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ON THE PERIODIC HURWITZ ZETA-FUNCTION WITH RATIONAL PARAMETER

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Abstract. The periodic Hurwitz zeta-function $\zeta(s,\alpha;\mathfrak{a})$, $s=\sigma+it$, is a generalization of the classical Hurwitz zeta-function, and is defined, for $\sigma>1$, by the series $\zeta(s,\alpha;\mathfrak{a})=\sum\limits_{m=0}^{\infty}\frac{a_m}{(m+\alpha)^s}$, where $\mathfrak{a}=\{a_m\}$ is a periodic sequence of complex numbers, and $0<\alpha\leqslant 1$ is a fixed parameter. In the paper, theorems on the approximation of analytic functions by discrete shifts $\zeta(s+ikh,\alpha;\mathfrak{a})$, $k=0,1,\dots$, $h>0$, with rational α are obtained.

1. Introduction

It is well known that some of zeta and L-functions are universal in the Voronin sense, i.e., their shifts approximate a wide class of analytic functions. A very extensive survey on this type of universality is given in [15]. In this note, we discuss universality theorems for the periodic Hurwitz zeta-function.

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Let $s = \sigma + it$ be a complex variable, and α, $0 < \alpha \leqslant 1$, be a fixed parameter. The classical Hurwitz zeta-function $\zeta(s, \alpha)$ was introduced in [5], and is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s},$$

and can be continued analytically to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1.

Properties of $\zeta(s, \alpha)$ including the universality depend on the arithmetic nature of the parameter α. In the case of transcendental or rational $\alpha \neq \frac{1}{2}, 1$, the universality of $\zeta(s, \alpha)$ was obtained by S.M. Gonek [4]. More precisely, if K is a compact subset of the strip $D = \left\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\right\}$ with connected complement, and $f(s)$ is a continuous function on K and analytic in the interior of K, then, for every $\varepsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau, \alpha) - f(s)| < \varepsilon\right\} > 0.$$

The latter inequality shows that there are infinitely many shifts $\zeta(s + i\tau, \alpha)$ approximating with accuracy ε a given analytic function. We also note that if at least one τ satisfies the inequality $\sup_{s \in K} |\zeta(s + i\tau, \alpha) - f(s)| < \varepsilon$, then, in view of the continuity with respect to τ, it follows that there infinitely many shifts τ satisfying the above inequality.

The universality of $\zeta(s, \alpha)$ with algebraic irrational α remains an open problem.

The type of universality from above is called continuous since the shifts τ are taken from the set of all real numbers. If τ is restricted to some discrete set, however, then the universality property is said to be of discrete type. A discrete universality theorem with transcendental or rational α was obtained in [1]. Let K and $f(s)$ be as above. Then, for $\alpha \neq \frac{1}{2}, 1$, $\varepsilon > 0$ and $h > 0$,

$$\liminf_{N \to \infty} \frac{1}{N+1} \# \left\{0 \leqslant k \leqslant N : \sup_{s \in K} |\zeta(s + ikh, \alpha) - f(s)| < \varepsilon\right\} > 0. \tag{1.1}$$

Define the set

$$L(\alpha, h, \pi) = \left\{(\log(m + \alpha) : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}), \frac{2\pi}{h}\right\}.$$

If the set $L(\alpha, h, \pi)$ is linearly independent over the field of rational numbers $\mathbb{Q}$, then it was proved in [9] that inequality (1.1) is true.

The periodic Hurwitz zeta-function, a natural generalization of the function $\zeta(s, \alpha)$, was introduced in [6]. Let $\mathfrak{a} = \{a_m : m \in \mathbb{N}_0\}$ be a periodic sequence of complex numbers with minimal period $q \in \mathbb{N}$. Then the periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{a})$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s, \alpha; \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s}.$$

In view of periodicity of the sequence $\mathfrak{a}$,

$$\zeta(s, \alpha; \mathfrak{a}) = \frac{1}{q^s} \sum_{l=0}^{q-1} a_l \zeta\left(s, \frac{\alpha+l}{q}\right). \tag{1.2}$$

Therefore, the function $\zeta(s, \alpha; \mathfrak{a})$ also has analytic continuation to the whole complex plane, except for a simple pole at the point $s = 1$ with residue

$$\hat{a} \overset{def}{=} \frac{1}{q} \sum_{l=0}^{q-1} a_l.$$

If $\hat{a}$=0, then the function $\zeta(s, \alpha; \mathfrak{a})$ is entire.

