FURTHER RESULTS ON A MULTIPLICATIVE TYPE FUNCTIONAL EQUATION

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Abstract. In the present paper the multiplicative type functional equation

$$f(xy)g(x+y) = h(xy+x)k(y),$$

derived from the pexiderized Davison equation (PD), is considered on different structures.

1. Introduction

The functional equation

$$\text{(D)} \qquad f(xy) + f(x+y) = f(xy+x) + f(y)$$

was introduced by T. M. K. Davison at the 17th ISFE (Oberwolfach, 1979) (see [2]). During the meeting W. Benz gave the continuous solution $f : \mathbb{R} \to \mathbb{R}$ of (D) for all $x, y \in \mathbb{R}$.

The general solution of (D) was given in [3] by R. Girgensohn and K. Lajkó:

Theorem 1.1. *The function $f : \mathbb{R} \to \mathbb{R}$ satisfies functional equation* (D) *for all $x, y \in \mathbb{R}$ if and only if f is of the form $f(x) = A(x) + b$, where $A : \mathbb{R} \to \mathbb{R}$ is an additive function and $b \in \mathbb{R}$ is an arbitrary constant.*

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In [3] the authors presented the general solution of the Pexiderized version

$$f(xy) + g(x + y) = h(xy + x) + k(y) \tag{PD}$$

of (D) for all $x, y \in \mathbb{R}$ and for all $x, y \in \mathbb{R}_+ := \{x | x > 0\}$:

Theorem 1.2. *The functions f, g, h, $k : \mathbb{R} \to \mathbb{R}$ satisfy* (PD) *for all $x, y \in \mathbb{R}$ if and only if they have the form $f(x) = A(x) + b_1$, $g(x) = A(x) + b_2$, $h(x) = A(x) + b_3$, $k(x) = A(x) + b_4$, where $A : \mathbb{R} \to \mathbb{R}$ is additive and b_1, b_2, b_3, $b_4 \in \mathbb{R}$ are constants with $b_1 + b_2 = b_3 + b_4$.*

Theorem 1.3. *The functions f, g, h, $k : \mathbb{R}_+ \to \mathbb{R}$ satisfy* (PD) *for all $x, y \in$ $\in \mathbb{R}_+$ if and only if they are of the form*

$$\begin{gathered} f(x) = A(x) + B(\log x) + b_1, \quad g(x) = A(x) + b_2, \\ h(x) = A(x) + B(\log x) + b_3, \quad k(x) = A(x) + B\left(\log \tfrac{x}{x+1}\right) + b_4, \end{gathered} \tag{1.1}$$

where A, $B : \mathbb{R} \to \mathbb{R}$ are additive and b_1, b_2, b_3, $b_4 \in \mathbb{R}$ are constants with $b_1 + b_2 = b_3 + b_4$.

Using (1.1) in Theorem 1.3, we easily get that

$$A(x) = g(x) - b_2, \quad B(\log x) = f(x) - g(x) - b_1 + b_2 \quad (x \in \mathbb{R}_+).$$

Thus the continuity (or measurability) of functions f, g implies that A, B are continuous (or measurable) on $\mathbb{R}_+$, too. This implies (see [1], [9]) that

$$A(x) = ax, \quad B(x) = bx \quad (x \in \mathbb{R}_+),$$

where $a, b \in \mathbb{R}$ are arbitrary constants.

Using these considerations together with Theorem 1.3, we get the following result.

Theorem 1.4. *The measurable* (*or continuous*) *functions f, g, h, $k : \mathbb{R}_+ \to \mathbb{R}$ satisfy* (PD) *for all $x, y \in \mathbb{R}_+$ if and only if they are of the form*

$$\begin{gathered} f(x) = ax + b \log x + b_1, \quad g(x) = ax + b_2, \\ h(x) = ax + b \log x + b_3, \quad k(x) = ax + b \log \tfrac{x}{x+1} + b_4, \end{gathered} \tag{1.2}$$

where a, b, b_1, b_2, b_3, $b_4 \in \mathbb{R}$ are constants with $b_1 + b_2 = b_3 + b_4$.

