.

Theorem 1.1. *For $j = 1, 2$, let $L(s; f_j)$ be an entire function of finite order having a convergent Dirichlet series representation of the form* (1.1) *in some right half-plane. If $L(s; f_1)$ and $L(s; f_2)$ share two distinct complex values counting multiplicities, then they are identical.*

To exclude trivialities we assume the functions $L(s; f_j)$ in the theorem to be non-constant. Notice that any such pair of functions $L(s; f_j)$ considered as meromorphic functions also share the value ∞. Nevertheless, the statement is best possible since $L(s; f_2)$ and $L(s; f_1)$ share their zeros whenever $f_2 = \lambda f_1$, where λ is a non-zero constant. An explicit family of entire functions defined as a Dirichlet series in some right half-plane is given by

$$L(s; f_\alpha) = \sum_{n \geq 1} \exp(-\alpha n) n^{-s},$$

so $f_\alpha(n) = \exp(-\alpha n)$, where α is any positive real number. In fact, these Dirichlet series converge in the whole complex plane as follows from a classical result due to Eugène Cahen [5] (resp. [21], §2.1, or [22], §9.14), namely that *if $\sum_{n\geq 1} f(n)$ converges, then the abscissa of convergence for $\sum_{n \geq 1} f(n) n^{-s}$ is equal to*

$$\limsup_{N \to \infty} \frac{\log \left| \sum_{n \geq N} f(n) \right|}{\log N}.$$

In our example, this limit superior equals $-\infty$ and therefore $L(s; f_\alpha)$ is an entire function. It is not difficult to estimate, for $s = r \exp(i\phi)$,

$$|L(s; f_\alpha)| \leq \left\{ \sum_{1 \leq n \leq r/\log r} + \sum_{n > r/\log r} \right\} \exp(-\alpha n) n^r \ll \exp(r \log r)$$

as $r \to \infty$, hence $L(s; f_\alpha)$ is of finite order. Later we shall study this family of Dirichlet series more closely to show that the statement of Theorem 1.1 is best possible, even if the shared value is not zero. The Riemann zeta-function $\zeta(s) = L(s; 1)$, where 1 denotes the arithmetical function constant 1, is analytic except for a simple pole at $s = 1$; we shall discuss such *almost* entire function briefly also in a later section.

2. Proof of the main result

Assume that $L(s; f_1)$ and $L(s; f_2)$ share two distinct complex values a and b counting multiplicities. Sharing the value a counting multiplicities, implies that

$$\ell_a(s) = \frac{L(s; f_1) - a}{L(s; f_2) - a}$$

defines an entire function of finite order. By Jacques Hadamard's theory of entire functions,

$$\ell_a(s) = \exp P_a(s) \tag{2.1}$$

for some polynomial P_a (see Titchmarsh [22], §8.24, or Weyl [23], p. 6).

Edmund Landau [11] proved that *given a non-vanishing function D, represented by a convergent Dirichlet series $D(s) = \sum_n b(n)n^{-s}$ for some right half-plane $\mathrm{Re}\, s > \sigma_0$, then also its reciprocal obeys a Dirichlet series representation $1/D(s) = \sum_u q(u)u^{-s}$ in the same half-plane.* Actually, we do not need this result in its full strength (the convergence in the complete zero-free half-plane): since every convergent Dirichlet series is zero-free in another right half-plane by the uniqueness theorem for Dirichlet series (see Apostol [2], Chapter 11, resp. Titchmarsh [22], §9.6), it follows that $L(s; f_2) - a$ is a non-vanishing Dirichlet series for all s having sufficiently large real part, hence, by Landau's theorem, its reciprocal has a Dirichlet series representation in the same range too. Defining $\epsilon : \mathbb{N} \to \mathbb{C}$ by $\epsilon(1) = 1$ and $\epsilon(n) = 0$ for all $n > 1$ (so that $\epsilon = \mu * 1$ as Dirichlet convolution with the Möbius μ-function; see Apostol [2], Chpater 2), we have $L(s; f_j) - a = L(s; f_j - a\epsilon)$ for $j = 1, 2$, and, by Landau's theorem,

$$L(s; f_2 - a\epsilon)^{-1} = L(s; g),$$

where $(f_2 - a\epsilon) * g = \epsilon$. This yields

$$\ell_a(s) = L(s; f_1 - a\epsilon)L(s; g),$$

valid for all s with sufficiently large real part. Since Dirichlet series form a ring, the right hand side is again Dirichlet series

$$L(s; f_1 - a\epsilon)L(s; g) = L(s; (f_1 - a\epsilon) * g) = \sum_{n\geq 1} g_a(n)n^{-s},$$

say. In view of

$$\sum_{n\geq 1} g_a(n)n^{-s} = \sum_{n\geq m_a} g_a(n)n^{-s},$$

where m_a is the minimum of all $n \in \mathbb{N}$ for which $g_a(n) \neq 0$, and in comparison with (2.1) it follows that

$$\begin{aligned} P_a(s) &= \log\left(\sum_{n\geq m_a} g_a(n)n^{-s}\right) = \\ &= \log g_a(m_a)m_a^{-s} + \log\left(1 + \sum_{n>m_a} \frac{g_a(n)}{g_a(m_a)}\left(\frac{m_a}{n}\right)^s\right). \end{aligned}$$

