BETA DISTRIBUTION ON ARITHMETICAL SEMIGROUPS

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Abstract. The sequences of distributions defined on an arithmetical semigroup are considered. We prove that any Beta distribution can occur as a limit law for such sequences.

1. Introduction and result

Let a, b be positive constants. The Beta law $\mathrm{B}(a, b)$ is two parameters distribution concentrated on the interval $u \in [0; 1]$ with distribution function

$$B(u; a, b) := \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int\limits_0^u \frac{\mathrm{d}t}{t^{1-a}(1-t)^{1-b}}.$$

For the distributions defined via arithmetical functions the convergence to the Beta law was considered in [6], [4], [2], [5], [3].

Authors of this paper have proved [1] that the one parameter Beta law $\mathrm{B}(a, 1-a)$ can be simulated by means of a sequence of the distributions defined on a multiplicative semigroup. In this paper we generalize this result for any two parameters Beta law $\mathrm{B}(a, b)$.

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To discuss this problem we need some notations. Let $\mathbb{G}$ be a commutative multiplicative semigroup with identity element a_0 and generated by a countable subset $\mathfrak{P}$ of prime elements. We assume that k, l, m, n are non-negative integers, $a, b, d \in \mathbb{G}$, $p \in \mathfrak{P}$ and a completely additive degree function $\partial : \mathbb{G} \to \mathbb{N} \cup \{0\}$ is defined so that $\partial(p) \geq 1$ for each prime p. In the sequel we assume that the semigroup $\mathbb{G}$ satisfies (see [10], [9]) the following

Axiom A*. *There exist constants $A > 0$, $q > 1$ and $0 \leq \nu < 1$ such that*

$$\mathbb{G}(n) := \#\{a \in \mathbb{G} : \partial(a) = n\} = Aq^n + \mathrm{O}(q^{\nu n}).$$

The prime number theorem in semingroup $\mathbb{G}$ (see [7], [8]) yields

$$\pi(n) := \#\{p \in \mathfrak{P} : \partial(p) = n\} = \frac{q^n}{n}(1 - (-1)^n I(\mathbb{G})) + \mathrm{O}(q^{\mu n})$$

with some $\max(1/2, \nu) < \mu < 1$. Here $I(\mathbb{G}) = 1$, if $Z(-1) = 0$, and $I(\mathbb{G}) = 0$ otherwise. Here

$$Z(z) := \sum_{n \geq 0} \mathbb{G}(n) \left(\frac{z}{q}\right)^n, \quad |z| < 1,$$

is the generating function which has an analytic continuation into the disc $|z| < q^{1-\nu}$ and $Z(z) \neq 0$ for $|z| \leq 1$ with the possible exception at the point $z = -1$.

Definition 1.1. Let $g : \mathbb{G} \to (0, \infty)$ be a multiplicative function such that $g(p^m) \leq C$ for $m \in \mathbb{N}$, any $p \in \mathfrak{P}$ and some $C > 0$. We say that g belongs to the class $\mathbb{M}(\varkappa, C, c)$, $\varkappa \geq 0$, if the function defined by

$$H(z) := \sum_{m \geq 1} \left(\frac{z}{q}\right)^m \sum_{\partial(p)=m}^{*} (g(p) - \varkappa), \quad |z| < 1,$$

has an analytic continuation into the disc $|z| < 1 + c$ for some $c > 0$.

For a multiplicative function $f : \mathbb{G} \to [0, \infty)$ and $v \geq 0$ set

$$\mathcal{T}_f(a, v) := \sum_{d|a,\, \partial(d) \leq v}^{*} f(d), \quad \mathcal{T}_f(a) := \mathcal{T}_f(a, \partial(a)).$$

Here and in the following the starred sum or product symbols mean that these operations are used over corresponding elements of the semigroup $\mathbb{G}$.

Definition 1.2. We say that a pair $(g; f)$ of the multiplicative functions belongs to the class $\mathcal{M}(\varkappa, \alpha, C_1, c_1)$ if $g \in \mathbb{M}(\varkappa, C_1, c_1)$ and $\frac{g}{\mathcal{T}_f} \in \mathbb{M}(\alpha, C_1, c_1)$.

For any $a \in \mathbb{G}$ and $t \in [0,1]$ set

$$X(a,t) := \frac{\mathcal{T}_f(a, \partial(a)t)}{\mathcal{T}_f(a)}.$$

When $\mathbb{G}(n) > 0$, we define

$$F_n(t;g,f) := \frac{q-1}{qG_n(g)} \sum_{\partial(a)\le n}^{*} g(a)X(a,t), \tag{1.1}$$

where $g:\ \mathbb{G} \to (0,\infty)$ is a multiplicative function and

$$G_n(g) := \sum_{\partial(a)=n}^{*} g(a).$$

Note that axiom A^* implies $\mathbb{G}(n) > 0$ for all sufficiently large n.