A continuous universality theorem for the function $\zeta(s, \alpha; \mathfrak{a})$ with transcendental α was obtained in [7]. The case of rational α was considered in [12]. In [11], the transcendence of the parameter α was replaced by the linear independence of the set

$$L(\alpha) = \{\log(m+\alpha) : m \in \mathbb{N}_0\}.$$

The aim of this note is to prove discrete universality theorems for the function $\zeta(s, \alpha; \mathfrak{a})$. For their statements, we use the following convenient notation. Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements, and by $H(K)$ with $K \in \mathcal{K}$ the class of continuous functions on K that are analytic in the interior of K.

The first discrete universality theorem for the function $\zeta(s, \alpha; \mathfrak{a})$ with transcendental α was obtained in [10].

Theorem 1 ([10])**.** *Suppose that α is a transcendental number and $h > 0$ is such that the number $\exp\left\{\frac{2\pi}{h}\right\}$ is rational. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{s \in K} |\zeta(s+ikh, \alpha; \mathfrak{a}) - f(s)| < \varepsilon \right\} > 0.$$

In [17], the hypothesis of Theorem 1 on the numbers α and h was replaced by the linear independence over $\mathbb{Q}$ for the set $L(\alpha, h, \pi)$. The present note is devoted to the discrete universality of the function $\zeta(s, \alpha; \mathfrak{a})$ with rational α.

In what follows, $\alpha = \frac{a}{b} \neq \frac{1}{2}$ with $a, b \in \mathbb{N}$, $a < b$, and $(a, b) = 1$. We recall that q is the period of the sequence $\mathfrak{a}$.

Theorem 2. *Suppose that $(lb+a, bq)=1$ for $l=0,1,\dots,q-1$. Let $K\in\mathcal{K}$ and $f(s)\in H(K)$. Then, for every $\varepsilon>0$ and $h>0$,*

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\left\{0\leqslant k\leqslant N:\sup_{s\in K}\left|\zeta\left(s+ikh,\frac{a}{b};\mathfrak{a}\right)-f(s)\right|<\varepsilon\right\}>0.$$

The condition that $(lb+a, bq)=1$ for $l=0,1,\dots,q-1$ is technical, and we believe that it can be removed.

Theorem 2 has the following modification.

Theorem 3. *Under hypotheses of Theorem 2, for every $h>0$, the limit*

$$\lim_{N\to\infty}\frac{1}{N+1}\#\left\{0\leqslant k\leqslant N:\sup_{s\in K}\left|\zeta\left(s+ikh,\frac{a}{b};\mathfrak{a}\right)-f(s)\right|<\varepsilon\right\}>0$$

exists for all but at most countably many $\varepsilon>0$.

Some composite functions of $\zeta\left(s,\frac{a}{b};\mathfrak{a}\right)$ also have the universality property. Denote by $H(G)$ the space of analytic functions in the region $G\subset\mathbb{C}$ endowed with the topology of uniform convergence on compacta. For $V>0$, let $D_V=\left\{s\in\mathbb{C}:\frac{1}{2}<\sigma<1, |t|<V\right\}$. For example, we have the statements.

Theorem 4. *Suppose that a, b, q and K, $f(s)$ are as in Theorem 2, $V>0$ is such that $K\subset D_V$, and $F:H(D_V)\to H(D_V)$ is a continuous operator such that, for every open set $G\subset H(D_V)$, the set $F^{-1}G$ is not empty. Then, for every $\varepsilon>0$ and $h>0$,*

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\left\{0\leqslant k\leqslant N:\sup_{s\in K}\left|F\left(\zeta\left(s+ikh,\frac{a}{b};\mathfrak{a}\right)\right)-f(s)\right|<\varepsilon\right\}>0.$$

For distinct complex numbers $a_1,\dots,a_v$, define

$$H_{a_1,\dots,a_v}(D_V)=\left\{g\in H(D_V):g(s)\neq a_j,\ j=1,\dots,v\right\}.$$

Theorem 5. *Suppose that a, b and q are as in Theorem 2, K is a compact set of D, $V>0$ is such that $K\subset D_V$, and that $F:H(D_V)\to H(D_V)$ is a continuous operator such that $H_{a_1,\dots,a_v}\subset F(H(D_V))$. For $v=1$, let $K\in\mathcal{K}$, $f(s)\in H(K)$ and $f(s)\neq a_1$ on K. For $v\geqslant 2$, let K be an arbitrary compact subset of D, and $f(s)\in H_{a_1,\dots,a_v}(D_V)$. Then the assertion of Theorem 4 is true.*

The set of the natural numbers a, b and q satisfying hypotheses of Theorems 2–5 is not empty. For example, if $a=3$, $b=8$ and $q=4$, then $(bl+a, qb)=1$ for $l=0,1,2,3$.

