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ON THE SUM OF THE RECIPROCALS OF THE MIDDLE PRIME FACTOR COUNTING MULTIPLICITY

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Abstract. We provide an estimate for the sum of the reciprocals of the middle prime factor counting multiplicity.

1. Introduction

Given an integer $n \geq 2$, write it as $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ where the primes p_j satisfy $p_1 < p_2 < \dots < p_k$ so that $k = \omega(n)$. For a real positive number $\beta < 1$, by letting $p^{\{\beta\}}(n) := p_{\max(1, \lfloor \beta(k+1) \rfloor)}$, the first author has shown in [6] that there exists a function $G(x, \beta)$ such that $G(x, \beta) - 1 = o(1)$ as $x \to \infty$ and

$$\sum_{1<n\leq x} \frac{1}{p^{\{\beta\}}(n)} = \frac{x}{\log x} \exp\left(\frac{(\log_2 x)^{1-\beta} (\log_3 x)^{\beta}}{\beta^{\beta} (1-\beta)^{1-2\beta}} \left(G(x,\beta) + O\left(\frac{1}{\log_3^2 x} \right) \right) \right).$$

Here and in what follows, $\log_k x$ denotes the k-th iterate of log evaluated at x, and we shall always assume that the input x is large enough so that the

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iterated logarithms are real and positive. For the case $\beta = 1$, we have that $p^{\{\beta\}}(n) = P(n)$, the largest prime factor of n, and it was studied by Erdős, Ivić and Pomerance [4]. For the case $\beta = 0$, $p^{\{\beta\}}(n) = p(n)$ the smallest prime factor of n. See [3, problem 9.6] for an estimate. Our goal is to extend the investigation for the middle prime factor to the case where the multiplicity of prime factors is taken into account. We write each integer n as $n = p_1 p_2 \cdots p_k$ with $p_1 \le p_2 \le \ldots \le p_k$, $k = \Omega(n)$ and we set $p^{(1/2)}(n) := p_{\lfloor \frac{k+1}{2} \rfloor}$. We prove the following.

Theorem 1.1. *As $x \to \infty$,*

$$\sum_{1<n\le x} \frac{1}{p^{(1/2)}(n)} = C \frac{x}{\sqrt{\log x}} \left(1 + O\left(\frac{1}{\log_2 x}\right)\right),$$

where $C := \dfrac{3}{2\Gamma(3/2)} \displaystyle\sum_p \frac{1}{p^2} g(p, 1/2) \prod_{3\le q\le p} \left(1 - \frac{2}{q}\right)^{-1}$ *and*

$g(p, 1/2) := \displaystyle\prod_{q<p} \left(1 - \frac{1}{2q}\right) \prod_q \left(1 - \frac{1}{2q}\right)^{-1} \left(1 - \frac{1}{q}\right)^{1/2}$ *where this last product runs over all primes q.*

2. The proof of Theorem 1.1

We have

$$\sum_{1<n\le x} \frac{1}{p^{(1/2)}(n)} = \sum_{p\le x} \frac{1}{p} \sum_{k\ge 1} \#\left\{1 < n \le x : \Omega(n) = k, p^{(1/2)}(n) = p\right\}. \tag{2.1}$$

In the case where $k \ge 3$, we write $k = 2k_0 + \delta$, where $k_0 := \left\lfloor \frac{k+1}{2} \right\rfloor$ and $\delta \in \{-1, 0\}$. Let $n = apb$, where $k = \Omega(n)$, $P(a) \le p$, $\Omega(a) = k_0 - 1$, $p = p^{(1/2)}(n)$, $p(b) \ge p$ and $\Omega(b) = k - k_0$. From here on, we define the sets $\mathcal{A} = \mathcal{A}_{p,k}$ and $\mathcal{B} = \mathcal{B}_{p,k}$ respectively by

