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SEEING THE INVISIBLE: AROUND GENERALIZED KUBERT FUNCTIONS

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Abstract. In this note, we shall make a survey on the generalized Kubert functions (also called functions with distribution property) emphasizing the use of the polylogarithm function in number theory. We shall reveal that the Bernoulli function part often appears in the form of a difference of the first polylogarithm function and elucidate the limiting behavior of functions in question as they approach a rational point on the unit circle. We shall clarify this elimination of Clausen function part in Riemann's posthumous fragment II and Mordell's Lambert-like series.

1. Introduction and preliminaries

Riemann left three (or four if we count the example, around 1854, of everywhere non-differentiable function) legacies in number theory. The biggest is surely his memoir [10] (1859) on the number of primes under a given quantity, his unique paper in number theory. Prior to this, *in his Habilitationsschrift*

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(1854), in which he developed the theory of Riemann integration, he gave an example of a discontinuos and integrable function, cf. Wang [16].

Dedekind [1] succeeded in elucidating the genesis of all the formulas in Part II of Riemann's posthumous fragment [11]. It has no text, only formulas dealing with the asymptotic behavior of those modular functions from Jacobi's [3], for which the variable tends to rational points on the unit circle. Part I has been analyzed notably by Wintner (1941) [17], which contains some far-reaching comments on Part II.

Dedekind did this task by introducing the most celebrated Dedekind eta-function defined by

$$\eta(z) = e^{\frac{\pi i z}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}),\ \mathrm{Im} z > 0. \tag{1.1}$$

Dedekind proved the general transformation formula, which contained the Dedekind sum

$$s(h,k) = \sum_{m=1}^{k-1} \left(\left(\frac{m}{k}\right)\right)\left(\left(\frac{hm}{k}\right)\right) = \sum_{m \bmod k} \left(\left(\frac{m}{k}\right)\right)\left(\left(\frac{hm}{k}\right)\right), \tag{1.2}$$

where $(h,k) = 1,\ k \in \mathbb{N}$, and where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & x \notin \mathbb{Z} \\ 0 & x \in \mathbb{Z} \end{cases} = -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin 2\pi n x}{n} \tag{1.3}$$

is the saw-tooth Fourier series, where the convergence is uniform in any interval free from integer points.

What Riemann did was to eliminate the singular part by taking the odd part

$$\sum_{2 \nmid n} a_n = \sum_{n} a_n - \sum_{2 | n} a_n. \tag{1.4}$$

The polylogarithm function $l_1(x)$ of order 1 is

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\cos(2\pi n x)}{n} + i \sum_{n=1}^{\infty} \frac{\sin(2\pi n x)}{n} &= l_1(x) = \\ = \sum_{n=1}^{\infty} \frac{e^{2\pi i n x}}{n} &= A_1(x) - \pi i \overline{B}_1(x), \end{aligned} \tag{1.5}$$

for $0 < x < 1$, where

$$A_1(x) = -\log 2|\sin \pi x| = \sum_{n=1}^{\infty} \frac{\cos(2\pi n x)}{n}, \tag{1.6}$$

is its real part, the Clausen function and the imaginary part is

$$l_1(x) - l_1(-x) = -2\pi i\overline{B}_1(x), \quad 0 < x < 1, \tag{1.7}$$

where

$$\bar{B}_1(x) = x - [x] - \frac{1}{2} = -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2\pi nx)}{n} = ((x)) \tag{1.8}$$

is the first periodic Bernoulli polynomial, where the second equality holds for $x \in \mathbb{Z}$. One of the earliest use of the polylogarithm function (inclding higher order ones) in this way is probably in Lehmer [6], Yamamoto [18] etc.

We give one typical example from Wang [15] and [16], the former of which generalized and shortened the 65 pages paper of de Reyna [2] to 16 pages.