From Theorem 5, it follows that certain elementary functions of the periodic Hurwitz zeta-functions, for example, $\sin\left(\zeta\left(s,\frac{a}{b};\mathfrak{a}\right)\right)$, have the discrete universality property.

2. Application of Dirichlet L-functions

Let χ be a Dirichlet character modulo k. The Dirichlet L-function $L(s,\chi)$ is defined , for $\sigma > 1$, by the Dirichlet series

$$L(s,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s},$$

or by the Euler product over primes

$$L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

If $\chi = \chi_0$ is the principal character modulo k, then $L(s,\chi_0)$ has analytic continuation to the whole complex plane, except for a simple pole at the point $s = 1$ with residue

$$\prod_{p|k} \left(1 - \frac{1}{p}\right),$$

and if $\chi \neq \chi_0$, then the function $L(s,\chi)$ is entire.

For us the formula

$$\zeta\left(s, \frac{a}{b}\right) = \frac{b^s}{\varphi(b)} \sum_{\chi(\operatorname{mod} b)} \overline{\chi}(a) L(s,\chi), \quad (a,b) = 1, \tag{2.1}$$

where $\varphi(m)$ denotes the Euler totient function, will be useful. Let $(lb+a, qb) = d_l$, $l = 0, 1, \dots, q-1$. Then the equalities (1.2) and (2.1) imply

$$\begin{aligned} \zeta\left(s, \frac{a}{b}; \mathfrak{a}\right) &= \frac{1}{q^s} \sum_{l=0}^{q-1} a_l \zeta\left(s, \frac{a/b + l}{q}\right) = \\ &= \frac{1}{q^s} \sum_{l=0}^{q-1} a_l \frac{(bq/d_l)^s}{\varphi(bq/d_l)} \sum_{\chi(\operatorname{mod} bq/d_l)} \overline{\chi}\left(\frac{bl+a}{d_l}\right) L(s,\chi). \end{aligned} \tag{2.2}$$

If $d_l = 1$ for all $l = 0, 1, \dots, q-1$, then the formula (2.2) gives

$$\zeta\left(s, \frac{a}{b}; \mathfrak{a}\right) = \frac{b^s}{\varphi(bq)} \sum_{l=0}^{q-1} a_l \sum_{\chi(\operatorname{mod} bq)} \overline{\chi}(bl+a) L(s,\chi). \tag{2.3}$$

Let, for brevity, $\varphi(bq) = r$, and

$$A_j = \sum_{l=0}^{q-1} a_l \overline{\chi_j}(bl + a),$$

where χ_j runs over the r Dirichlet characters modulo bq. Then, by (2.3),

$$\zeta\left(s, \frac{a}{b}; \mathfrak{a}\right) = \frac{b^s}{r} \sum_{j=1}^{r} A_j L(s, \chi_j). \tag{2.4}$$

Now, we remind some probabilistic joint value-distribution results for Dirichlet $L-$functions. Define

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \{s \in \mathbb{C} : |s| = 1\}$ for all primes p. The torus Ω, with product topology and pointwise multiplication, is a compact topological group. Therefore, denoting by $\mathcal{B}(X)$ the Borel $\sigma-$field of the space X, we have that, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H can be defined. This gives the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Let $\omega(p)$ be the projection of an element $\omega \in \Omega$ to the circle γ_p.

Let $\mathbb{P}$ be the set of all prime numbers. We divide the set of all positive numbers h into two parts. We say that h is of type 1 if the numbers $\exp\left\{\frac{2\pi m}{h}\right\}$ are irrational for all $m \in \mathbb{Z} \setminus \{0\}$, and h is of type 2 if it is not of type 1.

Let Ω_h be a closed subgroup of the group Ω generated by the element $\{p^{-ih} : p \in \mathbb{P}\}$. If h is of type 2, then there exists a minimal $m_0 \in \mathbb{N}$ such that the number $\exp\left\{\frac{2\pi m_0}{h}\right\}$ is rational. Suppose that

$$\exp\left\{\frac{2\pi m_0}{h}\right\} = \frac{m_1}{m_2}, \quad m_1, m_2 \in \mathbb{N},\ (m_1, m_2) = 1.$$

Extend the function $\omega(p), p \in \mathbb{P}$, to the $\mathbb{N}$ by the formula

$$\omega(m) = \prod_{\substack{p^l | m,\\ p^{l+1} \nmid m}} \omega^l(p), \quad m \in \mathbb{N}.$$

