Annales Univ. Sci. Budapest., Sect. Comp. 47 (2018) 239–248

CONVOLUTION SQUARE ROOT OF 1 AND THE PRIME NUMBER THEOREM

Harold G. Diamond (Urbana, Illinois, USA)

Communicated by Imre Kátai

(Received February 22, 2018; accepted March 6, 2018)

Abstract. Define $1^{*1/2}$, the *convolution square root of* 1, as the arithmetic function satisfying $1^{*1/2} * 1^{*1/2} = 1$ with $1^{*1/2}(1) = 1$. We discuss properties of this function and show how its summatory function is connected with the Prime Number Theorem.

1. Introduction

Let $*$ denote multiplicative convolution of arithmetic functions and let $1^{*1/2}$, which we call the *convolution square root of* 1, denote the solution of $1^{*1/2} * 1^{*1/2} = 1$ with $1^{*1/2}(1) = +1$. Alternatively, $1^{*1/2}$ is the arithmetic function whose generating function is the positive branch of the square root of the Riemann zeta function.

Analogously with the Dirichlet divisor problem approximation

$$\sum_{n\le x} 1 * 1(n) = x\log x + (2\gamma - 1)x + \cdots$$

($\gamma =$ Euler's constant), it is known [2], [4] that $N_{1/2}(x)$, the summatory function of $1^{*1/2}$, has the asymptotic expansion

$$\sum_{n\le x} 1^{*1/2}(n) = \frac{\pi^{-1/2}x}{\log^{1/2} x} + \frac{a_1 x}{\log^{3/2} x} + \frac{a_2 x}{\log^{5/2} x} + \cdots .$$

2010 Mathematics Subject Classification: Primary 11N05, 11N60, 11A41.

Key words and phrases: Arithmetic function, summatory function, convolution, square root, Prime Number Theorem.

https://doi.org/10.71352/ac.47.239

Our main result in this survey is an elementary proof of the Prime Number Theorem (PNT) under the assumption that we know the first two terms of the asymptotic expansion of $N_{1/2}(x)$. Also, we show a sense in which our assumption is minimal.

2. Main result and underlying idea

Theorem 2.1. *The condition*

$$N_{1/2}(x) := \sum_{n\le x} 1^{*1/2}(n) \;=\; \frac{\pi^{-1/2}\,x}{\log^{1/2} x} + \frac{k\,x}{\log^{3/2} x} + o\Big(\frac{x}{\log^{3/2} x}\Big), \tag{2.1}$$

for some constant k, implies the Prime Number Theorem.

(We do not need to know the value of k, as it gets washed out.)

We show that the first term of the expansion for $N_{1/2}(x)$ is necessary for the PNT. Could we have established the theorem using only the one term? We show by an example from Beurling generalized numbers [3] that we cannot prove the PNT without the second term of (2.1).

Our argument uses an analog of the Chebyshev identity for primes involving $1^{*1/2}$ and $\mu^{*1/2}$, the convolution inverse of $1^{*1/2}$ (see below). Our secret sauce is having a good elementary estimate for $\sum |\mu^{*1/2}|$.

3. Properties of $1^{*1/2}$ and $\mu^{*1/2}$

The function $1^{*1/2}$ is multiplicative and positive valued. One way of seeing this [1, §2.4] is to note that $1 = \exp\lambda$, with λ a function that is positive on prime powers and zero elsewhere and $\exp$ the exponential series with $*$ as multiplication. By homomorphic properties of $\exp$ analogous to ones on $\mathbb{R}$ we have $1^{*1/2} = \exp\{\lambda/2\}$. Now $\lambda/2$ has the same support and positivity as λ, so the assertions follow.

More classically, the generating function of $1^{*1/2}$ is the branch of $\zeta(s)^{1/2}$ that is positive on the real half-line $\{s > 1\}$. Starting with the Euler product for zeta, we find

$$\zeta(s)^{1/2} = \prod_p \left(1 - p^{-s}\right)^{-1/2} = \prod_p \sum_{\nu=0}^{\infty} \binom{-1/2}{\nu} (-1)^\nu p^{-\nu s}.$$

Since $\zeta(s)^{1/2}$ has an Euler product, $1^{*1/2}$ is multiplicative with

$$1^{*1/2}(p^\nu) = (-1)^\nu \binom{-1/2}{\nu} = \frac{1}{\nu!} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdots \frac{2\nu-1}{2} \text{ for } \nu \geq 1,$$

and so $1^{*1/2}$ is positive valued.