We consider a multiplicative function $f(a)$ defined on the semigroup with axiom A^*, provided the associated "divisors" function $\mathcal{T}_f(a)$ satisfies some analytic conditions. The aim of our paper is to show, that sequence (1.1) can be approximated by the Beta distribution with some positive parmeters. The main result is the following

Theorem. *Suppose that* $(g,f) \in \mathcal{M}(\varkappa,\alpha;C_1,c_1)$. *If* $0 < \alpha < \varkappa$, *then uniformly for* $0 \le t \le 1$

$$F_n(t;g,f) = B(t;\varkappa-\alpha,\alpha) + \mathrm{O}\left(\frac{1}{n^{\varkappa-\alpha}} + \frac{1}{n^{\alpha}} + \frac{(\ln n)^{\epsilon(\varkappa-\alpha)} + (\ln n)^{\epsilon(\alpha)}}{n}\right),$$

as $n \to \infty$. *Here* $\epsilon(1) = 1$ *and* $\epsilon(v) = 0$ *for* $v \ne 1$.

Unless otherwise indicated, we assume that the implicit constants in the $\ll$ or O() symbols depend at most on the parameters and constants involved in the definitions of the semigroup $\mathbb{G}$ and corresponding classes $\mathbb{M}()$ or $\mathcal{M}()$.

2. Preliminaries

We will need the estimate for the sum of the "shifted" multiplicative functions defined on $\mathbb{G}$

$$M_n(g,d) := \frac{1}{Aq^n} \sum_{\partial(a)=n}^{*} g(ad).$$

The following lemma yields the result of this type.

Lemma 2.1 ([1]). *Let $g:\ \mathbb{G}\to[0,\infty)$ be a multiplicative function such that $g\in\mathbb{M}(\varkappa,C,c)$ with some positive constants $\varkappa$, C and c. Then, uniformly for all $d\in\mathbb{G}$ and $n\geq 0$,*

$$M_n(g,d)=(A(n+1))^{\varkappa-1}\left(\frac{L(\varkappa,g)\tilde{g}(d)}{\Gamma(\varkappa)}+\mathrm{O}\left(\frac{\hat{g}(d)}{n+1}\right)\right),$$

where $L(\varkappa,g)$ and the multiplicative functions $\tilde{g}$ and $\hat{g}$ are defined by

$$\begin{aligned} L(\varkappa,g) &:= \prod_p{}^{*}\left(1-\frac{1}{q^{\partial(p)}}\right)^{\varkappa}\sum_{k\geq 0}\frac{g(p^k)}{q^{k\partial(p)}},\\ \tilde{g}(p^m) &:= \left(\sum_{k\geq 0}\frac{g(p^k)}{q^{k\partial(p)}}\right)^{-1}\sum_{k\geq 0}\frac{g(p^{k+m})}{q^{k\partial(p)}},\\ \hat{g}(p^m) &:= \left(1+\frac{c_1}{q^{2\partial(p)/3}}\right)\sum_{k\geq 0}\frac{g(p^{k+m})}{q^{2k\partial(p)/3}}. \end{aligned}$$

Here $c_1\geq 0$ is a constant, depending on $\varkappa$ and C.

Lemma 2.2 ([10] p.86). *Suppose that $\sigma\in\mathbb{R}$. Then*

$$\sum_{m=1}^{n} m^{\sigma}q^{m}=\frac{q}{q-1}n^{\sigma}q^{n}+\mathrm{O}\left(n^{\sigma-1}q^{n}\right).$$

Lemma 2.3. *For $0\leq t\leq 1$, $n\geq 1$, and γ, $\delta\in\mathbb{R}$ we have*

$$T_n(t;\gamma,\delta):=\sum_{k\leq nt}\frac{1}{(1+k)^{\gamma}(1+n-k)^{\delta}}=$$

$$=n^{1-\gamma-\delta}J(t;\gamma,\delta,n^{-1})+\mathrm{O}\left(\frac{1}{n^{\delta}}+\frac{1}{n^{\gamma}}\right), \tag{2.1}$$

where

$$J(t;\gamma,\delta,\eta):=\int_0^t\frac{\mathrm{d}v}{(\eta+v)^{\gamma}(\eta+1-v)^{\delta}}.$$

Moreover

$$T_n(t;\gamma,\delta)\ll\frac{1}{n^{\delta}}+\frac{1}{n^{\gamma}}+\frac{(\ln n)^{\epsilon(\delta)}+(\ln n)^{\epsilon(\gamma)}}{n^{\delta+\gamma-1}}. \tag{2.2}$$

In case $\delta<1$ and $\gamma<1$ we have

$$J(t;\gamma,\delta,n^{-1})=J(t;\gamma,\delta,0)+\mathrm{O}\left(n^{\gamma-1}+n^{\delta-1}+n^{-1}\right). \tag{2.3}$$

The implicit constants in the $\ll$ or $\mathrm{O}()$ symbols depend on γ and δ only.