Then it is known [1], see also [14], that

$$\Omega_h = \begin{cases} \Omega, & \text{if } h \text{ is of type 1,} \\ \{\omega \in \Omega : \omega(m_1) = \omega(m_2)\} & \text{if } h \text{ is of type 2.} \end{cases}$$

On $(\Omega_h, \mathcal{B}(\Omega_h))$, also there exists the probability Haar measure m_{Hh}. Denote by $\omega_h(p)$ the pth component of $\omega_h \in \Omega_h$. Now, on the probability space $(\Omega_h, B(\Omega_h), m_{Hh})$, define the $H^r(D)$-valued random element $\underline{L}(s, \omega_h, \underline{\chi})$, $\underline{\chi} = (\chi_1, \dots, \chi_r)$, by

$$\underline{L}(s, \omega_h, \underline{\chi}) = (L(s, \omega_h, \chi_1), \dots, L(s, \omega_h, \chi_r)),$$

where

$$L(s, \omega_h, \chi_j) = \prod_p \left(1 - \frac{\omega_h(p)\chi_j(p)}{p^s}\right)^{-1}, \quad j = 1, \dots, r,$$

and put

$$\underline{L}(s, \underline{\chi}) = (L(s, \chi_1), \dots, L(s, \chi_r)).$$

Moreover, for $A \in \mathcal{B}(H^r(D_V))$ and $h > 0$, we set

$$P_{N,V,h}(A) = \frac{1}{N+1} \# \left\{0 \leqslant k \leqslant N : \left(\underline{L}(s + ikh, \underline{\chi})\right) \in A\right\}$$

and

$$P_{\underline{L},V,h}(A) = m_{Hh} \left\{\omega_h \in \Omega_h : \underline{L}(s, \omega_h, \underline{\chi}) \in A\right\},$$

i.e., $P_{\underline{L},V,h}$ is the distribution of the random element $\underline{L}\left(s, \omega_h, \underline{\chi}\right)$. Let

$$S_V = \{g \in H(D_V) : g(s) \neq 0 \ \text{ or } \ g(s) \equiv 0\}.$$

Lemma 1. *For every $h > 0$, $P_{N,V,h}$ converges weakly to $P_{\underline{L},V,h}$ as $N \to \infty$. Moreover, the support of the measure $P_{\underline{L},V,h}$ is the set S_V^r.*

Proof. We note that a way of the proof of the lemma is the same as in case of $A \in \mathcal{B}(H(D))$.

In the case of h of type 1, the lemma for Dirichlet L-functions with non-equivalent characters was proved in [8]. The case of h of type 2 is considered similarly, see, for example, [14]. The method of the proof of limit theorems for zeta- or L-functions having the Euler product over primes is standard, therefore we give only a sketch of the proof of the lemma. First of all, using the linear independence over the field of rational numbers of the set $\{\log p : p \in \mathbb{P}\}$, a limit theorem with the limit measure m_{Hh} for probability measures on $(\Omega_h, \mathcal{B}(\Omega_h))$ is obtained. This theorem implies limit theorems in the space $(H(D_V), \mathcal{B}(H(D_V)))$ for certain absolutely convergent Dirichlet series. In the next step, it is proved that the latter Dirichlet series approximate in the mean the initial Dirichlet L-functions. This together with limit theorems for absolutely convergent Dirichlet series leads to the limit theorems with the same limit measure $P_{V,h}$ for $\underline{L}(s, \underline{\chi})$ and $\underline{L}(s, \omega_h, \underline{\chi})$. Finally, an application of the classical Birkhoff–Khintchine ergodic theorem shows that the limit measure $P_{V,h}$ coincides with $P_{\underline{L},V,h}$.

For the proof that the support of the measure $P_{\underline{L},V,h}$ is the set S_V^r, some classical properties of the series of independent random elements and of exponential functions are applied [8]. The problem reduces to an equation

$$\sum_{j=1}^{r} a_j \chi_j(m) = 0, \quad 1 \leqslant m \leqslant bq,$$

with some $a_j \in \mathbb{C}$. In [8], the non-equivalence of the characters is required for their linear independence over $\mathbb{C}$. In our case, the characters in (2.3), as in [1], [2], share the same modulus, therefore they are linearly independent over $\mathbb{C}$. Hence, it follows that $a_j = 0$ for $j = 1, \dots, r$, and this is sufficient to prove the second assertion of the lemma. ■

3. Discrete limit theorems for $\zeta\left(s, \frac{a}{b}; \mathfrak{a}\right)$

For $A \in B(H(D))$, define

$$Q_{N,V,h}(A) = \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \zeta\left(s + ikh, \frac{a}{b}; \mathfrak{a}\right) \in A \right\}.$$