$$\mathcal{A} = \{a \in \mathbb{N} : P(a) \le p, \Omega(a) = k_0 - 1\}, \mathcal{B} = \{b \in \mathbb{N} : p(b) \ge p, \Omega(b) = k - k_0\},$$

so that, from Mertens' theorem, Luca and Pomerance [5, Lem. 13] and (2.1), we have

$$\sum_{1<n\le x} \frac{1}{p^{(1/2)}(n)} = \sum_{p\le x} \frac{1}{p} \sum_{k\ge 3} \sum_{\substack{ab\le x/p \\ a\in\mathcal{A} \\ b\in\mathcal{B}}} 1 + O\left(\frac{x \log_2 x}{\log x}\right). \tag{2.2}$$

As De Koninck and Luca [2, eq. (2), eq. (3)] showed, we may also expect that integers n that have a middle prime factor larger than $\log x$ or that are such that $\Omega(n) \geq 10\log_2 x$ do not contribute much to (2.2), so that

$$\sum_{1<n\leq x} \frac{1}{p^{(1/2)}(n)} = \sum_{p\leq \log x} \frac{1}{p} \sum_{3\leq k\leq 10\log_2 x} \sum_{\substack{ab\leq x/p \\ a\in\mathcal{A} \\ b\in\mathcal{B}}} 1 + O\left(\frac{x\log_2 x}{\log x}\right). \tag{2.3}$$

Here and in what follows, we let $\mathcal{N}_{p,k}(x) := \sum_{\substack{ab\leq x/p \\ a\in\mathcal{A} \\ b\in\mathcal{B}}} 1 = \sum_{\substack{a\leq x/p \\ a\in\mathcal{A}}} \sum_{\substack{b\leq \frac{x}{ap} \\ b\in\mathcal{B}}} 1$. The following uses the same basic ideas as those presented in [6].

Proposition 2.1. *Uniformly for all real $r\in(0,3)$, integers $k\in[3, r\log_2 x]$ and primes $p\in[2,\log x]$, and setting $\lambda = k-k_0 = k - \left\lfloor \frac{k+1}{2}\right\rfloor$,*

$$\mathcal{N}_{p,k}(x) = \frac{x}{p\log x} \frac{g\left(p, \frac{\lambda-1}{\log_2 x}\right)}{\Gamma\left(1+\frac{\lambda-1}{\log_2 x}\right)} \frac{(\log_2 x)^{\lambda-1}}{(\lambda-1)!} \left(1 + O\left(\frac{(\log_2 p)^2}{\log_2 x}\right)\right) \sum_{\substack{a\leq x/p \\ a\in\mathcal{A}}} \frac{1}{a}.$$

Proof. First of all, one can easily see that $\lambda = k-k_0$ with $k_0 \leq R\log_2 x$, $0<R<2$. Moreover, since $p\leq \log x$ and $\Omega(a)\leq 3\log_2 x$, we have that $ap \leq (\log x)^{3\log_2 x+1}$ so that, by Alladi [1, thm. 6], we have

$$\sum_{\substack{b\leq \frac{x}{ap} \\ b\in\mathcal{B}}} 1 = \frac{x}{ap\log x} \frac{(\log_2 x)^{\lambda-1}}{(\lambda-1)!} \left(\frac{g\left(p, \frac{\lambda-1}{\log_2 x}\right)}{\Gamma\left(1+\frac{\lambda-1}{\log_2 x}\right)} + O\left(\frac{(\log_2 p)^2}{(\log p)^{\frac{\lambda-1}{\log_2 x}} \log_2 x}\right)\right) \times$$
$$\times \left(1 + O\left(\frac{(\log_2 x)^2}{\log x}\right)\right),$$

where $g(y,\mu) = \prod_{q<y}\left(1-\frac{\mu}{q}\right) \prod_{q} \left(1-\frac{\mu}{q}\right)^{-1} \left(1-\frac{1}{q}\right)^{\mu}$. Now, observe that $\Gamma\left(1+\frac{\lambda-1}{\log_2 x}\right) \asymp 1$ and that $g\left(p, \frac{\lambda-1}{\log_2 x}\right) \asymp (\log p)^{-\frac{\lambda-1}{\log_2 x}}$. Hence, it follows that