Let

$$\log\frac{2K}{\pi} = \sum_{p=1}^{\infty}\frac{4z^p}{p(1+z^p)}, \tag{1.9}$$

be one of elliptic modular functions, where p runs through odd integers (i.e. odd part)

Lemma 1.1.

$$\begin{aligned}\log\frac{2K}{\pi} &= \\ &= -\log(1-y) + \omega(y) + \log\frac{\pi}{Q} + \sum_{r=1}^{Q-1}(-1)^r\left(l_1\left(\frac{Mr}{Q}\right) - 2l_1\left(\frac{Mr}{2Q}\right)\right)\end{aligned} \tag{1.10}$$

for Q odd;

$$\begin{aligned}\log\frac{2K}{\pi} = -\frac{2\pi^2}{Q^2(1-y)} - \log(1-y) + \frac{\pi^2}{Q^2} + \log\frac{8\pi}{Q} + \omega(y) - \\ - \sum_{r=1}^{\frac{Q}{2}-1}(-1)^r\frac{2r}{Q}\left(2l_1\left(\frac{Mr}{2Q}\right) - 2l_1\left(\frac{-Mr}{2Q}\right) - l_1\left(\frac{Mr}{Q}\right) + l_1\left(\frac{-Mr}{Q}\right)\right),\end{aligned} \tag{1.11}$$

for Q even and $\frac{Q}{2}$ odd.

$$\begin{aligned}\log\frac{2K}{\pi} = -\log(1-y) + \log\frac{2\pi}{Q} + \omega(y) + 2\sum_{r=1}^{\frac{Q}{2}-1}(-1)^r\frac{2r}{Q}\times \\ \times\left(2l_1\left(\frac{Mr}{2Q}\right) - 2l_1\left(\frac{-Mr}{2Q}\right) - l_1\left(\frac{Mr}{Q}\right) + l_1\left(\frac{-Mr}{Q}\right)\right),\end{aligned} \tag{1.12}$$

for $\frac{Q}{2}$ even.

Theorem 1.1. (N.-L. Wang) *Let $\xi = \frac{M}{Q}$ be a rational number with M even and $Q > 1$ and let $z = ye^{\pi i\xi}$, $y \in [0, 1)$. Then we have*

$$\log\frac{2K}{\pi} = -\log(1-y) + \omega(y) + \log\frac{\pi}{Q} - \frac{\pi i}{2}\sum_{r=1}^{Q-1}(-1)^{r+\left[\frac{Mr}{Q}\right]}, \tag{1.13}$$

for Q odd, and

$$\log\frac{2K}{\pi} = -\frac{2\pi^2}{Q^2(1-y)} - \log(1-y) + \frac{\pi^2}{Q^2} + \log\frac{8\pi}{Q} + \omega(y) - \\ - \pi i\sum_{r=1}^{\frac{Q}{2}-1}\frac{2r}{Q}(-1)^{r+\left[\frac{Mr}{Q}\right]} \tag{1.14}$$

for Q even and $\frac{Q}{2}$ odd.

$$\log\frac{2K}{\pi} = -\log(1-y) + \log\frac{2\pi}{Q} + \omega(y) + 2\pi i\sum_{r=1}^{\frac{Q}{2}-1}\frac{2r}{Q}(-1)^{r+\left[\frac{Mr}{Q}\right]} \tag{1.15}$$

for $\frac{Q}{2}$ even; where $2\pi i\frac{M}{Q}$ is one of the values of $\log e^{2\pi i\frac{M}{Q}}$, and $\omega(y)$ is a continuation function on $\mathrm{I} = [0, 1]$ with $\omega(1) = 0$ which maybe different in different place.

There are more cases. In a forthcoming paper of Kanemitsu and Mehta [4], Mordell's Lambert-like series has been elucidated. In order to find the invisible in Mordell [8], we introduce the Lambert-like series

$$\tilde{g}(t, x) = \sum_{m=0}^{\infty}\tilde{h}(m, t)x^m = \sum_{n=1}^{\infty}a_n\left(\sum_{m=0}^{\infty}e^{2\pi i m\alpha}x^m\right)t^n, \tag{1.16}$$

where

$$\tilde{h}(m, t) = \sum_{n=1}^{\infty}a_n e^{2\pi i m\alpha}t^n, \tag{1.17}$$

for $|x| < 1$, $\alpha \notin \mathbb{Q}$.