The series on the right hand side converges for sufficiently large $\operatorname{Re} s$, however, for P_a being a polynomial the series has to be empty. Hence,

$$P_a = \log g_a(m_a)m_a^{-s} = -s\log m_a + \log g_a(m_a)$$

is either constant or linear (depending on m_a being equal to 1 or larger) and

$$\ell_a(s) = g_a(m_a)m_a^{-s}.$$

Concerning the other shared value b, we find the same way

$$\ell_b(s) := \frac{L(s; f_1) - b}{L(s; f_2) - b} = \exp P_b(s) = g_b(m_b)m_b^{-s}$$

with some constant or linear polynomial P_b. Hence

$$\begin{aligned} L(s; f_1) - a &= (L(s; f_2) - a) \cdot g_a(m_a)m_a^{-s}, \\ L(s; f_1) - b &= (L(s; f_2) - b) \cdot g_b(m_b)m_b^{-s}. \end{aligned}$$

Subtracting both equations yields

$$b - a = L(s; f_2)\Big(g_a(m_a)m_a^{-s} - g_b(m_b)m_b^{-s}\Big) + bg_b(m_b)m_b^{-s} - ag_a(m_a)m_a^{-s}.$$

If $m_a = m_b = 1$, then the latter equation simplifies to

$$b - a = L(s; f_2)(g_a(1) - g_b(1)) + bg_b(1) - ag_a(1).$$

Since the left hand side is constant, it follows again from the uniqueness of Dirichlet series representations (see Apostol [2], Chapter 11, resp. Titchmarsh [22], §9.6), that $g_a(1) = g_b(1)$, hence

$$b - a = (b - a)g,$$

where $g = g_a(1) = g_b(1)$. Obviously, this implies $g = 1$ and $L(s; f_1) = L(s; f_2)$.

If $m_a > 1 = m_b$, then

$$b - a = L(s; f_2)(g_a(m_a)m_a^{-s} - g_b(1)) + bg_b(1) - ag_a(m_a)m_a^{-s}$$

and we arrive at a contradiction by the uniqueness of Dirichlet series representations; in a similar way the case $m_a = 1 < m_b$ can be treated.

Finally, if both m_a and m_b are ≥ 2, then the Dirichlet series on the right hand side has vanishing constant term contradicting that the quantity on the left hand side is non-zero.

3. The results are best possible

Comparing with the proof of other (more restrictive) uniqueness theorems for Dirichlet series, the new tool in our reasoning is Landau's theorem [11] (see Bombieri & Ghosh [4] for an English version). This allows to prove uniqueness without using Nevanlinna theory at all. In previous proofs, for example, [20, 21] by the second named author as well as [6] by Steve Gonek et al., the characteristic function from Nevanlinna theory was used in order to bound the degree of the polynomial in Hadamard's factorization theorem.

An exception is the recent approach of Pei-Chu Hu & Bao Qin Li [13] which is elementary as well. They succeed in proving a variant of the above mentioned uniqueness theorem for the extended Selberg class. In order to show that their result is best possible they discusss the Dirichlet polynomials $L_1(s) = 1+2\cdot 4^{-s}$ and $L_2(s) = 1 + 3 \cdot 9^{-s}$ which, obviously, share the value 1. This also corrects a flaw in [20, 21] where Dirichlet polynomials were ignored. We observe that

$$\ell(s) = \frac{L_1(s) - 1}{L_2(s) - 1} = \tfrac{2}{3} \cdot \exp(s \log \tfrac{9}{4}).$$

In our opinion, this example of Dirichlet polynomials is somehow special since the shared value is not assumed at all and is as well the constant term in the series expansion. It follows from Theorem 1.1 that besides the value 1 no other complex value is shared (which could in this case also be shown by a more simple argument).