$$\sum_{\substack{b\leq \frac{x}{ap} \\ b\in\mathcal{B}}} 1 = \frac{x}{ap\log x} \frac{g\left(p, \frac{\lambda-1}{\log_2 x}\right)}{\Gamma\left(1+\frac{\lambda-1}{\log_2 x}\right)} \frac{(\log_2 x)^{\lambda-1}}{(\lambda-1)!} \left(1 + O\left(\frac{(\log_2 p)^2}{\log_2 x}\right)\right).$$

Using this estimate in $\mathcal{N}_{p,k}(x)$, we obtain the desired result. ■

The next part is to estimate the sum of the reciprocals for those integers $a \le x/p$ that are in the set $\mathcal{A}$.

Proposition 2.2. *Uniformly for* $0 < r < 3$, $3 \le k \le r \log_2 x$ *and* $2 \le p \le \log x$,

$$\sum_{\substack{a \le x/p \\ a \in \mathcal{A}}} \frac{1}{a} = \frac{1}{2^{k_0 - 1}} \prod_{3 \le q \le p} \left(1 - \frac{2}{q}\right)^{-1} + O\left(\frac{1}{3^{k_0 - 1}} \prod_{3 < q \le p} \left(1 - \frac{3}{q}\right)^{-1}\right).$$

Proof. We shall first obtain an estimate for the sum $\sum_{a \in \mathcal{A}} \frac{1}{a}$. Let $y = \pi(p)$, where $\pi(x)$ denotes the number of prime numbers not exceeding x. Then,

$$\sum_{a \in \mathcal{A}} \frac{1}{a} = \sum_{a_1 + a_2 + \cdots + a_y = k_0 - 1} \frac{1}{2^{a_1} 3^{a_2} 5^{a_3} \ldots p_y^{a_y}},$$

where p_j denotes the j-th prime number. Clearly,

$$\sum_{a \in \mathcal{A}} \frac{1}{a} = \frac{1}{2^{k_0 - 1}} \prod_{3 \le q \le p} \left(1 - \frac{2}{q}\right)^{-1} - \frac{1}{2^{k_0 - 1}} \sum_{\substack{\Omega(a) \ge k_0 \\ P(a) \le p \\ p(a) > 2}} \frac{2^{\Omega(n)}}{n}.$$

Proceeding in the same manner for this last sum, we have

$$\sum_{\substack{\Omega(a) \ge k_0 \\ P(a) \le p \\ p(a) > 2}} \frac{2^{\Omega(n)}}{n} \le \sum_{m \ge k_0} \left(\frac{2}{3}\right)^m \sum_{\substack{\Omega(a) \ge k_0 \\ P(a) \le p \\ p(a) > 3}} \frac{3^{\Omega(n)}}{n} \ll \left(\frac{2}{3}\right)^{k_0 - 1} \prod_{3 < q \le p} \left(1 - \frac{3}{q}\right)^{-1}.$$

Hence, we conclude that

$$\sum_{a \in \mathcal{A}} \frac{1}{a} = \frac{1}{2^{k_0 - 1}} \prod_{3 \le q \le p} \left(1 - \frac{2}{q}\right)^{-1} + O\left(\frac{1}{3^{k_0 - 1}} \prod_{3 < q \le p} \left(1 - \frac{3}{q}\right)^{-1}\right).$$

Since $p \le \log x$, $\Omega(a) \le 2 \log_2 x$ and $P(a) \le p$, we get $a \le (\log x)^{2 \log_2 x}$, so that $a \le x/p$ for any $a \in \mathcal{A}$, thus completing the proof of the proposition. ■