We have $1^{*1/2}(1) = 1 \neq 0$; thus $1^{*1/2}$ has a convolution inverse, which we call $\mu^{*1/2}$, i.e. $1^{*1/2} * \mu^{*1/2} = \delta$. Here $\delta(1) = 1$ and $\delta(n) = 0$ for all $n > 1$; δ is the identity for multiplicative convolution. It is easy to see that $\mu^{*1/2} * \mu^{*1/2} = \mu$, the Moebius function.

The generating function of $\mu^{*1/2}$ is

$$\zeta(s)^{-1/2} = \prod_p \left(1 - p^{-s}\right)^{1/2} = \prod_p \sum_{\nu=0}^{\infty} \binom{1/2}{\nu} (-1)^\nu p^{-\nu s}.$$

Thus $\mu^{*1/2}$ also is multiplicative, and since

$$\left|\binom{1/2}{\nu}\right| = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdots \frac{2\nu-3}{2} \leq (-1)^\nu \binom{-1/2}{\nu}, \ \nu \geq 2,$$

and the functions have equal absolute values for $\nu = 0, 1$, we have

(3.1) $$\left|\mu^{*1/2}(n)\right| \leq 1^{*1/2}(n), \quad n \geq 1.$$

4. Proof of the theorem

Define an operator L on arithmetic functions by $\mathrm{L}f(n) = f(n)\log n$. L is a derivation, so $\mathrm{L}(f * f) = 2f * \mathrm{L}f$. And, as usual, let Λ denote von Mangoldt's function. We show

(4.1) $$\Lambda = 2\,\mathrm{L}1^{*1/2} * \mu^{*1/2}.$$

Starting with $\mathrm{L}1 = \Lambda * 1$, Chebyshev's familiar identity for primes, and inserting $1^{*1/2} * 1^{*1/2}$ in place of 1, we find

$$2\,\mathrm{L}1^{*1/2} * 1^{*1/2} = \mathrm{L}\big(1^{*1/2} * 1^{*1/2}\big) = \Lambda * 1^{*1/2} * 1^{*1/2}.$$

Now convolve both sides by $\mu^{*1/2} * \mu^{*1/2}$ to get (4.1); then summing we obtain a Chebyshev-type formula

$$\psi(x) := \sum_{n \leq x} \Lambda(n) = \sum_{n \leq x} \big(2\,\mathrm{L}1^{*1/2} * \mu^{*1/2}\big)(n).$$

Our goal is to show $\psi(x) \sim x$.

Rewrite the ψ identity, converting the convolution sum first into a double sum over a region beneath a rectangular hyperbola and then into an iterated summation:

$$\psi(x) = 2\sum_{ij\le x} \mathrm{L}1^{*1/2}(i)\,\mu^{*1/2}(j) =$$

$$= 2\sum_{n\le x}\Big\{\sum_{i\le x/n} \mathrm{L}1^{*1/2}(i)\Big\}\mu^{*1/2}(n)\,. \tag{4.2}$$

In what follows, we shall apply the condition (2.1) to show that

$$2\sum_{i\le x/n} \mathrm{L}1^{*1/2}(i) \approx \sum_{k\le x/n} \big\{(1 + c'1^{*1/2} + c''\delta_1) * 1^{*1/2}\big\}(k). \tag{4.3}$$

The $1^{*1/2}$ factor in (4.3) will kill off $\mu^{*1/2}$ in (4.2) and we get

$$\psi(x) = \sum_{n\le x}\big\{1(n) + c'1^{*1/2}(n) + c''\delta(n)\big\} + o(x) \sim x$$

– assuming that the errors we made in "$\approx$" are small enough.

