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AN APPLICATION OF MAHLER'S METHOD TO CONTINUED FRACTIONS

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Abstract. Using Mahler's method we prove the transcendence of certain continued fractions.

1. Introduction and results

The aim of the present note is to study the transcendence of certain continued fractions and to give so a seemingly new application of the method developed by Mahler nearly ninety years ago for studying transcendence and algebraic independence of the values of functions satisfying some functional equations. An excellent introduction to the method and its later developments is given in Nishioka's book [6]. Typical easy examples of so-called Mahler functions are the series and products

$$\sum_{j=0}^{\infty} r(z^{d^j}), \quad \prod_{j=0}^{\infty} r(z^{d^j}),$$

where $d \geq 2$ is an integer and $r(z)$ a rational function satisfying certain natural conditions. These functions are solutions of simple Mahler type functional equations

$$f(z) = r(z) + f(z^d), \quad f(z) = r(z)f(z^d),$$

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respectively. There is a rather extensive list of papers where the transcendence and algebraic independence of the above type of series and products are studied by using Mahler's method, see [6], [1], [3] and the references there. In particular, note that the generating functions of some interesting sequences are special cases of the above functions, for example $\prod_{i=1}^{\infty}(1-z^{2^j})$ and $\prod_{i=1}^{\infty}(1+z^{2^j}+$ $+z^{2^{j+1}})$ are the generating functions of the Thue-Morse sequence on $\{-1,1\}$ and the Stern diatomic sequence, respectively, the paper [2] gives some algebraic independence results on these and related functions.

In the present note, we shall study the transcendence of the values of the (formal) continued fraction

$$f(z)=[r(z^{d^0}),r(z^{d^1}),r(z^{d^2}),\ldots],$$

where $r(z)=p(z)/q(z)$ is a rational function with coprime polynomials $p(z)$ and $q(z)$ satisfying $p(0)\neq 0, q(0)=0$ and $\deg p(z)\leq \deg q(z)=: k\geq 1$. This $f(z)$ satisfies a Mahler type functional equation

$$f(z)=\frac{1}{r(z)+f(z^d)}=\frac{q(z)}{p(z)+q(z)f(z^d)}$$

or equivalently

$$f(z)(r(z)+f(z^d))=1, \tag{1.1}$$

which gives a possibility to use Mahler's method in this context, too.

To state our result we denote $q(z)=z^m q_1(z), q_1(0)\neq 0$, where m is an integer satisfying $1\leq m\leq k$. The definition of $f(z)$ implies that

$$f(z)=\lim_{n\to\infty}[r(z^{d^0}),r(z^{d^1}),\ldots,r(z^{d^{n-1}})]=\lim_{n\to\infty}\frac{P_n(z)}{Q_n(z)},$$

where $P_n(z)$ and $Q_n(z)$ are polynomials with first nonzero terms $p(0)^{n-1}q_1(0)z^m$ and $p(0)^n$, respectively, leading to $f(0)=0$. The functional equation (1.1) has a unique power series solution

$$f(z)=\sum_{i=1}^{\infty}c_i z^i$$

with $f(0)=0$ and Theorem 1.7.1 of [6] implies that this series has a positive convergence radius R depending effectively on the coefficients of $p(z)$ and $q(z)$. Our main result reads as follows.

Theorem 1.1. *Assume that the polynomials $p(z)$ and $q(z)$ have algebraic coefficients and either $m>k/(d+1)$ or $1\leq m\leq k/(d+1)$ and simultaneously the polynomial $q_1(z)$ does not have any factor $z^d-\gamma,\gamma\neq 0$ (in $\mathbb{C}[z]$). If α is an algebraic number satisfying $|\alpha|<1, q(\alpha^{d^j})\neq 0, j=0,1,\ldots$, then $f(\alpha)$ is a well defined transcendental number.*

The choice $r(z) = 1/z$ in the above theorem gives the following corollary.

Corollary 1.1. *If β is an algebraic number satisfying $|\beta| > 1$, then $g(\beta) := [\beta, \beta^d, \beta^{d^2}, \ldots]$ is a transcendental number.*

Corollary 1.1 applies in particular to all $\beta = q \in \{2, 3, \ldots\}$, but in this case the result follows also immediately from Roth's theorem on the approximation of algebraic numbers by rationals. Further, we refer to [4] for some linear independence results on $g(q)$.

Before entering the proof of Theorem 1.1, we point out that a transcendence measure for $f(\alpha)$ could also be obtained by using the work [5] or its generalization [7].

2. Proof of Theorem 1.1

To apply Mahler's Theorem (Theorem 1.2 in [6]), we need to know that $f(z)$ is a transcendental function. By Theorem 1.3 of [6], it is enough to prove that $f(z)$ is not a rational function.