Combining the preceding two propositions with (2.3), we get for any fixed $r \in \left[\frac{2}{\log 2}, 3\right)$ that

$$\sum_{1<n\le x} \frac{1}{p^{(1/2)}(n)} = \frac{x}{\log x} \sum_{p\le \log x} \frac{1}{p^2} \prod_{3\le q\le p} \left(1-\frac{2}{q}\right)^{-1} \left(1+O\left(\frac{(\log_2 p)^2}{\log_2 x}\right)\right) \times \tag{2.4}$$

$$\times \sum_{3\le k\le r\log_2 x} \frac{1}{2^{k_0-1}} \frac{(\log_2 x)^{\lambda-1}}{(\lambda-1)!} \frac{g\left(p, \frac{\lambda-1}{\log_2 x}\right)}{\Gamma\left(1+\frac{\lambda-1}{\log_2 x}\right)} + E_1(x) + E_2(x) + E_3(x),$$

where

$$E_1(x) \ll \frac{x}{\log x} \sum_{p\le \log x} \frac{1}{p^2} \prod_{3<q\le p} \left(1-\frac{3}{q}\right)^{-1} \times$$

$$\times \sum_{3\le k\le r\log_2 x} \frac{1}{3^{k_0-1}} \frac{(\log_2 x)^{\lambda-1}}{(\lambda-1)!} \frac{g\left(p, \frac{\lambda-1}{\log_2 x}\right)}{\Gamma\left(1+\frac{\lambda-1}{\log_2 x}\right)},$$

$$E_2(x) \ll x \sum_{p\le \log x} \frac{1}{p^2} \prod_{3\le q\le p} \left(1-\frac{2}{q}\right)^{-1} \sum_{r\log_2 x<k\le 10\log_2 x} \frac{1}{2^{k_0-1}} +$$

$$+ x \sum_{p\le \log x} \frac{1}{p^2} \prod_{3<q\le p} \left(1-\frac{3}{q}\right)^{-1} \sum_{r\log_2 x<k\le 10\log_2 x} \frac{1}{3^{k_0-1}}$$

and $E_3(x) \ll \dfrac{x\log_2 x}{\log x}$. First observe that

$$E_2(x) \ll x \sum_{k>r\log_2 x} \frac{1}{2^{k_0-1}} \ll x \sum_{k>r\log_2 x} \frac{1}{2^{k/2}} \ll x 2^{-r\log_2 x/2} \ll \frac{x}{\log x}, \tag{2.5}$$

since $r \ge \frac{2}{\log 2}$. Noticing that $g\left(p, \frac{\lambda-1}{\log_2 x}\right) \ll \Gamma\left(1+\frac{\lambda-1}{\log_2 x}\right)$ uniformly for prime numbers $p \le \log x$ and integers $3 \le k \le r\log_2 x$, we obtain that

$$E_1(x) \ll \frac{x}{\log x} \sum_{3\le k\le r\log_2 x} \frac{1}{3^{k_0-1}} \frac{(\log_2 x)^{\lambda-1}}{(\lambda-1)!}.$$

When $k = 3$, we have $\lambda = k_0 = 1$, so that using Stirling's formula and the fact that $k_0 \asymp k/2 \asymp \lambda$, we get

$$E_1(x) \ll \frac{x}{\log x} \left(1 + \sum_{4\le k\le r\log_2 x} \left(\frac{e\log_2 x}{3(\lambda-1)}\right)^{\lambda-1}\right). \tag{2.6}$$

By definition, we have $\lambda = k - k_0 = k - \lfloor \frac{k+1}{2} \rfloor$, so that $\lambda \leq \frac{k}{2} + \frac{1}{2} \leq \frac{r}{2}\log_2 x + \frac{1}{2}$ and $\lambda \geq 2$. Each of those values of λ may appear at most twice in (2.6), so that, since the function f defined for $t \geq 1$ and $A > 1$ by $f(t) = \left(\frac{eA}{t}\right)^t$ is concave and reaches its global maximum when $t = A$, we obtain

$$E_1(x) \ll \frac{x}{\log x}\left(1 + \log_2 x \left(\frac{e \log_2 x}{3 \frac{\log_2 x}{3}}\right)^{\frac{\log_2 x}{3}}\right) \ll \frac{x}{(\log x)^{2/3}}. \tag{2.7}$$

