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ON THE LOCAL CONVERGENCE OF WEIGHTED-NEWTON METHODS UNDER WEAK CONDITIONS IN BANACH SPACES

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Abstract. In this paper, we consider the weighted-Newton methods developed in [18] and study their local convergence in Banach space. In the earlier study the Taylor expansion of higher order derivatives is used which may not exist or may be very expensive or impossible to compute. However, the hypotheses of present analysis are based on the first Fréchet-derivative only, thereby the applicability of methods is expanded. New analysis also provides radius of convergence, error bounds and estimates on the uniqueness of the solution. Such estimates are not provided in the approaches that use Taylor expansions of higher order derivatives. Order of convergence of the methods is calculated by using computational order of convergence or approximate computational order of convergence without using higher order derivatives. Numerical tests are performed on some problems of different nature that confirm the theoretical results.

1. Introduction

Let E_1, E_2 be Banach spaces and $D \subseteq E_1$ be closed and convex. In this study, we locate a solution α of the nonlinear equation

$$F(x) = 0, \tag{1.1}$$

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where $F : D \subseteq E_1 \to E_2$ is a Fréchet-differentiable operator. In computational sciences, many problems can be written in the form (1.1). See, for example [2, 4, 16, 19]. The solutions of such equations are rarely attainable in closed form. This shows why most methods for solving these equations are usually iterative in nature. The important part in the construction of an iterative method is to study its convergence analysis. In general, the convergence domain is small. Therefore, it is important to enlarge the convergence domain without additional hypotheses. Knowledge of the radius of convergence is useful because it gives us the degree of difficulty for obtaining initial points. Another important problem is to find more precise error estimates on $\|x_{n+1} - x_n\|$ or $\|x_n - \alpha\|$. Many authors have studied local and semilocal convergence analysis of iterative methods, see, for example [1, 2, 3, 4, 5, 6, 7, 11, 12, 13, 14, 15, 17].

The most basic method for approximating a simple solution α of equation (1.1) is the Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \quad \text{for each } n = 0, 1, 2 \ldots, \tag{1.2}$$

which has quadratic order of convergence. In order to attain the higher order of convergence, a number of modified Newton's or Newton-like methods have been proposed in the literature (see [1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 13, 17, 18, 19, 20]) and references therein. In particular, Sharma and Arora [18] have recently proposed methods of fourth and sixth order convergence for approximating solution of $F(x) = 0$, using the weighted-Newton scheme defined for each $n = 0, 1, \ldots$ by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= y_n - (3I - 2F'(x_n)^{-1}[y_n, x_n; F])F'(x_n)^{-1}F(x_n) \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ z_n &= y_n - (3I - 2F'(x_n)^{-1}[y_n, x_n; F])F'(x_n)^{-1}F(y_n), \\ x_{n+1} &= z_n - (3I - 2F'(x_n)^{-1}[y_n, x_n; F])F'(x_n)^{-1}F(z_n), \end{aligned} \tag{1.4}$$

where I is identity operator on D and $[.,.; F]$ is a divided difference of order one on D^2. The notable point of these methods is that they use only single derivative and also single inverse operator which makes them computationally more efficient than other existing higher order methods. To prove the order of convergence the authors used Taylor expansions and hypotheses requiring the derivatives up to sixth order although only the first order derivative appears in the methods. It is quite clear that these hypotheses restrict the application of methods to functions which are not six times Fréchet-differentiable. As a motivational example, let us define a function g on $D = [-\frac{1}{2}, \frac{5}{2}]$ by

$$g(t) = \begin{cases} t^3 \ln t^2 + t^5 - t^4, & t \neq 0 \\ 0, & t = 0. \end{cases} \tag{1.5}$$

Successive differentiation yields

$$g'(t) = 3t^2 \ln t^2 + 5t^4 - 4t^3 + 2t^2,$$

$$g''(t) = 6t \ln t^2 + 20t^3 - 12t^2 + 10t$$

and

$$g'''(t) = 6 \ln t^2 + 60t^2 - 24t + 22.$$

Obviously, g''' is unbounded on D. Notice that the proofs of convergence in [18] use Taylor expansions. So, the hypothesis requiring derivatives up to sixth order in the expansions is not applicable. Keeping this in mind, here we study the local convergence of the methods (1.3) and (1.4) using the hypotheses only on the first Fréchet-derivative by taking advantage of the Lipschitz continuity of the first Fréchet-derivative. Moreover, our results are presented in the more general setting of a Banach space.