Theorem 1.2. *Let $\alpha \notin \mathbb{Q}$, β real, $|x| < 1$. Then*

$$\sum_{n=1}^{\infty}\frac{e^{2\pi i\beta n}}{n}\left(\sum_{m=0}^{\infty}e^{2\pi i mn\alpha}x^m\right) = \\ = \sum_{n=1}^{\infty}\frac{e^{2\pi i n\beta}}{n}\frac{1}{1 - xe^{2\pi i n\alpha}} = \sum_{m=0}^{\infty}\tilde{h}(m, 1)x^m, \tag{1.18}$$

where

$$\tilde{h}(m,1) = -\log 2 \mid \sin \pi\theta(m) \mid -\pi i \bar{B}_1\left(\theta(m)\right), \tag{1.19}$$

where

$$\theta(m) = m\alpha + \beta \notin \mathbb{Z}. \tag{1.20}$$

In view of (1.7), we may consider the imaginary part of $\tilde{h}(m,t)$, which is the function $h(m,t)2i \operatorname{Im} \tilde{h}(m,t) =$ considered by Mordell.

On the other hand, the Clausen function appears in certain crucial places. We quote a typical example from Kanemitsu–Kuzumaki [5].

Theorem 1.3. *Let $m = 5$, $K = \mathbf{Q}(\zeta_5)$ and $k = 2,3$. Then $K^+ = \mathbf{Q}(\sqrt{5})$, and*

$$\begin{aligned}
\zeta_{\mathbf{Q}(\zeta_5)}(2) &= \frac{8\pi^6}{375\sqrt{5}} \left\{ A_2\left(\frac{1}{5}\right)^2 - A_2\left(\frac{2}{5}\right)^2 \right\}, \\
\zeta_{\mathbf{Q}(\zeta_5)}(3) &= -\frac{64\pi^8}{5^{7\frac{1}{2}}}\, \zeta(3) \left\{ A_3\left(\frac{1}{5}\right) - A_3\left(\frac{2}{5}\right) \right\}, \\
\zeta_{\mathbf{Q}(\sqrt{5})}(2) &= \frac{2\pi^4}{75\sqrt{5}}, \\
\zeta_{\mathbf{Q}(\sqrt{5})}(3) &= -\frac{4\pi^2}{3\sqrt{5}}\, \zeta(3) \left\{ A_3\left(\frac{1}{5}\right) - A_3\left(\frac{2}{5}\right) \right\}.
\end{aligned}$$

2. Kubert identities

As we have seen, the invisible part, the Clausen function plays an important role. In the hope of introducing more general Dedekind sum by replacing the saw-tooth Fourier series in (1.2) by Kubert functions, we study the class of (generalized) Kubert functions $f(x)$. They are periodic of period 1 and satisfy the (*generalized*) *Kubert relation* (Walum [14])

$$\sum_{r \bmod m} f\left(x + \frac{r}{m}\right) = \sum_{r=0}^{m-1} f\left(\frac{mx+r}{m}\right) = \theta(m) f(mx), \tag{2.1}$$

for every positive integer m, where $\theta(m)$ is a function in m only and the sum is over all residue classes modulo m. In most cases, however, we may choose the least non-negative residues modulo m as the mid term in (2.1).

(2.1) is often expressed as

$$\sum_{r \bmod m} f\left(\frac{x+r}{m}\right) = \sum_{r=0}^{m-1} f\left(\frac{x+r}{m}\right) = \theta(m) f(x). \tag{2.2}$$

Here $\frac{x+r}{m}$ varies all the solutions y of $my = x$ in $\mathbb{Q}/\mathbb{Z}$ or $\mathbb{R}/\mathbb{Z}$ but the situation is set up so that it makes sense for $x \in (0,1)$ or $x \in (0,\infty)$. (2.1) is slightly more general than the defining condition of other authors who restrict to the case $\theta(m) = m^{1-s}$. Milnor [7] defines the relation $(*_s)$:

$$(*_s) \qquad \sum_{r=0}^{m-1} f\left(\frac{x+r}{m}\right) = m^{1-s} f(x). \tag{2.3}$$

Example 2.1. There are many examples of Kubert functions.

(i) The simplest is the Bernoulli polynomial. The nth *Bernoulli polynomial* $B_n(x)$ is defined by the Taylor expansion

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n, \quad |z| < 2\pi \tag{2.4}$$

and the nth Bernoulli number B_n is the value $B_n(0)$. The nth Bernoulli polynomial satisfies the Kubert identity

$$(*_n) \qquad \sum_{r=0}^{m-1} B_n\left(\frac{x+r}{m}\right) = m^{1-n} B_n(x),$$

which has been known as the *multiplication formula* in the theory of Bernoulli polynomials.