We shall provide another class of examples of Dirichlet series sharing exactly one arbitrary complex value. Given c, define arithmetical functions $f_1, f_2 : \mathbb{N} \to \mathbb{C}$ by setting

$$f_1(1) \neq c\,, \quad f_2(1) \neq c\,, \qquad \text{and} \qquad f_2(n) = \gamma f_1(n) \quad \text{for} \quad n \geq 2,$$

where

$$\gamma = \frac{f_1(1) - c}{f_2(1) - c}.$$

If the corresponding Dirichlet series $L(s; f_j)$ converge throughout $\mathbb{C}$, then

$$\ell(s) = \frac{L(s; f_1) - c}{L(s; f_2) - c} = \gamma$$

by the same reasoning as above, and, consequently, $L(s; f_1)$ and $L(s; f_2)$ share the value c. For instance, defining f_1 by $f_1(n) = \exp(-\alpha n)$ for $n \geq 2$ with some positive real number α and $c \neq \exp(-\alpha)$ yields an uncountable family of pairs of entire Dirichlet series sharing a complex value.

Finally, let us mention that the statement of the theorem is false if we drop the condition of finite order. For instance, the Dirichlet series

$$\exp\big(\pm 2^{-s}\big) = \sum_{m\geq 0} \frac{(-1)^m}{m!} 2^{-ms}$$

share the complex values $0, \pm 1$ counting multiplicities.

4. Further results involving poles and swapping values

By a similar reasoning as above it is not difficult to show that *if two entire functions $L(s; f_j)$ of finite order having a convergent Dirichlet series representation of the form (1.1) in some right half-plane and $L(s; f_1)$ is attaining a complex value a if and only if $L(s; f_2)$ is attaining another complex value b CM and vice versa, then $f_1(n) = -f_2(n)$ for $n \geq 2$ and $f_1(1) + f_2(1) = a + b$.* This follows simply by swapping the roles of the values $a \leftrightarrow b$ in the proof above. More precisely, the method of proof from the previous section leads to

$$\frac{L(s; f_1) - a}{L(s; f_2) - b} = \frac{L(s; f_1) - b}{L(s; f_2) - a} = -1.$$

Hence, $L(s; f_1) = a + b - L(s; f_2)$ from which the statement easily follows.

It is also not difficult to obtain results for functions represented by Dirichlet series having a pole at $s = 1$ of equal order and being analytic elsewhere (as some elements in the extended Selberg class) by an almost identical reasoning. In this case, of course, these functions share the value ∞ as well. A greater obstacle to consider more general functions is our assumption on meromorphicity in the whole complex plane. Emil Grosswald & Franz Josef Schnitzer [7] considered Euler products similar to the one for the Riemann zeta-function ζ, namely

$$\zeta^*(s) = \prod_{n\geq 1} (1 - q_n^{-s})^{-1},$$

where the q_ns are arbitrary real numbers satisfying $p_n \leq q_n \leq p_{n+1}$ with p_n denoting the nth prime number in ascending order. These products have a convergent Dirichlet series representation in the half-plane $\mathrm{Re}\, s > 1$; moreover Grosswald & Schnitzer showed the remarkable result that *any such function $\zeta^*(s)$ can be continued analytically to the right half-plane except for a simple pole at $s = 1$ and shares there the value zero CM with $\zeta(s)$* (so they share the Riemann hypothesis with ζ). In general these functions ζ^* have the imaginary axis as natural boundary and this indicates that our assumption to deal with entire or *almost* entire functions having a Dirichlet series representation somewhere is indeed relevant and cannot be dropped easily.

5. Last but not least: Read the Classics!

George Pólya refers to Alfred Pringsheim [19] and not to Jacques Hadamard [8], however, Pólya's reference to Pringsheim's article (page 327) is about a special case of what is nowadays well-known as Hadamard's factorization theorem. Actually, as a motivation for his treatise, Pringsheim wrote:

> "In several short articles, which appeared within the last two years in the Münchener Sitzungsberichten [Sitzungsberichte der Bayerischen Akademie der Wissenschaften zu München, Mathematisch-Physikalische Klasse*], I have made the attempt to derive certain main theorems from the theory of *entire transcendental functions of finite order* in a completely *elementary* way."†

Pringsheim stresses his contribution in pointing out the *elementary* approach. The origins of those certain main theorems, however, are not mentioned explicitly, the reader may guess to find them in the four papers listed in a footnote including Hadamard's path-breaking article [8]. Another source for Pringsheim is Henri Poincaré's paper [17] where the converse to Hadamard's theorem was proven.