We summarize the contents of the paper. The local convergence including radius of convergence, computable error bounds and uniqueness results of methods (1.3) and (1.4) are presented in Section 2. Finally, in Section 3 numerical examples are performed to verify the theoretical results.

2. Local convergence analysis

In what follows we present the local convergence analysis of methods (1.3) and (1.4). Let $a_0 > 0$, $a > 0$, $b \geq 0$ be given parameters. It is convenient for the local convergence analysis to produce some functions and parameters. Define functions $\phi_1(t)$ and $p(t)$ on interval $[0, \frac{1}{a_0})$ by

$$\phi_1(t) = \frac{at}{2(1 - a_0 t)},$$

$$p(t) = \frac{2}{1 - a_0 t}\Big(a_0 + a_1\big(1 + \phi_1(t)\big)\Big)t$$

and parameter

$$q_1 = \frac{2}{2a_0 + a} < \frac{1}{a_0}. \tag{2.1}$$

Then, we have that $\phi_1(q_1) = 1$ and $0 \le \phi_1(t) \le 1$ for each $t \in [0, q_1)$. Furthermore, define the functions $\phi_2(t)$ and $\psi_2(t)$ on interval $[0, \frac{1}{a_0})$ by

$$\phi_2(t) = \left(1 + \frac{b(1+p(t))}{1-a_0 t}\right)\phi_1(t)$$

and

$$\psi_2(t) = \phi_2(t) - 1.$$

We have that $\psi_2(0) = -1 < 0$ and $\psi_2(q_1) = \frac{b(1+p(q_1))}{1-a_0 q_1} > 0$. It follows from the intermediate theorem that function ψ_2 has zeros in the interval $(0, q_1)$. Let q_2 be the smallest such zero. Finally, define functions $\phi_3(t)$ and $\psi_3(t)$ in the interval $(0, \frac{1}{a_0})$ by

$$\phi_3(t) = \left(1 + \frac{b(1+p(t))}{1-a_0 t}\right)\phi_2(t)$$

and

$$\psi_3(t) = \phi_3(t) - 1.$$

We have that $\psi_3(0) = -1 < 0$ and $\psi_3(q_2) = \frac{b(1+p(q_2))}{1-a_0 q_2} > 0$. From the intermediate theorem it follows that function ψ_3 has zeros in the interval $(0, q_2)$. Denote by q_3 the smallest such zero of function ψ_3 in interval $(0, q_2)$. Set:

$$q = \min\{q_i\}, \quad i = 1, 2, 3. \tag{2.2}$$

Then we have

$$0 < q \le q_1. \tag{2.3}$$

Then, for each $t \in [0, q)$

$$0 \le \phi_1(t) \le 1, \tag{2.4}$$

$$0 \le \phi_2(t) \le 1 \tag{2.5}$$

and

$$0 \le \phi_3(t) \le 1. \tag{2.6}$$

Let $B(v, \rho)$ and $\bar{B}(v, \rho)$ be the open and closed balls in E_1, respectively with center $v \in E_1$ and of radius $\rho > 0$. Let also $\mathcal{L}(E_1, E_2)$ be the set of bounded linear operators between E_1 and E_2.

Next, we present the local convergence analysis of method (1.4) using the preceding notations.

Theorem 2.1. *Let $F : D \subseteq E_1 \to E_2$ be a Fréchet-differentiable operator and let $[.,.; F] : E_1 \times E_1 \to \mathcal{L}(E_1, E_2)$ be a divided difference operator of F. Suppose that there exist $\alpha \in D$, $a_0 > 0$, $a > 0$, $a_1 > 0$, $a_2 > 0$ and $b \geq 1$ such that for each $x, y \in D$*

$$F(\alpha) = 0,\ F'(\alpha)^{-1} \in \mathcal{L}(E_2, E_1), \tag{2.7}$$

$$\|F'(\alpha)^{-1}\big(F'(x) - F'(\alpha)\big)\| \leq a_0\|x - \alpha\|, \tag{2.8}$$

$$\|F'(\alpha)^{-1}\big(F'(x) - F'(y)\big)\| \leq a\|x - y\|, \tag{2.9}$$

$$\|F'(\alpha)^{-1}F'(x)\| \leq b, \tag{2.10}$$

$$\|F'(\alpha)^{-1}\big([x, y : F] - F'(\alpha)\big)\| \leq a_1(\|x - \alpha\| + \|y - \alpha\|), \tag{2.11}$$