To make things honest, we calculate each side of (4.3) explicitly. First, summation by parts using (2.1) yields

$$\sum_{n\le x} 1^{*1/2}(n)\,\log n = \frac{x\log^{1/2} x}{\sqrt{\pi}} + \frac{c'x}{\log^{1/2} x} + o\Big(\frac{x}{\log^{1/2} x}\Big)\,.$$

For the main term of the approximation, instead of $\sum 1^{*1/2} * 1$, it is convenient to use in the sequel the related expression

$$I := \int_1^x dN_{1/2} * dt = \iint_{st\le x} dN_{1/2}(s)\,dt = \int_1^x N_{1/2}(x/t)\,dt.$$

By another application of (2.1) and easy estimates we find

$$I = \frac{2}{\sqrt{\pi}}\,x\log^{1/2} x + c''x + \frac{c'''x}{\log^{1/2} x} + o\Big(\frac{x}{\log^{1/2} x}\Big)\,.$$

Also, trivially,

$$\sum_{n\le x}(1^{*1/2} * 1^{*1/2})(n) = \sum_{n\le x} 1(n) = \lfloor x\rfloor,$$

and

$$\sum_{n\le x} 1^{*1/2}(n) = \frac{x}{\sqrt{\pi}\log^{1/2} x} + o\Big(\frac{x}{\log^{1/2} x}\Big).$$

Thus, with suitable choices of c_2, c_3, we have

$$\Delta(x) := 2\sum_{n\le x} \mathrm{L}1^{*1/2}(n) - \int_1^x dN_{1/2} * dt - c_2\lfloor x\rfloor - c_3 N_{1/2}(x) = \\ = o\Big(\frac{x}{\log^{1/2} x}\Big).$$

From the formula for ψ, the estimate for Δ, and the relation $1^{*1/2} * \mu^{*1/2} = \delta$ we find

$$\psi(x) = \sum_{n\le x} (2\,\mathrm{L}1^{*1/2} * \mu^{*1/2})(n) = \\ = \int_1^x dt \;+\; c_2 N_{1/2}(x) + c_3 + E(x) \;=\; x + o(x) + E(x),$$

where, by the Δ estimate and (3.1),

$$E(x) := \sum_1^x \Delta(x/n)\,\mu^{*1/2}(n) \ll \sum_1^x o\Big(\frac{x/n}{\log^{1/2} x/n}\Big)\,1^{*1/2}(n).$$

Using the bound for $N_{1/2}$ again and making one further simple estimate, we find $E(x) = o(x)$, and hence $\psi(x) \sim x$. ■

Remark 4.1. Could we have proved the PNT by applying the preceding argument directly to the classical Chebyshev formula? For $\sum 1$ we have an exact value, which would make the proof unconditional, and further, we can accurately approximate $\sum \mathrm{L}1$ by an expression $\sum(1*1+k\,1)$. We would have

$$\psi(x) = \sum_{n\le x}\Big\{\sum_{m\le x/n} \mathrm{L}1(m)\Big\}\,\mu(n) \approx \\ \approx \sum_{n\le x}\Big\{\sum_{m\le x/n} (1*1+k\,1)(m)\Big\}\,\mu(n) = x + O(1).$$

Unfortunately, we cannot justify the preceding approximation: *a priori* we know only $\sum_{n\le x}\mu(n) \ll x$, which would give rise to an $O(x)$ error term for $\psi(x)$; our PNT proof succeeded by using the better bound

$$\sum_{n\le x} \big|\mu^{*1/2}(n)\big| \;\le\; \sum_{n\le x} 1^{*1/2}(n) \ll x\log^{-1/2} x.$$

5. Estimate of $\sum 1^{*1/2}$

We establish the approximation for $N_{1/2}$ by using Perron's formula [1, §7.5], [5, §5.1], [6, §II.2], assuming a modest zero-free region for $\zeta(s)$, the Riemann zeta function. For non-integral x,

$$N_{1/2}(x) = \frac{1}{2\pi i} \int_{\mathcal{C}} x^s \zeta(s)^{1/2} ds/s\,, \tag{5.1}$$

with $\mathcal{C}$ a vertical line to the right of 1 in $\mathbb{C}$. Of course, with this technology, our proof of the PNT ceases to be elementary.