Similarly, the error related to the term $O\left(\frac{(\log_2 p)^2}{\log_2 x}\right)$ in (2.4) is smaller than $C\frac{x}{\sqrt{\log x}}\frac{1}{\log_2 x}$ for a positive constant C and x large enough. Using this bound along with (2.5) and (2.7), we obtain that, for any fixed real number r such that $\frac{2}{\log 2} \leq r < 3$,

$$\sum_{1<n\leq x} \frac{1}{p^{(1/2)}(n)} = \frac{x}{\log x} \sum_{p \leq \log x} \frac{S}{p^2} \prod_{3 \leq q \leq p} \left(1 - \frac{2}{q}\right)^{-1} + O\left(\frac{x}{\sqrt{\log x}} \frac{1}{\log_2 x}\right), \tag{2.8}$$

where $S = S(x,p,r) := \displaystyle\sum_{3 \leq k \leq r \log_2 x} \frac{1}{2^{k_0 - 1}} \frac{(\log_2 x)^{\lambda - 1}}{(\lambda - 1)!} \frac{g\left(p, \frac{\lambda-1}{\log_2 x}\right)}{\Gamma\left(1 + \frac{\lambda - 1}{\log_2 x}\right)}$. To estimate S, it may be easier to write it over $\lambda - 1$.

Proposition 2.3. *For any fixed real number r such that $\frac{2}{\log 2} \leq r < 3$, we have*

$$S = \frac{3}{2} \sum_{1 \leq m \leq \lambda_1 - 2} \frac{1}{m!} \left(\frac{\log_2 x}{2}\right)^m \frac{g\left(p, \frac{m}{\log_2 x}\right)}{\Gamma\left(1 + \frac{m}{\log_2 x}\right)} + O(1),$$

where $\lambda_1 := \lfloor r \log_2 x \rfloor - \left\lfloor \frac{\lfloor r \log_2 x \rfloor + 1}{2} \right\rfloor$.

Proof. As noted before, since $\lambda = \lfloor \frac{k+1}{2} \rfloor$, each possible value of λ when $3 \leq k \leq r \log_2 x$ may be attained only once in the case where $k = 3$ and possibly the case $k = \lfloor r \log_2 x \rfloor$, and twice for every other values of λ. Define λ_1 to be the largest possible value for λ, so that

$$\lambda_1 := \lfloor r \log_2 x \rfloor - \left\lfloor \frac{\lfloor r \log_2 x \rfloor + 1}{2} \right\rfloor = \frac{r}{2} \log_2 x + O(1).$$

We have that $\lambda = 1$ when $k = 3$ and $\lambda \geq 2$ for $k \geq 4$. Hence, when $k = 3$, $\lambda = 1$ and $k_0 = 1$, so that the summand is equal to 1. When $k = \lfloor r \log_2 x \rfloor$, we have $k_0 - 1 \asymp \frac{r}{2}\log_2 x$, so that the summand is $\ll 1$ by Stirling's formula. Moreover, we have $\frac{1}{2^{k_0-1}} = 4\frac{1}{2^k}2^{\lambda-1}$. Since

$$\lambda = k - k_0 \iff \lambda = k - \left\lfloor \frac{k+1}{2} \right\rfloor \iff k = 2\lambda \text{ or } k = 2\lambda+1,$$

it follows that

$$S = 4\sum_{2\le\lambda<\lambda_1} \frac{(2\log_2 x)^{\lambda-1}}{(\lambda-1)!}\frac{g\left(p, \frac{\lambda-1}{\log_2 x}\right)}{\Gamma\left(1+\frac{\lambda-1}{\log_2 x}\right)}\left(\frac{1}{2^{2\lambda}}+\frac{1}{2^{2\lambda+1}}\right)+O(1).$$

Hence, by setting $m = \lambda - 1$, we obtain the desired result. ■

To estimate the above sum, we start by omitting the g and Γ functions.