Lemma 2.1. *If the polynomials $p(z)$ and $q(z)$ satisfy the assumptions of Theorem* 1.1 (*apart from the algebraicity of their coefficients*), *then $f(z)$ is not a rational function.*

Proof. Assume that a rational function $f(z) = a(z)/b(z)$ with coprime polynomials $a(z)$ and $b(z)$ satisfies (1.1). Then

$$a(z)(p(z)b(z^d) + q(z)a(z^d)) = q(z)b(z)b(z^d). \tag{2.1}$$

Thus necessarily $\deg a(z) = \deg b(z) =: D \geq 1$. Further, since $a(z)$ and $b(z)$ as well as $p(z)$ and $q(z)$ are coprime, (2.1) implies

$$b(z^d) \mid q(z)a(z), \quad q(z) \mid a(z)b(z^d). \tag{2.2}$$

Thus $dD \leq D + k$ and $k \leq (d+1)D$, and therefore

$$\frac{k}{d+1} \leq D \leq \frac{k}{d-1}. \tag{2.3}$$

As we saw above, $f(0) = 0$, hence $a(0) = 0$. Let us write $a(z) = z^u a_1(z)$, where $a_1(0) \neq 0$. By (2.1), we then have

$$z^u a_1(z)(p(z)b(z^d) + z^{m+du} q_1(z)a_1(z^d)) = z^m q_1(z)b(z)b(z^d).$$

Thus $u = m$ and

$$a_1(z)(p(z)b(z^d) + z^{(d+1)m}q_1(z)a_1(z^d)) = q_1(z)b(z)b(z^d). \tag{2.4}$$

Now $a_1(z^d)$ and $b(z^d)$ are coprime and therefore

$$b(z^d) \mid z^{(d+1)m}q_1(z)a_1(z).$$

Since $b(0) \neq 0$, this implies

$$b(z^d) \mid q_1(z)a_1(z). \tag{2.5}$$

Assuming now $m > k/(d+1)$ and using (2.5), we get

$$dD \leq k + D - 2m, \quad (d-1)D \leq k - 2m < k - \frac{2k}{d+1} = \frac{(d-1)k}{d+1}, \quad D < \frac{k}{d+1},$$

which is not possible, by (2.3).

In the case $1 \leq m \leq k/(d+1)$, we denote

$$b(z) = c\prod_{i=1}^{D}(z - \beta_i)$$

with nonzero $c, \beta_1, \ldots, \beta_D$. If

$$z^d - \beta_i = (z - \beta_{i1}) \cdots (z - \beta_{id}),$$

then the extra assumption in this case means, by (2.5), that at least one of β_{ij} is a zero of $a_1(z)$ for each i. This is possible only if $\deg a_1(z) \geq D$, but then we have a contradiction, since $\deg a_1(z) = D - m < D$. ■

Remark 2.1. The extra assumption given in Lemma 2.1 in the case $1 \leq m \leq$ $\leq k/(d+1)$ is needed, since otherwise the functional equation (1.1) may have rational solutions. In the case $d = 2$ and $k = 3$ we give the following two examples. If

$$r(z) = \frac{z^2 + z + 1}{z(z^2 + 1)},$$

then the function $f(z) = z/(z+1)$ satisfies (1.1), and if

$$r(z) = \frac{1 - 2z - 2z^2}{2z(2z^2 - 1)},$$

then (1.1) has a solution $f(z) = 2z/(2z - 1)$.

Proof of Theorem 1.1. Its statement follows immediately from Mahler's Theorem, if $R \geq 1$, and the same holds for all $0 < |\alpha| < R$ in the case $R < 1$. If $R < 1$ and α is an algebraic number satisfying $R \leq |\alpha| < 1$, we can see that $f(\alpha)$ is defined and transcendental as follows. Let n denote the smallest positive integer for which $\beta := \alpha^{d^n}$ satisfies $|\beta| < R$. Then $f(\beta)$ is transcendental by Mahler's Theorem and therefore all numbers

$$f(\alpha^{d^{n-1}}) = \frac{1}{r(\alpha^{d^{n-1}}) + f(\beta)} = [r(\alpha^{d^{n-1}}) + f(\beta)],$$

$$f(\alpha^{d^{n-2}}) = \frac{1}{r(\alpha^{d^{n-2}}) + f(\alpha^{d^{n-1}})} = [r(\alpha^{d^{n-2}}), r(\alpha^{d^{n-1}}) + f(\beta)],$$

...

$$f(\alpha) = \frac{1}{r(\alpha) + f(\alpha^d)} = [r(\alpha), r(\alpha^d), \dots, r(\alpha^{d^{n-1}}) + f(\beta)]$$

are well defined and transcendental. Note here that, at each step, the transcendence of $f(\alpha^{d^j})$ and our assumption $q(\alpha^{d^{j-1}}) \neq 0$ implies that the denominator in the fraction above is defined and nonzero and so $f(\alpha^{d^{j-1}})$ is defined and transcendental. ∎

Remark 2.2. In the case $R < 1$, it is not possible to extend the definition of $f(z)$ to all $|z| < 1$ by the above considerations, namely with transcendental z some of the denominators above could vanish.

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