2. Positive solution of a multiplicative type functional equation stemming from (PD)

Let us write (PD) in the following multiplicative form:

$$f(xy)g(x + y) = h(xy + x)k(y) \tag{2.1}$$

for functions $f,\ g,\ h,\ k : \mathbb{R}$ (or $\mathbb{R}_+$) $\to \mathbb{R}_+$ for all $x, y \in \mathbb{R}$ or for all $x, y \in \mathbb{R}_+$.

Taking the logarithm of (2.1), we get the functional equation

$$\log\left(f(xy)\right) + \log\left(g(x+y)\right) = \log\left(h(xy+x)\right) + \log\left(k(y)\right)$$

for all $x, y \in \mathbb{R}$ or for all $x, y \in \mathbb{R}_+$.

Thus the functions $F,\ G,\ H,\ K : \mathbb{R}$ (or $\mathbb{R}_+$) $\to \mathbb{R}$ defined by

$$F = \log \circ f, \quad G = \log \circ g, \quad H = \log \circ h, \quad K = \log \circ k$$

satisfy functional equation (PD).

Using Theorems 1.2, 1.3 and 1.4 and that

$$f = \exp \circ F, \quad g = \exp \circ G, \quad h = \exp \circ H, \quad k = \exp \circ K,$$

we get immediately the following results.

Theorem 2.1. *The functions* $f,\ g,\ h,\ k : \mathbb{R} \to \mathbb{R}_+$ *satisfy* (2.1) *for all* $x, y \in \mathbb{R}$ *if and only if*

$$f(x) = c_1 \exp\left(A(x)\right), \quad g(x) = c_2 \exp\left(A(x)\right),$$

$$h(x) = c_3 \exp\left(A(x)\right), \quad k(x) = c_4 \exp\left(A(x)\right),$$

where $A : \mathbb{R} \to \mathbb{R}$ *is an additive function and* $c_1,\ c_2,\ c_3,\ c_4 \in \mathbb{R}_+$ *are constants with* $c_1 c_2 = c_3 c_4$.

Theorem 2.2. *The functions* $f,\ g,\ h,\ k : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfy* (2.1) *for all* $x, y \in \mathbb{R}_+$ *if and only if they are of the form*

$$\begin{gathered} f(x) = c_1 \exp\left(A(x) + B(\log x)\right), \quad g(x) = c_2 \exp\left(A(x)\right), \\ h(x) = c_3 \exp\left(A(x) + B(\log x)\right), \\ k(x) = c_4 \exp\left(A(x) + B\left(\log \frac{x}{x+1}\right)\right), \end{gathered} \tag{2.2}$$

where $A,\ B : \mathbb{R} \to \mathbb{R}$ *are additive and* $c_1,\ c_2,\ c_3,\ c_4 \in \mathbb{R}_+$ *are constants with* $c_1 c_2 = c_3 c_4$.

Theorem 2.3. *The measurable* (*or continuous*) *functions* $f,\ g,\ h,\ k : \mathbb{R}_+ \to \to \mathbb{R}_+$ *satisfy* (2.1) *for all* $x, y \in \mathbb{R}_+$ *if and only if*

$$\begin{gathered} f(x) = c_1 \exp\left(ax + b\log x\right), \quad g(x) = c_2 \exp\left(ax\right), \\ h(x) = c_3 \exp\left(ax + b\log x\right), \quad k(x) = c_4 \exp\left(ax + b\log \frac{x}{x+1}\right), \end{gathered} \tag{2.3}$$

where $a,\ b \in \mathbb{R}$ *and* $c_1,\ c_2,\ c_3,\ c_4 \in \mathbb{R}_+$ *are constants with* $c_1 c_2 = c_3 c_4$.

3. Nonnegative solutions of (2.1)

Now let us assume, that the nonnegative measurable functions f, g, h, k satisfy (2.1) for all $x, y \in \mathbb{R}_+$ and none of the functions h, k are almost everywhere zero. Does it follow that they are positive everywhere on $\mathbb{R}_+$?

In order to give an affirmative answer we will use the following result (see Járai, Lajkó, Mészáros [8], Remark 4).