Moreover, on the probability space $(\Omega_h, \mathcal{B}(\Omega_h), m_{Hh})$, define two $H(D_V)$-valued random elements

$$\zeta_1(s, \omega_h) = \frac{b^s \overline{\omega}_h(b)}{r}$$

and

$$\zeta_2(s, \omega_h) = \sum_{j=1}^{r} A_j L(s, \omega_h, \chi_j),$$

and set

$$\zeta\left(s, \frac{a}{b}, \omega_h; \mathfrak{a}\right) = \zeta_1(s, \omega_h) \zeta_2(s, \omega_h).$$

Denote by $P_{\zeta,V,h}$ the distribution of the random element $\zeta\left(s, \frac{a}{b}, \omega_h; \mathfrak{a}\right)$, i.e.,

$$P_{\zeta,V,h}(A) = m_{Hh}\left\{ \omega_h \in \Omega_h : \zeta\left(s, \frac{a}{b}, \omega_h; \mathfrak{a}\right) \in A \right\}, \quad A \in \mathcal{B}(H(D_V)).$$

Lemma 2. *Suppose that a, b and q are as in Theorem 2. Then $Q_{N,V,h}$ converges weakly to the measure $P_{\zeta,V,h}$ as $N \to \infty$. Moreover, the support of the measure $P_{\zeta,V,h}$ is the whole space $H(D_V)$.*

Proof. The function

$$\zeta_1(s) = \frac{b^s}{r}$$

is a Dirichlet polynomial. Therefore, we find by a standard method (the case of h of the type 2 is discussed in [14]) that

$$\frac{1}{N+1}\#\{0 \leqslant k \leqslant N : \zeta_1(s+ikh) \in A\}, \quad A \in \mathcal{B}(H(D_V)), \tag{3.1}$$

converges weakly to the distribution of the random element $\zeta_1(s, \omega_h)$ as $N \to \infty$.

For the proof of a limit theorem for the function

$$\zeta_2(s) = \sum_{j=1}^{r} A_j L(s, \chi_j),$$

we will apply Lemma 1. Let the function $u : H^r(D_V) \to H(D_V)$ be given by the formula

$$u(g_1, \dots, g_r) = \sum_{j=1}^{r} A_j g_j, \quad g_1, \dots, g_r \in H(D_V).$$

Then, clearly, the function u is continuous. Moreover, for $A \in \mathcal{B}(H(D_V))$,

$$\begin{aligned}
Q_{2,N,V,h}(A) &\stackrel{def}{=} \frac{1}{N+1}\#\{0 \leqslant k \leqslant N : \zeta_1(s+ikh) \in A\} = \\
&= \frac{1}{N+1}\#\{0 \leqslant k \leqslant N : u\left(L(s+ikh, \chi_1), \dots, L(s+ikh, \chi_r)\right) \in A\} = \\
&= \frac{1}{N+1}\#\left\{0 \leqslant k \leqslant N : (L(s+ikh, \chi_1), \dots, L(s+ikh, \chi_r)) \in u^{-1}A\right\} = \\
&= P_{N,V,h}(u^{-1}A) = P_{N,V,h}u^{-1}(A).
\end{aligned}$$

Therefore, the continuity of the function u, Theorem 5.1 of [3] and Lemma 1 show that $Q_{2,N,V,h}$ converges weakly to the measure $P_{\underline{L},V,h}u^{-1}$ as $N \to \infty$. We observe that the measure $P_{\underline{L},V,h}u^{-1}$ is the distribution of the random element $\zeta_2(s, \omega_h)$. Actually, for $A \in \mathcal{B}(H(D_V))$,

$$\begin{aligned}
P_{\underline{L},V,h}u^{-1}(A) = P_{\underline{L},V,h}(u^{-1}A) &= m_{Hh}\left\{\omega_h \in \Omega_h : \underline{L}(s, \omega_h, \underline{\chi}) \in u^{-1}A\right\} = \\
&= m_{Hh}\left\{\omega_h \in \Omega_h : u\left(\underline{L}(s, \omega_h, \underline{\chi})\right) \in A\right\} = \\
&= m_{Hh}\{\omega_h \in \Omega_h : \zeta_2(s, \omega_h) \in A\}.
\end{aligned}$$

Now, the weak convergence of the measures (3.1) and $Q_{2,N,V,h}$, and a modified Cramér–Wold method imply the weak convergence for

$$\frac{1}{N+1}\#\{0 \leqslant k \leqslant N : (\zeta_1(s+ikh), \zeta_2(s+ikh)) \in A\}, \quad A \in B(H^2(D_V)),$$

to the distribution of the random element $(\zeta_1(s,\omega_h),\zeta_2(s,\omega_h))$ as $N\to\infty$. From this, using the function $u_1: H^2(D_V)\to H(D_V)$ given by

$$u_1(g_1,g_2)=g_1g_2, \qquad g_1,g_2\in H(D_V),$$

we easily find that, in view of (2.4), $Q_{N,V,h}$ converges weakly to the distribution of the random element $\zeta_1(s,\omega_h)\zeta_2(s,\omega_h)=\zeta\left(s,\frac{a}{b},\omega_h;\mathfrak{a}\right)$ as $N\to\infty$.