The *periodic Bernoulli polynomial* $\bar{B}_n(x)$ is defined by

$$\bar{B}_n(x) = B_n(x - [x]), \tag{2.5}$$

where $[x]$ indicates the greatest integer not exceeding x.

The nth periodic Bernoulli polynomial satisfies the Kubert identity $(*_n)$. We have the *Fourier expansion* (Hurwitz, 1890)

$$\bar{B}_n(x) = -\frac{n!}{(2\pi i)^n} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{e^{2\pi i k x}}{k^n}. \tag{2.6}$$

(ii) The digamma function $\psi(x) = \frac{\Gamma'}{\Gamma}(x)$ satisfies a modified Kubert relation

$$(*_1) \qquad m^{-1} \sum_{r=0}^{m-1} \psi\left(\frac{x+r}{m}\right) = \psi(x) - \log m$$

and the gamma function satisfies the Gauss multiplication formula.

(iii) The Hurwitz zeta-function and the polylogarithm functions satisfy the Kubert relation. These as well as the Bernoulli polynomials will be expounded as examples of the Lipschitz–Lerch transcendent elsewhere.

A more general setting of the *uniform map* f has been considered by Sun [13] which reads

$$\sum_{r=0}^{m-1} f\left(\frac{x+r}{m}, my\right) = f(x,y). \tag{2.7}$$

Here the domain of definition of f is $X \times Y \subset \mathbb{C} \times \mathbb{C}$ and it is assumed that $\{(\frac{x+r}{m}, my) \big| r \bmod m\} \subset X \times Y$.

In the case where f is *of variables separable type*

$$f(x,y) = g(x)h(y), \tag{2.8}$$

it is proved ([13]) that (2.7) amounts to Theorem 2.3, i.e. θ is *completely multiplicative* or totally multiplicative, i.e. $\theta(mn) = \theta(m)\theta(n)$ for all $m, n \in \mathbb{N}$.

Theorem 2.2. ([13]) *If f is a non-zero solution of* (2.7) *of variables separable type, then the uniform map condition* (2.7) *amounts to*

$$\sum_{r=0}^{m-1} g\left(\frac{x+r}{m}\right) = \theta(m)g(x), \tag{2.9}$$

i.e. the generalized Kubert relation (2.1) *and the function θ in* (2.17) *is completely multiplicative.*

We shall prove Theorem 2.4 below which is a generalization of Sun's theorem.

On [7, p.281], it is remarked that it suffices to assume (2.10) to hold for all prime values of m, to cover the general case. First we remark that the proof of [14, Theorem 2.1], which seems to be due to Carlitz, will give simultaneously a proof of the fact that θ in (2.1) is completely multiplicative as well as this primality assertion.

Theorem 2.3. ([7], [14])

(i) *If f is a non-zero solution of* (2.1), *then θ is completely multiplicative.*

(ii) *If* (2.1) *holds true for all prime values of m, then it is true for all positive integral values of m.*

Proof. For arbitrary m and n, we transform

$$S := \sum_{r=0}^{mn-1} f\left(x + \frac{r}{mn}\right). \tag{2.10}$$

by writing $r = an + b$. We see that a, b run through $0 \le a < m$ and $0 \le b < n$, respectively. Hence (2.10) becomes the double sum

$$S = \sum_{b=0}^{n-1} \sum_{a=0}^{m-1} f\left(x + \frac{b}{mn} + \frac{a}{m}\right). \tag{2.11}$$

Since the inner sum in (2.11) is $\theta(m) f\left(mx + \frac{b}{n}\right)$ in case (i) or if m is a prime in case (ii), it follows that

$$S = \sum_{r=0}^{mn-1} f\left(x + \frac{r}{mn}\right) = \theta(m)\theta(n) f(mnx), \tag{2.12}$$

in case (i) or if m is a prime in case (ii). In the case of (i), since $S = \theta(mn) f(mnx)$, we choose x so that $f(mnx) \neq 0$, and we conclude that θ is completely multiplicative.