Theorem 3.1. *Suppose that measurable functions $f_1 : X \to \mathbb{C}$, $f_2 : Y \to \mathbb{C}$, $g_1 : U \to \mathbb{C}$, $g_2 : V \to \mathbb{C}$ satisfy functional equation*

$$f_1(x)f_2(y) = g_1\left(G_1(x,y)\right)g_2\left(G_2(x,y)\right)H(x,y) \tag{3.1}$$

for all $(x,y) \in X \times Y$, where X, Y, U, V are nonvoid open intervals, G_1, G_2 and H are given functions, such that H is nowhere zero on $X \times Y$, the mapping $(x,y) \mapsto G(x,y) = (G_1(x,y), G_2(x,y))$ is a $\mathcal{C}^1$-diffeomorphism of $X \times Y$ onto $U \times V$ with inverse $(u,v) \mapsto F(u,v) = (F_1(u,v), F_2(u,v))$, such that all the partial derivatives of functions G_1, G_2, F_1, F_2 vanish nowhere on their domain and if on one side none of the functions are almost everywhere zero, then all the functions are everywhere nonzero.

Theorem 3.2. *If the nonnegative measurable functions f, g, h, $k : \mathbb{R}_+ \to \mathbb{R}$ satisfy functional equation (2.1) for all $x, y \in \mathbb{R}_+$ such that h, k are positive on some subsets of $\mathbb{R}_+$ with positive Lebesgue measure, then f, g, h, k are positive everywhere on $\mathbb{R}_+$.*

Proof. Using the symmetry of the left-hand side of (2.1) in x and y, we get

$$h(xy+x)k(y) = h(xy+y)k(x)$$

for all $x, y \in \mathbb{R}_+$. On the other hand, by replacing x by $\frac{x}{y+1}$, we find that h and k satisfy functional equation

$$h\left(x\right)k\left(y\right) = h\left(\frac{xy}{y+1} + y\right)k\left(\frac{x}{y+1}\right) \tag{3.2}$$

for all $x, y \in \mathbb{R}_+$, i.e. functional equation (3.1) for the unknown functions $f_1 = g_1 = h$, $f_2 = g_2 = k$, $X = Y = U = V = \mathbb{R}_+$ and for the given functions

$$G_1\left(x,y\right) = \frac{xy}{y+1} + y,\ G_2\left(x,y\right) = \frac{x}{y+1},\ H\left(x,y\right) = 1\ \left((x,y) \in \mathbb{R}_+^2\right).$$

Observe that H is nowhere zero on $\mathbb{R}_+^2$ and the mapping

$$(x,y) \to G\left(x,y\right) = \left(G_1\left(x,y\right), G_2\left(x,y\right)\right) = \left(\frac{xy}{y+1} + y, \frac{x}{y+1}\right)$$

is $\mathcal{C}^1$-diffeomorphism of $\mathbb{R}_+^2$ onto $\mathbb{R}_+^2$ with inverse

$$(u,v) \to F(u,v) = (F_1(u,v), F_2(u,v)) = \left(\frac{uv}{v+1} + v, \frac{u}{v+1}\right).$$

The partial derivatives

$$\frac{\partial G_1}{\partial x} = \frac{y}{y+1}, \frac{\partial G_1}{\partial y} = \frac{x}{(y+1)^2}, \frac{\partial G_2}{\partial x} = \frac{1}{y+1}, \frac{\partial G_2}{\partial y} = -\frac{x}{(y+1)^2},$$

$$\frac{\partial F_1}{\partial u} = \frac{v}{v+1}, \frac{\partial F_1}{\partial v} = \frac{u}{(v+1)^2}, \frac{\partial F_2}{\partial u} = \frac{1}{v+1}, \frac{\partial F_2}{\partial v} = -\frac{u}{(v+1)^2},$$

vanish nowhere on $\mathbb{R}_+^2$ and further none of the functions h, k are almost everywhere zero. All assumptions of Theorem 3.1 are satisfied, which implies that the functions h, k are everywhere nonzero on $\mathbb{R}_+$, then by equation (2.1) we get that functions f, g are everywhere nonzero, too.