$$\|F'(\alpha)^{-1}\big([x, y; F] - F'(x)\big)\| \leq a_2\|x - y\| \tag{2.12}$$

and

$$\bar{B}(\alpha, q) \subset D, \tag{2.13}$$

where the radius q is defined by (2.2). Then, the sequence $\{x_n\}$ produced by method (1.4) for $x_0 \in B(\alpha, q) - \{\alpha\}$ is well defined, remains in $B(\alpha, q)$ for each $n = 0, 1, \ldots$ and converges to α. Moreover, the following error bounds

$$\|y_n - \alpha\| \leq \phi_1(\|x_n - \alpha\|)\|x_n - \alpha\| < \|x_n - \alpha\| < q, \tag{2.14}$$

$$\|z_n - \alpha\| \leq \phi_2(\|x_n - \alpha\|)\|x_n - \alpha\| < \|x_n - \alpha\| < q \tag{2.15}$$

and

$$\|x_{n+1} - \alpha\| \leq \phi_3(\|x_n - \alpha\|)\|x_n - \alpha\|, \tag{2.16}$$

are satisfied, where the "ϕ" functions are defined previously. Furthermore, for $T \in [q, \frac{2}{a_0})$ the limit point α is the only solution of equation $F(x) = 0$ in $\bar{B}(\alpha, T) \cap D$.

Proof. We will show the estimates (2.14)–(2.16) using mathematical induction. From (2.1), (2.8) and the hypotheses $x_0 \in B(\alpha, q) - \{\alpha\}$, we get that

$$\|F'(\alpha)^{-1}\big(F(x_0) - F(\alpha)\big)\| \leq a_0\|x_0 - \alpha\| < a_0 q < 1. \tag{2.17}$$

Using (2.17) and the Banach Lemma on invertible operators (see [4, 15]), it follows that $F'(x_0)^{-1} \in \mathcal{L}(E_2, E_1)$ and

$$\|F'(x_0)^{-1}F'(\alpha)\| \le \frac{1}{1 - a_0\|x_0 - \alpha\|} < \frac{1}{1 - a_0 q}. \tag{2.18}$$

Hence, y_0 is well defined by the first step of method (1.4) for $n = 0$. Then, we have by equations (2.1), (2.4), (2.9) and (2.18) that

$$\begin{aligned} \|y_0 - \alpha\| &\le \|x_0 - \alpha - F'(x_0)^{-1}F(x_0)\| \le \\ &\le \|F'(x_0)^{-1}F'(\alpha)\| \Big\| \int_0^1 F'(\alpha)^{-1}[F'(\alpha + \tau(x_0 - \alpha)) - \\ &\quad - F'(x_0)](x_0 - \alpha)] \Big\| d\tau \le \frac{a\|x_0 - \alpha\|^2}{2(1 - a_0\|x_0 - \alpha\|)} = \\ &= \phi_1(\|x_0 - \alpha\|)\|x_0 - \alpha\| < \|x_0 - \alpha\| < q, \end{aligned} \tag{2.19}$$

which shows (2.14) for $n = 0$ and $y_0 \in B(\alpha, q)$.

Next, with linear operator $A_0 = 3I - 2F'(x_n)^{-1}[y_n, x_n; F]$, by using (2.8), (2.11) and (2.18), we obtain

$$\begin{aligned} \|A_0\| &= \|3I - 2F'(x_0)^{-1}[y_0, x_0; F]\| \le \\ &\le 1 + \|2F'(x_0)^{-1}\big(F'(x_0) - [y_0, x_0; F]\big)\| \le \\ &\le 1 + 2\|F'(x_0)^{-1}F'(\alpha)\|\|F'(\alpha)^{-1}(F'(x_0) - [y_0, x_0; F])\| \le \\ &\le 1 + 2\|F'(x_0)^{-1}F'(\alpha)\|\|F'(\alpha)^{-1}\big(F'(x_0) - F'(\alpha) + \\ &\quad + F'(\alpha) - [y_0, x_0; F]\big)\| \le \\ &\le 1 + 2\|F'(x_0)^{-1}F'(\alpha)\|\Big(\|F'(\alpha)^{-1}(F'(x_0) - F'(\alpha))\| + \\ &\quad + \|F'(\alpha)^{-1}(F'(\alpha) - [y_0, x_0; F])\|\Big) \le \\ &\le 1 + \frac{2}{1 - a_0\|x_0 - \alpha\|}\Big(a_0\|x_0 - \alpha\| + a_1\big(\|x_0 - \alpha\| + \\ &\quad + \|y_0 - \alpha\|\big)\Big) \le \\ &\le 1 + \frac{2}{1 - a_0\|x_0 - \alpha\|}\Big(a_0\|x_0 - \alpha\| + a_1\big(\|x_0 - \alpha\| + \\ &\quad + \phi_1(\|x_0 - \alpha\|)\|x_0 - \alpha\|\big)\Big) \le \\ &\le 1 + \frac{2}{1 - a_0\|x_0 - \alpha\|}\Big(a_0 + a_1\big(1 + \phi_1(\|x_0 - \alpha\|)\big)\Big) \times \\ &\quad \times \|x_0 - \alpha\| = \\ &= 1 + p(\|x_0 - \alpha\|). \end{aligned} \tag{2.20}$$