Proposition 2.4. *As* $x \to \infty$,

$$T := \sum_{1\le m\le\lambda_1-2} \frac{1}{m!}\left(\frac{\log_2 x}{2}\right)^m = \sqrt{\log x}\left(1+O\left(\frac{1}{\sqrt{\log x}\,\log_2 x}\right)\right).$$

Note that this estimate alone does not directly tell us how to estimate the sum S.

Proposition 2.5. *Uniformly for* $2 \le p \le \log x$,

$$S = \frac{3}{2}\frac{g(p,1/2)}{\Gamma(3/2)}\sqrt{\log x}\left(1+O\left(\left(\sum_{q<p}\frac{1}{2q-1}\right)^2\frac{1}{\log_2 x}\right)\right).$$

Proof. For $A > 10$ and $t \ge 1$, consider the function $f(t) := \frac{1}{\sqrt{t}}\left(\frac{eA}{t}\right)^t$. This function is concave and reaches its global maximum at t_0, where

$$(2.9)\quad t_0 = \frac{\log_2 x}{2} - \frac{1}{2} + O\left(\frac{1}{\log_2 x}\right) \quad\text{and}\quad t_0(\log t_0 - \log_3 x + \log 2) = -\frac{1}{2},$$

so that

$$(2.10)\qquad f(t_0) = \sqrt{\frac{2\log x}{\log_2 x}}\left(1+O\left(\frac{1}{\log_2 x}\right)\right).$$

We will use that global maximum to estimate S. Let $\xi = (\log_2 x)^{-3/8}$ and write

$$T = \sum_{1\le m\le t_0(1-\xi)}\frac{1}{m!}\left(\frac{\log_2 x}{2}\right)^m + \sum_{t_0(1+\xi)<m\le\lambda_1-2}\frac{1}{m!}\left(\frac{\log_2 x}{2}\right)^m + \\ + \sum_{t_0(1-\xi)<m\le t_0(1+\xi)}\frac{1}{m!}\left(\frac{\log_2 x}{2}\right)^m =: T_1+T_2+T_3.$$

From Stirling's formula, we have

$$T_1 \ll \sum_{1\le m\le t_0(1-\xi)} f(m) \ll t_0 f\left(t_0\left(1-\xi\right)\right) \ll f\left(t_0\left(1-\xi\right)\right)\log_2 x$$

and

$$T_2 \ll \sum_{t_0(1+\xi)<m\le \lambda_1-2} f(m) \ll f\left(t_0\left(1+\xi\right)\right)\log_2 x.$$

Notice that we have, for any $v\in[-\xi,\xi]$, $f\left(t_0\left(1+v\right)\right)\asymp f\left(t_0\right)\exp\left(-\frac{t_0v^2}{2}\right)$. Hence, from (2.10), we have

$$T_1 \ll f\left(t_0\right)\log_2 x \exp\left(-\frac{\left(\log_2 x\right)^{1/4}}{2}\right) = o\left(\frac{1}{\sqrt{\log_2 x}} f\left(t_0\right)\right) \ll \frac{\sqrt{\log x}}{\log_2 x}$$

and $T_2 = o\left(\frac{\sqrt{\log x}}{\log_2 x}\right)$. Moreover, from Proposition 2.4, we conclude that

$$T_3 = \sqrt{\log x}\left(1+O\left(\frac{1}{\log_2 x}\right)\right).$$

Observe that the integers m that satisfy $\left(1-\xi\right)t_0 < m \le \left(1+\xi\right)t_0$ are the main contributors to the sum T. Write $S = S_1+S_2+S_3+O(1)$ with