The main contribution to the Perron integral arises from the half-order pole of $\zeta(s)^{1/2}$ at $s = 1$. We evaluate the integral by deforming the contour $\mathcal{C}$ to extend a bit to the left of the line $\{\Re s = 1\}$, with a loop taken about $s = 1$ to avoid crossing the half-line $\{s = \sigma \le 1\}$. Applying the Hankel loop integral formula [5, Appendix C3], [6, §II.5] we obtain (2.1) in the form

$$N_{1/2}(x) = \frac{x}{\pi^{1/2} \log^{1/2} x} + \frac{(2-\gamma)\,x}{4\pi^{1/2} \log^{3/2} x} + o\Big(\frac{x}{\log^{3/2} x}\Big).$$

Remark 5.1. Usual proofs of the PNT contain a tauberian element. Fundamental to our method is the identity

$$\sum_{n \le x} 1^{*1/2} * 1^{*1/2}(n) = \lfloor x \rfloor,$$

an entanglement of two copies of $1^{*1/2}$. Our tauberian step is to "undo" the convolution to obtain the asymptotic formula (2.1) for $N_{1/2}(x)$.

Could we have established (2.1) by a reasonably simple elementary argument? This seems unlikely. Among other reasons, as we have just seen, the formula leads to an easy proof of the PNT.

6. Suitability of our hypothesis

To conclude, we consider the relation of the terms of (2.1) with the PNT. It is convenient to speak in the context of a Beurling generalized (g-) number system $\mathcal{N}$ [3]; *a fortiori* our statements hold for the rational integers. We assume that the counting function of $\mathcal{N}$ satisfies

$$N(x) = \alpha x + O(x/\log^2 x).$$

We call α the density of $\mathcal{N}$; in the classical case $N(x) = \lfloor x \rfloor$ and $\alpha = 1$. With $*$ denoting multiplicative convolution of measures on $[1, \infty)$, define $N_{1/2}$ by

$$dN = dN_{1/2} * dN_{1/2}, \quad dN_{1/2}\{1\} = +1.$$

The first term of (2.1) *follows from the truth of the PNT.*

By the PNT, the zeta function (defined as $\int_{1-}^{\infty} x^{-s} dN(x)$ on $\{\sigma > 1\}$) has a continuation as a nonzero function on the closed half plane $H = \{\sigma \geq 1\}$. The function $(s-1)\,\zeta(s)$ is analytic on the interior of H and continuous on H, and $\phi(s) := \sqrt{(s-1)\,\zeta(s)}$ inherits the same properties with $\phi(1) = \alpha^{1/2}$. For $\sigma > 1$ we have

$$\zeta(s)^{1/2} = \int_{1-}^{\infty} x^{-s} dN_{1/2}(x) = \frac{1}{\sqrt{s-1}}\, \phi(s),$$

and so, by a generalization of the Wiener-Ikehara theorem of Delange [1, §7.4],

$$N_{1/2}(x) \sim \frac{\phi(1)\, x}{\Gamma(1/2)\, \log^{1/2} x} = \frac{\sqrt{\alpha}\, x}{\sqrt{\pi \log x}}. \quad \blacksquare$$

Without the second term in (2.1) *the PNT can fail.*

We show this via the following (continuous!) g-number example, based on Example 13.8 of [3]. Define a "wobbly g-prime counting function" by

$$\pi_w(x) := \int_1^x \frac{1 - \cos(\log t)}{\log t}\, dt, \quad x \geq 1, \tag{6.1}$$

and the corresponding weighted prime power counting function by

$$\Pi_w(x) := \pi_w(x) + \Pi_B(x) \tag{6.2}$$

with

$$\Pi_B(x) := \frac{1}{2}\pi_w(x^{1/2}) + \frac{1}{3}\pi_w(x^{1/3}) + \cdots.$$

From the identity [1, Lemma 8.13], [5, §6.2, Problem 23]

$$\mathrm{li}(x) := \mathrm{P.V.}\int_0^x \frac{dt}{\log t} = \int_1^x \frac{1 - t^{-1}}{\log t}\, dt + \log\log x + \gamma$$

(with P.V. the principal value integral) and a small calculation we see that

$$\pi_w(x)/\mathrm{li}(x) = 1 - \sin(\log x + \pi/4)/\sqrt{2} + o(1), \quad x \to \infty;$$

clearly, the PNT does not hold for π_w.