Notice that for each $\tau \in [0,1]$, $\|\alpha + \tau(x_0 - \alpha) - \alpha\| = \tau\|x_0 - \alpha\| < q$. That is $\alpha + \tau(x_0 - \alpha) \in B(\alpha, q)$. We can write

$$F(x_0) = F(x_0) - F(\alpha) = \int_0^1 F'(\alpha + \tau(x_0 - \alpha))(x_0 - \alpha)d\tau. \tag{2.21}$$

Then, using (2.10) and (2.19), we get that

$$\begin{aligned} \|F'(\alpha)^{-1}F(x_0)\| &= \left\| \int_0^1 F'(\alpha)^{-1}F'(\alpha + \tau(x_0 - \alpha))(x_0 - \alpha)d\tau \right\| \leq \\ &\leq b\|x_0 - \alpha\|. \end{aligned} \tag{2.22}$$

Similarly, we obtain that

$$\|F'(\alpha)^{-1}F(y_0)\| \leq b\|y_0 - \alpha\|, \tag{2.23}$$

$$\|F'(\alpha)^{-1}F(z_0)\| \leq b\|z_0 - \alpha\|. \tag{2.24}$$

Using the second step of method (1.4) and relations (2.5), (2.18), (2.19), (2.20), (2.23), it follows that

$$\begin{aligned} \|z_0 - \alpha\| \leq& \|y_0 - \alpha\| + \|A_0\|\|F'(x_0)^{-1}F(y_0)\| = \\ =& \|y_0 - \alpha\| + \|A_0\|\|F'(x_0)^{-1}F(\alpha)\|\|F'(\alpha)^{-1}F(y_0)\| \leq \\ \leq& \|y_0 - \alpha\| + \big(1 + p(\|x_0 - \alpha\|)\big)\frac{b\|y_0 - \alpha\|}{1 - a_0\|x_0 - \alpha\|} \leq \\ \leq& \left(1 + \frac{b\big(1 + p(\|x_0 - \alpha\|)\big)}{1 - a_0\|x_0 - \alpha\|}\right)\|y_0 - \alpha\| \leq \\ \leq& \left(1 + \frac{b\big(1 + p(\|x_0 - \alpha\|)\big)}{1 - a_0\|x_0 - \alpha\|}\right)\phi_1(\|x_0 - \alpha\|)\|x_0 - \alpha\| \leq \\ \leq& \phi_2(\|x_0 - \alpha\|)\|x_0 - \alpha\| < \|x_0 - \alpha\| < q. \end{aligned} \tag{2.25}$$

This proves (2.15) for $n = 0$ and $z_0 \in B(\alpha, q)$. Then using equations (2.1), (2.6), (2.24) and (2.25), we obtain

$$\begin{aligned} \|x_1 - \alpha\| \leq& \|z_0 - \alpha\| + \|A_0\|\|F'(x_0)^{-1}F(z_0)\| = \\ =& \|z_0 - \alpha\| + \|A_0\|\|F'(x_0)^{-1}F(\alpha)\|\|F'(\alpha)^{-1}F(z_0)\| \leq \\ \leq& \|z_0 - \alpha\| + \big(1 + p(\|x_0 - \alpha\|)\big) \times \frac{b\|z_0 - \alpha\|}{1 - a_0\|x_0 - \alpha\|} \leq \\ \leq& \left(1 + \frac{b\big(1 + p(\|x_0 - \alpha\|)\big)}{1 - a_0\|x_0 - \alpha\|}\right)\|z_0 - \alpha\| \leq \\ \leq& \left(1 + \frac{b\big(1 + p(\|x_0 - \alpha\|)\big)}{1 - a_0\|x_0 - \alpha\|}\right)\phi_2(\|x_0 - \alpha\|)\|x_0 - \alpha\| \leq \\ \leq& \phi_2(\|x_0 - \alpha\|)\|x_0 - \alpha\| < \|x_0 - \alpha\|, \end{aligned} \tag{2.26}$$