Proof. Easy to check, that

$$(2.4)\qquad \int_0^1 \frac{dv}{(n^{-1}+v)^\gamma (n^{-1}+1-v)^\delta} \ll_{\gamma,\delta} n^{\delta-1} + n^{\gamma-1} + \ln^{\epsilon(\gamma)} n + \ln^{\epsilon(\delta)} n.$$

The first assertion in this lemma follows from Euler-Maclaurin summation formula and (2.4). The inequality (2.2) follows from (2.1) and (2.4). Formula (2.3) follows from Lagrange mean value theorem applying the estimate (2.4) (see in [3] and [1]). ■

3. Proof of Theorem

Assumptions of the Theorem imply, that the multiplicative functions $h := g/\mathcal{T}_f \in \mathbb{M}(\alpha, C_1, c_1)$ and $f \cdot h \in \mathbb{M}(\varkappa - \alpha, C_1, c_1)$. We have

$$(3.1)\qquad F_n(t; g, f) = S_n(t) + R_n(t),\ t \in [0;1],$$

where

$$S_n(t) := \frac{q-1}{q\,G_n(g)} \sum_{0\le m\le n} \sideset{}{^*}\sum_{\partial(a)=m} \frac{g(a)\mathcal{T}_f(a,nt)}{\mathcal{T}_f(a)},$$

$R_n(0) = 0$ and for $t \in (0,1]$

$$R_n(t) := \frac{q-1}{q\,G_n(g)} \sideset{}{^*}\sum_{\partial(a)\le n} g(a) \sideset{}{^*}\sum_{\partial(a)=m} \frac{\mathcal{T}_f(a,nt) - \mathcal{T}_f(n,\partial(a)t)}{\mathcal{T}_f(a)} \ll$$

$$\ll \frac{1}{G_n(g)} \sideset{}{^*}\sum_{\partial(d)\le nt} f(d) \sum_{k\le \partial(d)(1-t)/t} \sideset{}{^*}\sum_{\partial(a)=k} h(ad).$$

Applying Lemma 2.1 for the inner sum we have

$$(3.2)\qquad \sideset{}{^*}\sum_{\partial(a)=k} h(ad) = \frac{A^\alpha q^k}{(1+k)^{1-\alpha}} \left(\frac{L(\alpha,h)\tilde{h}(d)}{\Gamma(\alpha)} + \mathrm{O}\left(\frac{\hat{h}(d)}{1+k} \right) \right).$$

An easy calculation shows that

$$\tilde{h}(p^m) = h(p^m) + \mathrm{O}\left(\frac{1}{\mathcal{T}_f(p^m) q^{\partial(p)}} \right),$$

$$\hat{h}(p^m) = h(p^m) + \mathrm{O}\left(\frac{1}{\mathcal{T}_f(p^m) q^{2\partial(p)/3}} \right).$$

Note, that these relations imply $f\tilde{h},\ f\hat{h} \in \mathbb{M}(\varkappa-\alpha, C_1, c_1)$.

Using (3.2) we get

$$R_n(t) \ll \frac{1}{G_n(g)} \sum_{\partial(d)\le tn}^{*} f(d)\hat{h}(d) \sum_{k\le \partial(d)(1-t)/t} \frac{q^k}{(1+k)^{1-\alpha}}.$$

By Lemma 2.2

$$R_n(t) \ll \frac{q^n q^{-nt}}{G_n(g)(1+n(1-t))^{1-\alpha}} \sum_{m\le nt} \sum_{\partial(a)=m}^{*} f(a)\hat{h}(a). \tag{3.3}$$

Since $g \in \mathbb{M}(\varkappa, C_1, c_1)$, Lemma 2.1 yields

$$G_n(g) = \frac{A^\alpha q^n}{(1+n)^{1-\varkappa}} \left(\frac{L(\varkappa, h)}{\Gamma(\varkappa)} + \mathrm{O}\left(\frac{1}{1+n}\right)\right). \tag{3.4}$$

For the inner sum in (3.3) we can apply Lemma 2.1, with $d = a_0$. Then employing Lemma 2.2 again and having in mind (3.4) we obtain

$$R_n(t) \ll (1+n)^{1-\varkappa} \left((1+n(1-t))^{\alpha-1}(1+nt)^{\varkappa-\alpha-1}\right).$$

Thus (3.1) becomes

$$F_n(t,g,f) = S_n(t) + \mathrm{O}\left(n^{\alpha-\varkappa} + n^{-\alpha} + n^{-1}\right), \tag{3.5}$$

uniformly for $t \in [0,1]$.