It remains to find the support of the measure $P_{\zeta,V,h}$. Let g be an arbitrary element of $H(D_V)$, and G be its any open neighbourhood. Since the function u is continuous, we have that the set $u^{-1}G$ is open as well. If $K\subset D_V$ is a compact subset with connected complement, then, by the Mergelyan theorem [16], for every $\varepsilon>0$, there exists a polynomial $p=p(s)$ such that

$$\sup_{s\in K}|f(s)-p(s)|<\varepsilon.$$

It is well known, see, for example, [13], that the approximation in the space $H(D_V)$ can be restricted to that on compact sets with connected complements. Therefore, we can choose the polynomial $p(s)$ to lie in the set G. The region D_V is bounded, and the non-vanishing of $p(s)$ can be controlled by its constant term. Hence, it is not difficult to see that there exist $g_1,\dots,g_r\in S_V$ such that

$$u(g_1,\dots,g_r)=p.$$

Really, since $\alpha\neq\frac{1}{2}$, we have that $b\geqslant 3$, hence, $r\geqslant 2$. Therefore, at least two of the coefficients A_j in (2.4) are distinct from zero. Actually, if only one $A_j\neq 0$, then

$$\zeta\left(s,\frac{a}{b};\mathfrak{a}\right)=\frac{b^sA_j}{r}L(s,\chi_j).$$

Writing the left-hand side as a Dirichlet series, we have for $\sigma>1$ that

$$\sum_{m=1}^{\infty}\frac{c_m}{m^s}=\frac{A_j}{r}\sum_{m=1}^{\infty}\frac{\chi(m)}{m^s},$$

where $c_m=0$ for $m\in B\overset{def}{=}\mathbb{N}\setminus\bigcup_{k=0}^{\infty}\{a+bk\}$. Thus, there exists a prime number $p\in B$ such that $(p,bq)=1$. Therefore, $\chi(p)\neq 0$, and, by the uniqueness theorem for Dirichlet series, this contradicts a property of the set B. Thus, without loss of generality, we may suppose that $A_1\neq 0$ and $A_2\neq 0$. Then we can find $C\in\mathbb{C}$ with sufficiently large $|C|$ such that, for $s\in D_V$,

$$g_1(s)=\frac{p(s)+C}{A_1}\neq 0$$

and

$$g_2(s)=-\frac{C+A_3+\dots+A_r}{A_2}\neq 0.$$

If $g_3(s) = \cdots = g_r(s) = 1$, then $g_1, \ldots, g_r \in S_V$, and

$$\sum_{j=1}^{r} A_j g_j(s) = p(s).$$

This shows that

$$(u^{-1}\{p\}) \cap S_V^r \neq \varnothing.$$

Since $p(s)$ lies in G, hence,

$$(u^{-1}G) \cap S_V^r \neq \varnothing.$$

Therefore, there exists $g_1 \in S_V^r$ such that $g_1 \in u^{-1}G$, i.e., $u^{-1}G$ is an open neighbourhood of an element of the set S_V^r. By Lemma 1, the set S_V^r is the support of the measure $P_{\underline{L},V,h}$. Hence, $P_{\underline{L},V,h}(u^{-1}G) > 0$. Therefore,

$$P_{\underline{L},V,h} u^{-1}(G) = P_{\underline{L},V,h}(u^{-1}G) > 0.$$

Since g and G are arbitrary, this shows that the support of $P_{\underline{L},V,h} u^{-1}$ is the whole $H(D_V)$, and the same assertion is true for the random element $\zeta_2(s, \omega_h)$.