$$S_j := \frac{3}{2}\sum_{m\in I_j}\frac{1}{m!}\left(\frac{\log_2 x}{2}\right)^m \frac{g\left(p,\frac{m}{\log_2 x}\right)}{\Gamma\left(1+\frac{m}{\log_2 x}\right)} \qquad \left(j=1,2,3\right),$$

where the intervals I_j are defined by $I_1 = [1,\left(1-\xi\right)t_0)$, $I_2 = (\left(1+\xi\right)t_0,\lambda_1-2]$ and $I_3 = (\left(1-\xi\right)t_0,\left(1+\xi\right)t_0]$. To estimate S_1 and S_3, we may use the fact that $g\left(p,\frac{m}{\log_2 x}\right) \ll \Gamma\left(1+\frac{m}{\log_2 x}\right)$ which gives, as $x\to\infty$ and uniformly for $2\le p\le \log x$,

$$S_1 \ll T_1 = o\left(\frac{\sqrt{\log x}}{\log_2 x}\right) \quad \text{and} \quad S_2 \ll T_2 = o\left(\frac{\sqrt{\log x}}{\log_2 x}\right),$$

so that

$$S = S_3 + o\left(\frac{\sqrt{\log x}}{\log_2 x}\right). \tag{2.11}$$

For S_3, note that, from (2.9), $\frac{m}{\log_2 x} = \frac{t_0(1+v)}{\log_2 x} = \frac{1}{2}+O\left(\xi\right)$, where $-\xi<v\le\xi$. Moreover $\xi\log_2 p \le \left(\log_2 x\right)^{-3/8}\log_3 x = o(1)$ as $x\to\infty$. Hence, there exists a constants K_1 such that, uniformly for $2\le p\le\log x$ and $m\in I_3$,

$$\frac{g\left(p,\frac{m}{\log_2 x}\right)}{g\left(p,\frac{1}{2}\right)} = 1+\Delta\,\mathcal{K}+\frac{\Delta^2}{2}\mathcal{K}^2+O\left(\mathcal{K}\frac{1}{\log_2 x}\right), \tag{2.12}$$

where $\Delta := \frac{m}{\log_2 x} - \frac{1}{2}$ and $\mathcal{K} := K_1 - \sum_{q<p} \frac{2}{2q-1}$. On the other hand, we have from (2.9) that there exist constant D_1 and D_2 such that

$$\frac{1}{\Gamma\left(1+\frac{m}{\log_2 x}\right)} = \frac{1}{\Gamma(3/2)}\left(1 + D_1\Delta + D_2\Delta^2 + O\left(\frac{1}{\log_2 x}\right)\right). \tag{2.13}$$

Combining (2.12) and (2.13) in S_3, we get

$$S_3 = \frac{3}{2}\frac{g(p,1/2)}{\Gamma(3/2)} T_3\left(1 + O\left(\sum_{q<p}\frac{1}{2q-1}\frac{1}{\log_2 x}\right)\right) + S_4 + S_5,$$

where

$$S_4 := \frac{3g(p,1/2)}{2\Gamma(3/2)}(D_1 + \mathcal{K})\sum_{m\in I_3}\frac{1}{m!}\left(\frac{\log_2 x}{2}\right)^m\left(\frac{m}{\log_2 x} - \frac{1}{2}\right)$$

and

$$S_5 := \frac{3g(p,1/2)}{2\Gamma(3/2)}\left(D_2 + \mathcal{K}\left(D_1 + \frac{\mathcal{K}}{2}\right)\right)\sum_{m\in I_3}\frac{1}{m!}\left(\frac{\log_2 x}{2}\right)^m\left(\frac{m}{\log_2 x} - \frac{1}{2}\right)^2.$$