To prove (ii), we appeal to (2.12), which means that $\sum_{r=0}^{mn-1} f\left(x + \frac{r}{mn}\right) = \theta(m)\theta(n) f(mnx) = \theta(mn) f(mnx)$ holds by (i) if m, n are primes, i.e. that (2.1) is true for their product mn, which implies the validity of (2.1) for all values of m. ∎

Lemma 2.1. *Suppose f is a non-zero uniform map of variable separable type satisfying* (2.8) *and that*

$$f(y_0) \neq 0, \tag{2.13}$$

for some y_0 in the domain of h, then

$$f(my_0) \neq 0, \tag{2.14}$$

for all $m \in \mathbb{N}$.

Proof. (2.25) reads

$$h(my_0) \sum_{r=0}^{m-1} g\left(\frac{x+r}{m}\right) = g(x)h(y_0). \tag{2.15}$$

For some x_0, we must have $g(x_0) \neq 0$, in which case the right-hand side is non-zero, whence so is the left-hand side and the assertion follows. ∎

Theorem 2.4. (Generalization of Sun's theorem)

(i) *Suppose f is a non-zero solution of* (2.7) *of variables separable type $f(x, y) = g(x)h(y)$. For a fixed y_0 such that $h(y_0) \neq 0$, let*

$$\theta(m) = \frac{h(y_0)}{h(my_0)} \neq 0 \tag{2.16}$$

whose non-vanishingness follows from Lemma 2.1. Then (2.7) amounts to

$$\sum_{r=0}^{m-1} g\left(\frac{x+r}{m}\right) = \theta(m)g(x), \tag{2.17}$$

and the function θ in (2.17) is completely multiplicative.

(ii) *It suffices to assume the validity of*

$$g(x)h(y) = h(mny) \sum_{r=0}^{mn-1} g\left(\frac{x+r}{mn}\right). \tag{2.18}$$

for all relatively prime pairs m, n and the validity of (2.25) for prime power values of m.

Proof. For arbitrary m and n, we transform the right-hand-side of

$$g(x)h(y_0) = h(mny_0) \sum_{r=0}^{mn-1} g\left(\frac{x+r}{mn}\right), \tag{2.19}$$

which is (2.25) with modulus mn. By writing $r = an + b$, we see that a, b run through $0 \le a < m$ and $0 \le b < n$, respectively. Hence the right-hand side of (2.19) becomes the double sum

$$g(x)h(y_0) = h(mny_0) \sum_{b=0}^{n-1} \sum_{a=0}^{m-1} g\left(\frac{x+b}{mn} + \frac{a}{m}\right) \tag{2.20}$$

whose innermost sum is

$$\frac{1}{h(my_0)} h(my_0) \sum_{a=0}^{m-1} g\left(\frac{x+b}{mn} + \frac{a}{m}\right) = \frac{1}{h(my_0)} h(y_0) g\left(\frac{x}{n} + \frac{b}{n}\right). \tag{2.21}$$

Substituting this in (2.20) and arguing in the same way as above, we conclude that

$$\begin{aligned} g(x)h(y_0) &= h(mny_0)\frac{1}{h(my_0)h(ny_0)} h(y_0)h(ny_0) \sum_{b=0}^{n-1} g\left(\frac{x}{n} + \frac{b}{n}\right) = \\ &= h(mny_0)\theta(m)\theta(n)g(x) \end{aligned} \tag{2.22}$$

Choosing an x such that $g(x) \ne 0$, we infer that (2.22) implies

$$h(y_0) = h(mny_0)\theta(m)\theta(n), \tag{2.23}$$

which is the multiplicativity of θ.

Note that (2.22) may be written as

$$g(x)\theta(mn) = \theta(m) \sum_{b=0}^{n-1} g\left(\frac{x}{n} + \frac{b}{n}\right) \tag{2.24}$$

which amounts to (2.17) on account of multiplicativity and non-vanishingness of θ, which proves assertion (i).

To prove (ii) we note that Uunder condition (2.8), we note that (2.7) reads

$$h(my) \sum_{r=0}^{m-1} g\left(\frac{x+r}{m}\right) = g(x)h(y). \tag{2.25}$$

We appeal to (2.22) and note that (2.19) leads to

$$\sum_{r=0}^{mn-1} g\left(\frac{x+r}{mn}\right) = \theta(m)\theta(n)g(x) \tag{2.26}$$

valid for all primes m, n. Using multiplicativity of θ, (2.26) amounts to the validity of (2.25) for the product of two primes. By the UFD property of integers, this leads to the validity of (2.25) for all $m \in \mathbb{N}$, completing the proof. ■

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