By the construction, $\{\omega_h(p) : p \in \mathbb{P}\}$ is a sequence of independent random variables. If $p \mid b$, then $p \mid qb$, thus, $\chi_j(p) = 0$. Hence,

$$L(s, \omega_h, \chi_j) = \prod_{p \nmid b} \left(1 - \frac{\omega_h(p)\chi_j(p)}{p^s}\right)^{-1}, \quad j = 1, \ldots, r.$$

From this, it follows that the random elements $\zeta_1(s, \omega_h)$ and $\zeta_2(s, \omega_h)$ are independent. Since the random element $\zeta_1(s, \omega_h)$ is not degenerated at zero, and the support of the random element is $H(D_V)$, we obtain that the support of the random element $\zeta_1(s, \omega_h)\zeta_2(s, \omega_h)$ is $H(D_V)$. The lemma is proved. ∎

Lemma 3. *Under hypotheses of Theorem 4 on a, b, q and the operator F,*

$$Q_{N,F,V,h}(A) \overset{def}{=} \frac{1}{N+1} \# \left\{0 \leqslant k \leqslant N : F\left(\zeta\left(s + ikh, \frac{a}{b}; \mathfrak{a}\right)\right) \in A\right\}, \qquad A \in \mathcal{B}(H(D_V)),$$

converges weakly to the measure $P_{\zeta,V,h}F^{-1}$ as $N \to \infty$. Moreover, the support of $P_{\zeta,V,h}F^{-1}$ is the whole of $H(D_V)$.

Proof. Since, for $A \in \mathcal{B}(H(D_V))$,

$$\begin{aligned} Q_{N,F,V,h}(A) &= \frac{1}{N+1} \# \left\{0 \leqslant k \leqslant N : \zeta\left(s + ikh, \frac{a}{b}; \mathfrak{a}\right) \in F^{-1}A\right\} = \\ &= Q_{N,V,h}(F^{-1}A) = Q_{N,V,h}F^{-1}(A), \end{aligned}$$

the continuity of F, Lemma 2 and Theorem 5.1 of [3] show that $Q_{N,F,V,h}$ converges weakly to $P_{\zeta,V,h}F^{-1}$ as $N \to \infty$.

Let $g \in H(D_V)$ be arbitrary , and let G be any open neighbourhood of g. The continuity of F shows that $F^{-1}G$ is an open set as well. By the hypothesis of Theorem 4, the set $F^{-1}G$ is non-empty, therefore, it is an open neighbourhood of a certain element $g_1 \in H(D_V)$. By the second part of Lemma 2, g_1 is an element of the support of the measure $P_{\zeta,V,h}$. Therefore,

$$P_{\zeta,V,h}F^{-1}(G) = P_{\zeta,V,h}(F^{-1}G) > 0.$$

Since g and G are arbitrary, this shows that the support of $P_{\zeta,V,h}$ is the whole $H(D_V)$. ■

Lemma 4. *Under hypotheses of Theorem* 5 *on* a, b, q *and the operator* F, *the support of the measure* $P_{\zeta,V,h}F^{-1}$ *includes the closure of the set* $H_{a_1,\dots,a_v}(D_V)$.

Proof. Since $H_{a_1,\dots,a_v}(D_V) \subset F(H(D_V))$, for each $g \in H_{a_1,\dots,a_r}(D_V)$, there exists an element $g_1 \in H(D_V)$ such that $F(g_1) = g$. Therefore, for every open neighbourhood G of the element g, we have that $F^{-1}G$ is an open neighbourhood of an element of $H(D_V)$. Hence, in view of Lemma 2, $P_{\zeta,V,h}(F^{-1}G) > 0$, and

$$P_{\zeta,V,h}F^{-1}(G) = P_{\zeta,V,h}(F^{-1}G) > 0.$$

This shows that g is an element of the support of the measure $P_{\zeta,V,h}F^{-1}$. Consequently, the set $H_{a_1,\dots,a_v}(D_V)$ and its closure lie in the support of $P_{\zeta,V,h}F^{-1}$. ■

4. Proofs of universality theorems

Proof of Theorem 2. By the Mergelyan theorem, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{4.1}$$

We take $V > 0$ such that $K \subset D_V$, and define the set

$$G = \left\{ g \in H(D_V) : \sup_{s\in K} |g(s) - p(s)| < \frac{\varepsilon}{2} \right\}.$$

By Lemma 2, G is an open neighbourhood of the element $p(s)$ of the support of the measure $P_{\zeta,V,h}$. Therefore,

$$P_{\zeta,V,h}(G) > 0. \tag{4.2}$$

Moreover, Lemma 2 together with the equivalent of weak convergence of probability measures in terms of open sets [3, Theorem 2.1], implies the inequality

$$\liminf_{N\to\infty} Q_{N,V,h}(G) \geqslant P_{\zeta,V,h}(G).$$

This, the definitions of $Q_{N,V,h}$ and G, (4.1) and (4.2) prove the theorem. ■

Proof of Theorem 3. We apply similar arguments to those of the proof of Theorem 2, however, in place of equivalent of weak convergence of probability measures in terms of open sets, we apply the equivalent in terms of continuity sets [3, Theorem 2.1]. We recall that the set $A \in B(H^r(D_V))$ is a continuity set of the measure $P_{\zeta,V,h}$ if $P_{\zeta,V,h}(\partial A) = 0$, where ∂A denotes the boundary of A.