Thus the nonnegativity of functions implies that f, g, h, $k : \mathbb{R}_+ \to \mathbb{R}$ are everywhere positive on $\mathbb{R}_+^2$. ■

Now we can easily prove the following result for equation (2.1).

Theorem 3.3. *If the nonnegative measurable (or continuous) functions f, g, h, $k : \mathbb{R}_+ \to \mathbb{R}$ satisfy* (2.1) *for all $x, y \in \mathbb{R}_+$ such that h, k are positive on some Lebesgue measurable subsets of positive Lebesgue measure, then they have the form* (2.3)*, i.e.*

$$f(x) = c_1 \exp(ax + b\log x), \quad g(x) = c_2 \exp(ax),$$

$$h(x) = c_3 \exp(ax + b\log x), \quad k(x) = c_4 \exp\left(ax + b\log\frac{x}{x+1}\right),$$

where a, $b \in \mathbb{R}$ and c_1, c_2, c_3, $c_4 \in \mathbb{R}_+$ are constants with $c_1c_2 = c_3c_4$.

Proof. Theorem 3.2 implies that functions f, g, h, k are positive everywhere on $\mathbb{R}_+$ and then Theorem 2.3 gives (2.3) for these functions, which completes the proof. ■

4. Nonnegative solutions of (2.1) satisfying almost everywhere

Now let us assume, that the nonnegative measurable functions f, g, h, k satisfy (2.1) for almost all $x, y \in \mathbb{R}_+$ and none of the functions h, k are almost everywhere zero. Does it follow that they are positive almost everywhere on $\mathbb{R}_+$?

In order to give an affirmative answer we will use the following result (see Járai, Lajkó, Mészáros [8], Theorem 2).

Theorem 4.1. *Suppose that measurable functions $f_1 : X \to \mathbb{C}$, $f_2 : Y \to \mathbb{C}$, $g_1 : U \to \mathbb{C}$, $g_2 : V \to \mathbb{C}$ satisfy functional equation*

$$f_1(x)f_2(y) = g_1\left(G_1(x,y)\right) g_2\left(G_2(x,y)\right) H(x,y) \tag{4.1}$$

for almost all $(x,y) \in X \times Y$ (with respect to the plane Lebesgue measure), where X, Y, U, V are nonvoid open intervals, G_1, G_2 and H are given functions, such that H is nowhere zero on $X \times Y$, the mapping $(x,y) \mapsto G(x,y) = (G_1(x,y), G_2(x,y))$ is a $\mathcal{C}^1$-diffeomorphism of $X \times Y$ onto $U \times V$ with inverse $(u,v) \mapsto F(u,v) = (F_1(u,v), F_2(u,v))$, such that all the partial derivatives of functions G_1, G_2, F_1, F_2 vanish nowhere on their domain. Then either one of the functions f_1 and f_2 and one of the functions g_1 and g_2 is zero almost everywhere or all of them are almost everywhere nonzero.

Theorem 4.2. *If the nonnegative measurable functions f, g, h, $k : \mathbb{R}_+ \to \mathbb{R}$ satisfy functional equation (2.1) for almost all $x, y \in \mathbb{R}_+$ such that h, k are positive on some subsets of $\mathbb{R}_+$ with positive Lebesgue measure, then f, g, h, k are positive almost everywhere on $\mathbb{R}_+$.*

Proof. Similarly to the proof of Theorem 3.2 we can prove that all assumptions of Theorem 4.1 are satisfied. This implies that the functions h, k are almost everywhere nonzero on $\mathbb{R}_+$, then by equation (2.1) we get that functions f, g are almost everywhere nonzero, too.

Thus the nonnegativity of functions implies that f, g, h, $k : \mathbb{R}_+ \to \mathbb{R}$ are almost everywhere positive on $\mathbb{R}^2_+$. ∎

To get the nonnegative measurable solutions of (2.1) satisfying almost everywhere, we need the following result of A. Járai (see [4], [5], [6], [7]).