It remains to evaluate the sum $S_n(t)$. Changing order of summation we obtain

$$S_n(t) = \frac{q-1}{q\,G_n(g)} \sum_{\partial(d)\le nt}^{*} f(d) \sum_{m=0}^{n-\partial(d)} \sum_{\partial(a)=m}^{*} h(ad).$$

Hence (3.2) gives

$$S_n(t) = \Phi_n(t) + \mathrm{O}\left(r(n,t)\right), \tag{3.6}$$

where

$$\Phi_n(t) := \frac{(q-1)A^\alpha L(\alpha,h)}{q\,G_n(g)\Gamma(\alpha)} \sum_{\partial(d)\le nt}^{*} f(d)\tilde{h}(d) \sum_{m=0}^{n-\partial(d)} \frac{q^m}{(1+m)^{1-\alpha}}$$

and

$$r(n,t) := \frac{1}{G_n(g)} \sum_{\partial(d)\le nt}^{*} f(d)\hat{h}(d) \sum_{m=0}^{n-\partial(d)} \frac{q^m}{(1+m)^{2-\alpha}}.$$

Consider the remainder term in (3.6). By Lemma 2.2

$$r(n,t) \ll r_1(n,t) := \frac{1}{G_n(g)} \sum_{m \le nt} \frac{q^{n-m}}{(1+n-m)^{2-\alpha}} \sum_{\partial(d)=m}^{*} f(d)\hat{h}(d)\,.$$

Further using (3.4) and Lemma 2.1 , we deduce

$$r_1(n,t) \ll (1+n)^{1-\varkappa} T_n(t; 1-\varkappa+\alpha, 2-\alpha)\,.$$

Finally (2.2) yields

$$r(n,t) \ll r_1(n,t) \ll \frac{1}{n^{\alpha}} + \frac{(\ln n)^{\epsilon(\alpha)}}{n}. \tag{3.7}$$

The main term in (3.6) by Lemma 2.2 becomes

$$\Phi_n(t) = \frac{A^{\alpha} L(\alpha,h)}{\Gamma(\alpha) G_n(g)} \sum_{\partial(d) \le nt}^{*} \frac{f(d)\tilde{h}(d) q^{n-\partial(d)}}{(1+n-\partial(d))^{1-\alpha}} + \mathrm{O}\left(r_1(n,t)\right).$$

The latter sum we can write

$$\sum_{m \le nt} \frac{q^{n-m}}{(1+n-m)^{1-\alpha}} \sum_{\partial(d)=m}^{*} f(d)\tilde{h}(d).$$

Therefore (3.4) and Lemma 2.1 imply

$$\begin{aligned} \Phi_n(t) \quad &= \quad \frac{\Gamma(\varkappa) L(\alpha,h) L(\varkappa-\alpha, f\tilde{h})}{\Gamma(\alpha)\Gamma(\varkappa-\alpha) L(\varkappa,g)} \frac{T_n(t;1-\varkappa+\alpha,1-\alpha)}{(1+n)^{\varkappa-1}} + \\ &+ \quad \mathrm{O}\left(\frac{T_n(t;2-\varkappa+\alpha,1-\alpha)}{(1+n)^{\varkappa-1}} + r_1(n,t) \right). \end{aligned}$$

The routine calculation yields that $L(\alpha,h)L(\varkappa-\alpha,f\tilde{h}) = L(\varkappa,g)$ (see e.g. [3]). Now we may apply (3.7) and Lemma 2.3 to obtain

$$\begin{aligned} \Phi_n(t) \quad &= \quad \frac{\Gamma(\varkappa)}{\Gamma(\alpha)\Gamma(\varkappa-\alpha)} J(t;1-\varkappa+\alpha,1-\alpha,n^{-1}) + \\ &+ \quad \mathrm{O}\left(\frac{1}{n^{\varkappa-\alpha}} + \frac{1}{n^{\alpha}} + \frac{(\ln n)^{\epsilon(\varkappa-\alpha)} + (\ln n)^{\epsilon(\alpha)}}{n} \right). \end{aligned}$$

This estimate, (2.3), (3.6) and (3.5) complete the proof of Theorem. ■

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