Define the set

$$\hat{G}_\varepsilon = \left\{ g \in H(D_V) : \sup_{s\in K} |g(s) - f(s)| < \varepsilon \right\},$$

where $V > 0$ is such that $K \subset D_V$. Since $\partial\hat{G}_\varepsilon$ lies in the set

$$\left\{ g \in H(D_V) : \sup_{s\in K} |g(s) - f(s)| = \varepsilon \right\},$$

we have that $\partial\hat{G}_{\varepsilon_1} \cap \partial\hat{G}_{\varepsilon_2} = \varnothing$ for different positive ε_1 and ε_2. From this, it follows that the set $\hat{G}_\varepsilon$ is a continuity set of the measure $P_{\zeta,V,h}$ for all but at most countably many $\varepsilon > 0$. Then, in view of Lemma 2 and the equivalent of weak convergence of probability measures in terms of continuity sets, we obtain that the limit

$$\lim_{N\to\infty} Q_{N,V,h}(G_\varepsilon) = P_{\zeta,V,h}(G_\varepsilon). \tag{4.3}$$

exists for all but at most countably many $\varepsilon > 0$. Let G_ε be the same set as in the proof of Theorem 2. Then by (4.1), we find that $G_\varepsilon \subset \hat{G}_\varepsilon$. Thus, $P_{\zeta,V,h}(\hat{G}_\varepsilon) \geqslant P_{\zeta,V,h}(G_\varepsilon) > 0$, and the theorem follows from (4.3). ■

Proof of Theorem 4. The theorem is obtained in the same way as Theorem 2, by using Lemma 3. ■

Proof of Theorem 5. Let $v = 1$. By the Mergelyan theorem, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{4}. \tag{4.4}$$

Since $f(s) \neq a_1$, on K, we have that $p(s) \neq a_1$ on K as well if $\varepsilon > 0$ is small enough. Therefore, we can define a continuous branch of the logarithm by $\log(p(s)-a_1)$ which will be analytic in the interior of K. Applying the Mergelyan theorem once more, we find a polynomial $q(s)$ such that

$$\sup_{s\in K}\left|(p(s)-a_1)-e^{q(s)}\right|<\frac{\varepsilon}{4}. \tag{4.5}$$

Then the function $g_{a_1}(s)\overset{def}{=}a_1+e^{q(s)}\in H(D_V)$, and $g_{a_1}(s)\neq a_1$, where $V>0$ is chosen to satisfy $K\subset D_V$. By Lemma 4, $g_{a_1}(s)$ is an element of the support of the measure $P_{\zeta,V,h}F^{-1}$. Therefore, setting

$$G_{1,\varepsilon}=\left\{g\in H(D_V):\sup_{s\in K}|g(s)-g_{a_1}(s)|<\frac{\varepsilon}{2}\right\},$$

we have that $P_{\zeta,V,h}F^{-1}(G_{1,\varepsilon})>0$. Hence, be Lemma 4,

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\left\{0\leqslant k\leqslant N:\sup_{s\in K}\left|F\left(\zeta\left(s+ikh,\frac{a}{b};\mathfrak{a}\right)\right)-g_{a_1}(s)\right|<\frac{\varepsilon}{2}\right\}>0.$$

Combining this with (4.4) and (4.5) gives the assertion of the theorem.

Now let $v\geqslant 2$. Define

$$G_{2,\varepsilon}=\left\{g\in H(D_V):\sup_{s\in K}|g(s)-f(s)|<\varepsilon\right\}.$$

The function $f(s)$ belongs to the set $H_{a_1,\dots,a_v}(D_V)$. Therefore, by Lemma 4, $f(s)$ is an element of the support of $P_{\zeta,V,h}F^{-1}$. This implies $P_{\zeta,V,h}F^{-1}(G_{2,\varepsilon})>$ >0. Therefore, by Lemma 5,

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\left\{0\leqslant k\leqslant N:\sup_{s\in K}\left|F\left(\zeta\left(s+ikh,\frac{a}{b};\mathfrak{a}\right)\right)-f(s)\right|<\varepsilon\right\}\geqslant$$
$$\geqslant P_{\zeta,V,h}F^{-1}(G_{2,\varepsilon})>0.$$

The theorem is proved. ■

Obviously, Theorems 4 and 5 have modifications of type of Theorem 3.

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