Theorem 4.3. *Let Z be a regular topological space, Z_i $(i = 1, 2, \dots, n)$ be topological spaces and T be a first countable topological space. Let Y be an open subset of $\mathbb{R}^k$, X_i an open subset of $\mathbb{R}^{r_i}$, $r_i \in \mathbb{Z}$, $(i = 1, 2, \dots, n)$ and D an open subset of $T \times Y$. Let furthermore $T' \subset T$ be a dense subset, $F : T' \to Z$, $g_i \colon D \to X_i$ and $H \colon D \times Z_1 \times \dots \times Z_n \to Z$. Suppose that the function f_i is almost everywhere defined on X_i (with respect to the r_i-dimensional Lebesgue measure) with values in Z_i $(i = 1, 2, \dots n)$ and the following conditions are satisfied:*

1. *for all $t \in T'$ and for almost all $y \in D_t = \{y \in Y | (t,y) \in D\}$*

$$F(t) = H(t, y, f_1(g_1(t,y)), \dots, f_n(g_n(t,y)));$$

2. *for each fixed* y *in* Y, *the function* H *is continuous in the other variables;*

3. f_i *is Lebesgue measurable on* $\mathbb{R}^{r_i}$ $(i = 1, 2, \dots, n)$;

4. g_i *and the partial derivative* $\frac{\partial g_i}{\partial y}$ *are continuous on* D $(i = 1, 2, \dots, n)$;

5. *for each* $t \in T$ *there exist a* y *such that* $(t, y) \in D$ *and the partial derivative* $\frac{\partial g_i}{\partial y}$ *has the rank* r_i *at* $(t, y) \in D$ $(i = 1, 2, \dots, n)$.

Then there exists a unique continuous function $\widetilde{F}$ *such that* $F = \widetilde{F}$ *almost everywhere on* T, *and if* F *is replaced by* $\widetilde{F}$ *then the functional equation is satisfied almost everywhere on* D.

Using Theorems 4.2 and 4.3 we can prove the following result.

Theorem 4.4. *If the nonnegative measurable functions* f, g, h, $k : \mathbb{R}_+ \to \mathbb{R}$ *satisfy* (2.1) *for almost all* $(x, y) \in \mathbb{R}_+^2$ *such that they are positive on some subsets of* $\mathbb{R}_+$ *with positive Lebesgue measure, then there exist unique continuous functions* $\tilde{f}$, $\tilde{g}$, $\tilde{h}$, $\tilde{k} : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\tilde{f} = f$, $\tilde{g} = g$, $\tilde{h} = h$ *and* $\tilde{k} = k$ *almost everywhere on* $\mathbb{R}_+$, *and if* f, g, h, k *are replaced by* $\tilde{f}$, $\tilde{g}$, $\tilde{h}$, $\tilde{k}$, *respectively, then* (2.1) *is satisfied everywhere on* $\mathbb{R}_+^2$.

Proof. Theorem 4.2 shows that the functions f, g, h, k are positive almost everywhere on $\mathbb{R}_+$.

First we prove that there exists a unique continuous function $\tilde{h}$ which is equal to h almost everywhere on $\mathbb{R}_+$ and replacing h by $\tilde{h}$, equation (2.1) is satisfied almost everywhere on $\mathbb{R}_+^2$.

With the substitution $t = xy + x$ we get from (2.1) the equation

$$h(t) = \frac{f\left(\frac{ty}{y+1}\right) g\left(\frac{t}{y+1} + y\right)}{k(y)} \tag{4.2}$$

which is satisfied for almost all $(t, y) \in \mathbb{R}_+^2$.

By Fubini's Theorem it follows that there exists $T' \subseteq \mathbb{R}_+$ of full measure such that for all $t \in T'$ equation (4.2) is satisfied for almost every $y \in \left\{y \in \mathbb{R}_+ | (t, y) \in \mathbb{R}_+^2\right\} = \mathbb{R}_+$.

Let us define the functions g_1, g_2, g_3, H in the following way:

$$g_1(t, y) = \frac{ty}{y+1}, \quad g_2(t, y) = \frac{t}{y+1} + y, \quad g_3(t, y) = y$$

$$H(t, y, z_1, z_2, z_3) = \frac{z_1 z_2}{z_3}$$

and let us now apply Theorem 4.3 of Járai to (4.2) with the following casting: $h(t) = F(t)$, $f(t) = f_1(t)$, $g(t) = f_2(t)$, $k(t) = f_3(t)$, $Z = Z_i = \mathbb{R}_+$, $T = Y = X_i = \mathbb{R}_+$ $(i = 1, 2, 3)$.