Let N_w denote the counting function of "integers" generated by π_w. The zeta function for N_w satisfies

$$\zeta_w(s) := \int_{1-}^{\infty} x^{-s} dN_w(x) = \exp\Big\{ \int_1^{\infty} x^{-s} d\Pi_w(x) \Big\} = \zeta_A(s)\,\zeta_B(s),$$

where

$$\zeta_A(s) = \exp\Big\{ \int_1^{\infty} x^{-s} d\pi_w(x) \Big\},$$

$$\zeta_B(s) = \exp\Big\{ \int_1^{\infty} x^{-s} d\Pi_B(x) \Big\} = \zeta_A(2s)^{1/2}\,\zeta_A(3s)^{1/3} \cdots .$$

We claim that

$$\zeta_A(s) = \frac{\sqrt{(s-1)^2+1}}{s-1}. \tag{6.3}$$

To see this write

$$\log \zeta_A(s) = \int_1^{\infty} x^{-s}\Big(1 - \frac{1}{2}\big(x^i + x^{-i}\big)\Big)\frac{dx}{\log x}$$

and differentiate this formula with respect to s. The resulting integral is easy to evaluate; we find

$$\zeta'(s)/\zeta(s) = \frac{-1}{s-1} + \frac{1/2}{s-1-i} + \frac{1/2}{s-1+i}.$$

One further integration yields (6.3).

Also, ζ_B is analytic and nonzero on the half plane $\{\Re s > 1/2\}$, because $\Pi_B(x) \ll x^{1/2}$ and ζ_B is an exponential. It follows that

$$\zeta_w(s)^{1/2} = (s-1)^{-1/2}\,\{(s-1)^2+1\}^{1/4}\,\zeta_B(s)^{1/2},$$

with cuts along lines from $-\infty$ to 1, $1+i$, and $1-i$.

For application of the Perron formula, we show that

$$\zeta_w(\sigma+it)^{1/2}=1+O(1/t^2)\ \text{ on }\ \{1/2\le\sigma\le 2,\ |t|\ge 2\}.$$

To see this, for $n=1,\,2,\,\ldots,$ rewrite (6.3) as

$$\zeta_A(ns)^{1/2n}=\exp\left\{\frac{1}{2n}\log\left(1+\frac{1}{(ns-1)\{\sqrt{(ns-1)^2+1}+(ns-1)}\right)\right\}.$$

Another small calculation shows that

$$\zeta_A(ns)^{1/2n}=1+O\left(\frac{1}{n^3t^2}\right)$$

holds uniformly in the stated regions. Multiplying together these estimates gives the claimed formula for $\zeta_w(s)^{1/2}$.

Now apply the Perron formula

$$N_{1/2}(x)=\frac{1}{2\pi i}\int_{\mathcal{C}} x^s(s-1)^{-1/2}\,\{(s-1)^2+1\}^{1/4}\,\zeta_B(s)^{1/2}\,\frac{ds}{s}\,,$$

with the contour again shifted, this time to the vertical line $\{\sigma=2/3\}$, but with loops taken about 1 and $1\pm i$. By three further applications of the Hankel formula, we find

$$N_{1/2}(x)=\frac{c\,x}{\log^{1/2}x}+bx\cos(\log x+\theta)\log^{-5/4}x+o\big(x\log^{-5/4}x\big)$$

with real constants c, b, and θ. Note that $x\log^{-3/2}x=o(x\log^{-5/4}x)$, so the second term in (2.1) does not hold. ■

References

[1] **Bateman, P.T. and H.G. Diamond,** *Analytic Number Theory. An Introductory Course*, World Scientific, Singapore, 2004. Reprinted, with minor changes, in Monographs in Number Theory, Vol. 1, 2009.

[2] **Diamond, H.G.,** Interpolation of the Dirichlet divisor problem, *Acta Arith.*, **XIII** (1967), 151–168.

[3] **Diamond, H.G. and W.B. Zhang,** *Beurling Generalized Numbers*, Math. Surveys and Monographs, 213. American Mathematical Society, Providence, RI, 2016.

[4] **Dixon, R.D.,** On a generalized divisor problem, *J. Indian Math. Soc. N.S.*, **28** (1964), 187–195.

[5] **Montgomery, H.L. and R.C. Vaughan,** *Multiplicative Number Theory I., Classical Theory*, Cambridge studies in advanced math. 97, University Press, Cambridge, 2007.

[6] **Tenenbaum, G.,** *Introduction to Analytic and Probabilistic Number Theory*, Cambridge studies in advanced math. 46, University Press, Cambridge, 1995.

H.G. Diamond
University of Illinois
Urbana IL 61801 U.S.A.
hdiamond@illinois.edu