In his historical survey "on the birth of Nevanlinna theory", Olli Lehto [12] provides a valuable overview of this branch of complex analysis. The motivation for studying the value distribution of analytic functions in general at that time was Émile Picard's great theorem from 1879 that *every non-constant entire function attains every complex value with at most one exception* and subsequent research such as, for example, Émile Borel's work on the order of entire functions and Karl Weierstraß' factorization theorem. Hadamard's contribution is considered as fundamental as well as Poincaré's theorem (mentioned above) which stressed "the more general viewpoint of considering a-points rather than zeros is essential" ([12], p. 8). For the class of number-theoretical relevant Dirichlet series and zeta-functions this idea had been revived in Edmund Landau's invited talk [10] at the occasion of the fifth International Mathematical Congress held at Cambridge in 1912:

> "*Now let me discuss some different investigations about* $\zeta(s)$. *Given an analytic function, the points for which this function is* 0 *are*

*a journal of the Bavarian Academy of Sciences

†This is the authors' translation of the German original: ,,In einigen kleineren Aufsätzen, welche im Laufe der letzten zwei Jahre in den Münchener Sitzungsberichten erschienen sind, habe ich den Versuch gemacht, gewisse Hauptsätze aus der Theorie der *ganzen transcendenten Funktionen von endlicher Ordnung* in vollkommen *elementarer* Weise zu begründen."

> *very important; however, of equal interest are those points where the function assumes a given value a. It is easy to prove that $\zeta(s)$ takes any value a. But where do the roots of $\zeta(s) = a$ lie?*"[‡]

According to Lehto, "the Nevanlinna theory came into being through the work he did in the years 1922-24" ([12], p. 5). Unfortunately, neither Pólya's article [18] nor the joint work of Rolf and his one year elder brother Frithiof Nevanlinna [15] on the Riemann zeta-function are mentioned. The latter piece of work from 1924 includes a new proof of the Riemann-von Mangoldt formula for the number of zeta zeros relying on methods from Nevanlinna theory (which can be extended to asymptotic formulae for the number of a-points; see [21]). According to Walter Hayman, "Nevanlinna theory was greatly influenced by Rolf's discussions with Frithiof, which continued all their lives. They would walk up and down each side of a big square, talking mathematics" ([9], p. 421). One may say that the initial point for Rolf Nevanlinna's work on value-distribution was the use of Jensen's formula and the definition of the related Nevanlinna functions, namely the proximity function m and the counting function N, carrying all information about the characteristics of a meromorphic function f. In fact, Pólya's reasoning was restricted to entire functions with respect to the question of shared values, but Nevannlinna could apply his deep second fundamental theorem and deduce his five point theorem. Hayman wrote: "The corresponding result for entire functions of finite order and 3 values a_ν had previously been obtained by Pólya [1921]" ([9], p. 427). Another biographer, Rolf Nevanlinna's doctoral student and first Fields medalist Lars Ahlfors, wrote about inventing Jensen's formula and other methods from potential theory that

> "It has been said that this step marks the birth of what was to become and still is called Nevanlinna theory in all its facets and variations. Nevanlinna's first version of the Second Main Theorem dealt with the distribution of three values, and it was observed by Collingwood and Littlewood that the theorem and its proof carry over to the more general situation of any q values, thereby leading to the Defect Relation. Collingwood's and Littlewood's merit should not be underrated, but it was Nevanlinna who found the key that opened the gate." ([1], p. III)

Nevanlinna himself cited Pólya's work and wrote

[‡]This is the authors' translation of the German original: "*Ich komme jetzt zu einigen anderen Untersuchungen über $\zeta(s)$. Es sind bei einer analytischen Funktion die Punkte, an denen sie* 0 *ist, zwar sehr wichtig; ebenso interessant sind aber die Punkte, an denen sie einen bestimmten Wert a annimmt. Zu beweisen, dass $\zeta(s)$ jeden Wert a annimmt, ist ein leichtes. Wo liegen aber die Wurzeln von $\zeta(s) = a$?*"

> "In the special case of entire functions of finite [order] (one of the five values then is infinity) this theorem had earlier been found by Mr Pólya under the additional restriction that the shared values appear with the same multiplicities."[§]

Actually, Nevanlinna also discusses some further details of Pólya's article (when three complex values are shared counting multiplicities and Picard exceptional values). It is apparent from his presentation that he was deeply motivated by Pólya's paper.

We conclude with a quotation of Ralph Boas: "Hardy is supposed to have said once that Pólya had brilliant ideas but didn't follow them up. There was some truth in this unkind remark. The *Collected Papers* [of Pólya] include many brief contributions that contain the germs of substantial theories that were redeveloped later by others. Nevertheless, it would be unreasonable to complain, considering that at the height of his career Pólya was publishing two or three major papers in analysis every year, and doing the same thing in probability" ([3], p. 576).

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[§] Again this is the authors' free translation of the German original: ,,Im speziellen Fall ganzer Funktionen von endlichem Geschlecht (einer der fünf Werte ist dann unendlich) wurde dieser Satz früher von Herrn PÓLYA gefunden, allerdings unter der einschränkenden Annahme, dass die gemeinsamen Stellen auch mit gleichen Multiplizitäten auftreten." ([16], p. 368)

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