The first assumption of Theorem 4.3 with respect to (4.2) holds.

In the case of fixed y, the function H is continuous in the other variables, so the second assumption holds too.

Because the functions in (4.2) are measurable, the third assumption is trivial.

The functions g_i and the partial derivatives

$$D_2g_1(t,y) = \frac{t}{(y+1)^2}, \quad D_2g_2(t,y) = -\frac{t}{(y+1)^2} + 1, \quad D_2g_3(t,y) = 1$$

are continuous, so the fourth assumption holds, too.

For each $t \in \mathbb{R}_+$ there exist a $y \in \mathbb{R}_+$ such that $(t, y) \in D$ and the partial derivatives don't equal zero in (t, y), so they have the rank 1. Thus the last assumption is satisfied in Theorem 4.3.

So we get from Theorem 4.3 that there exists unique continuous function $\tilde{h}$ which is almost everywhere equal to h on $\mathbb{R}_+$ and f, g, $\tilde{h}$, k satisfy equation (4.2) almost everywhere, which is equivalent to the equation

$$f(xy)g(x+y) = \tilde{h}(xy+x)k(y)$$

for almost all $(x, y) \in \mathbb{R}_+^2$. Furthermore $\tilde{h}$ is positive for almost all $x \in \mathbb{R}_+$.

By a similar argument we can prove the same for the functions f, g and k, i.e. there exist continuous functions $\tilde{f} : \mathbb{R}_+ \to \mathbb{R}$, $\tilde{g} : \mathbb{R}_+ \to \mathbb{R}$ and $\tilde{k} : \mathbb{R}_+ \to \mathbb{R}$ which are almost everywhere equal to f, g and k on $\mathbb{R}_+$, respectively, and the functional equation

$$\tilde{f}(xy)\tilde{g}(x+y) = \tilde{h}(xy+x)\tilde{k}(y) \tag{4.3}$$

is satisfied almost everywhere on $\mathbb{R}_+^2$.

Both sides of (4.3) define continuous functions on $\mathbb{R}_+^2$, which are equal to each other on a dense subset of $\mathbb{R}_+^2$, therefore we obtain that (4.3) is satisfied everywhere on $\mathbb{R}_+^2$.

Applying Theorem 3.2 for equation (4.3), one can show that if the nonnegative continuous functions $\tilde{f}$, $\tilde{g}$, $\tilde{h}$ and $\tilde{k} : \mathbb{R}_+ \to \mathbb{R}$ satisfy functional equation (4.3) for all $(x, y) \in \mathbb{R}_+^2$, such that they are positive almost everywhere on $\mathbb{R}_+$, then they are positive everywhere on $\mathbb{R}_+$. ■

Now we can easily prove the following result for equation (2.1).

Theorem 4.5. *If the nonnegative measurable (or continuous) functions f, g, h, $k:\mathbb{R}_+\to\mathbb{R}$ satisfy* (2.1) *for almost all $x,y\in\mathbb{R}_+$ such that they are positive on some Lebesgue measurable subsets of positive Lebesgue measure, then they have the form*

$$f(x)=c_1\exp\left(ax+b\log x\right),\quad g(x)=c_2\exp\left(ax\right)\quad a.a.\ x\in\mathbb{R}_+,$$

$$h(x)=c_3\exp\left(ax+b\log x\right)\quad a.a.\ x\in\mathbb{R}_+,$$

$$k(x)=c_4\exp\left(ax+b\log\frac{x}{x+1}\right)\quad a.a.\ x\in\mathbb{R}_+,$$

where a, $b\in\mathbb{R}$ and c_1, c_2, c_3, $c_4\in\mathbb{R}_+$ are constants with $c_1c_2=c_3c_4$.

Proof. Using Theorems 4.4 and 4.2, we get immediately the statement of Theorem 4.5. ■

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