For S_4 and S_5, we have to estimate $\frac{1}{2}\sum_{m\in I_3}\frac{1}{(m-j)!}\left(\frac{\log_2 x}{2}\right)^{m-j}$, where $j=1$ or $j=2$. As with T_3,

$$\frac{1}{2}\sum_{m\in I_3}\frac{1}{(m-j)!}\left(\frac{\log_2 x}{2}\right)^{m-j} = \frac{\sqrt{\log x}}{2}\left(1 + O\left(\frac{1}{\log_2 x}\right)\right).$$

Hence, uniformly for $2 \le p \le \log x$,

$$\frac{S_4}{g(p,1/2)} \ll \sum_{q<p}\frac{1}{2q-1}\frac{\sqrt{\log x}}{\log_2 x}.$$

For S_5, we have

$$\frac{S_5}{\frac{3g(p,1/2)}{2\Gamma(3/2)}} = \left(D_2 + \mathcal{K}\left(D_1 + \frac{\mathcal{K}}{2}\right)\right)\times$$
$$\times\left(\frac{1}{4}\sum_{m\in I_3}\frac{1}{(m-2)!}\left(\frac{\log_2 x}{2}\right)^{m-2} - \frac{1}{2}\sum_{m\in I_3}\frac{1}{(m-1)!}\left(\frac{\log_2 x}{2}\right)^{m-1} + \right.$$
$$\left. + \frac{1}{4}T_3\right),$$

so that we get

$$\frac{S_5}{g(p,1/2)} \ll \left(\sum_{q<p} \frac{1}{2q-1}\right)^2 \frac{\sqrt{\log x}}{\log_2 x}.$$

Hence, from (2.11), we obtain

$$S = \frac{3}{2}\frac{g(p,1/2)}{\Gamma(3/2)}\sqrt{\log x}\left(1 + O\left(\left(\sum_{q<p}\frac{1}{2q-1}\right)^2 \frac{1}{\log_2 x}\right)\right),$$

which completes the proof of Proposition 2.5. ∎

Using this last estimate for S in (2.8), we get

$$\sum_{1<n\le x} \frac{1}{p^{(1/2)}(n)} = \frac{3}{2\Gamma(3/2)}\frac{x}{\sqrt{\log x}} \sum_{p\le \log x} \frac{1}{p^2} \prod_{3\le q\le p}\left(1-\frac{2}{q}\right)^{-1} g(p,1/2) \times$$
$$\times \left(1 + O\left(\left(\sum_{q<p}\frac{1}{2q-1}\right)^2 \frac{1}{\log_2 x}\right)\right). \tag{2.14}$$

What is left to estimate is the sum over the primes $p \le \log x$. Since

$$\prod_{3\le q\le p}\left(1-\frac{2}{q}\right)^{-1} = \exp\left(-\sum_{3\le q\le p}\log\left(1-\frac{2}{q}\right)\right) \asymp (\log p)^2,$$

$g(p,1/2) \asymp \frac{1}{\sqrt{\log p}}$ and $\sum_{q<p} \frac{1}{2q-1} \asymp \log_2 p$, we obtain that

$$\sum_{p>\log x} \frac{1}{p^2} \prod_{3\le q\le p}\left(1-\frac{2}{q}\right)^{-1} g(p,1/2)\left(1 + O\left(\frac{(\log_2 p)^2}{\log_2 x}\right)\right) \ll \frac{1}{\sqrt{\log x}}.$$

Hence, it follows from (2.14) that

$$\sum_{1<n\le x}\frac{1}{p^{(1/2)}(n)} = C\frac{x}{\sqrt{\log x}}\left(1 + O\left(\frac{1}{\log_2 x}\right)\right),$$

where

$$C := \frac{3}{2\Gamma(3/2)}\sum_{p}\frac{1}{p^2} g(p,1/2)\prod_{3\le q\le p}\left(1-\frac{2}{q}\right)^{-1},$$

which completes the proof of Theorem 1.1. ∎

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