which proves (2.16) for $n=0$ and $x_1 \in B(\alpha, q)$. Then, substitute x_0, y_0, z_0, x_1 by x_n, y_n, z_n, x_{n+1} in the preceding estimates to obtain (2.14)-(2.16). Then, from the estimates $\|x_{n+1}-\alpha\| \leq c\|x_n-\alpha\| < q$, where $c = \phi_3(\|x_0-\alpha\|) \in [0,1)$, we deduce that $\lim_{n\to\infty} x_n = \alpha$ and $x_{n+1} \in B(\alpha, q)$.

Finally, we show the uniqueness part. Let $Q = \int_0^1 F'(\beta + t(\alpha-\beta))dt$ for some $\beta \in \bar{B}(\alpha, q)$ with $F(\beta) = 0$. Using (2.13), we get that

$$\begin{aligned} \|F'(\alpha)^{-1}(Q - F'(\alpha)\| &\leq \int_0^1 a_0\|\beta + t(\alpha-\beta) - \alpha\|dt \leq \\ &\leq \int_0^1 a_0(1-t)\|\alpha-\beta\|dt \leq \\ &\leq \frac{a_0}{2}T < 1. \end{aligned} \tag{2.27}$$

It follows from (2.27) that Q^{-1} exists. Then, from the estimate $0 = F(\alpha) - -F(\beta) = Q(\alpha-\beta)$, we conclude that $\alpha = \beta$. ∎

Remark 2.1. (i) Method (1.4) remains the same when we use the conditions of Theorem 2.1 instead of stronger conditions used in [18]. Let $\{w_n\}$ be any iterative method. Define the computational order of convergence (COC) [20] by

$$\mathrm{COC} = \log\left\|\frac{w_{n+2}-\alpha}{w_{n+1}-\alpha}\right\| \Big/ \log\left\|\frac{w_{n+1}-\alpha}{w_n-\alpha}\right\|, \quad \text{for each } n = 1, 2, ... \tag{2.28}$$

and the approximate computational order of convergence (ACOC) [9], by

$$\mathrm{ACOC} = \log\left\|\frac{w_{n+2}-w_{n+1}}{w_{n+1}-w_n}\right\| \Big/ \log\left\|\frac{w_{n+1}-w_n}{w_n-w_{n-1}}\right\|, \quad \text{for each } n = 1, 2, ... \tag{2.29}$$

This way we obtain a practical order of convergence.

(ii) In order to present the corresponding results for method (1.3), we simply restrict to the definition of functions $\phi_1(t)$, $\phi_2(t)$ and parameters q_1 and q_2. Moreover, we define

$$q = \min\{q_1, q_2\}. \tag{2.30}$$

Hence, in view of the proof of Theorem 2.1, we arrive at

Theorem 2.2. *Suppose that the hypotheses of Theorem* 2.1 *are satisfied with* q *now defined by* (2.30)*. Then, the conclusions of Theorem* 2.1 *hold* (*except* (2.16)) *for method* (1.3) *replacing method* (1.4).

3. Numerical results

In this section, the theoretical results proved in section 2 are tested through numerical experimentation. We use the divided difference given by $[x, y;\, F] = \frac{1}{2}(F'(x) + F'(y))$ or $[x, y;\, F] = \int_0^1 (F'(y + \tau(x - y))d\tau$. By (2.8) and (2.10), it can easily be seen that we can choose $b(t) = 1 + a_0 t$ or even $b(t) = 2$, since $t \in [0, \frac{1}{a_0})$.

Example 3.1. Let us revisit the example (1.5) given at the introduction of this paper. In this case, we have $\alpha = 1, a_0 = a = 146.66290$, $a_1 = a_2 = \frac{a_0}{2}$ and $b = 2$. The calculated values of parameters q_1, q_2, q_3 and q are displayed in Table 1. Theorems 2.1 and 2.3 guarantee the convergence of methods to $\alpha = 1$ provided that $x_0 \in B(\alpha, q)$.

Method (1.4)	Method (1.3)
$q_1 = 0.0045456$	$q_1 = 0.0045456$
$q_2 = 0.0016457$	$q_2 = 0.0016457$
$q_3 = 0.00073460$	–
$q = 0.00073460$	$q = 0.0016457$

Table 1. Numerical results

Example 3.2. Let $E_1 = E_2 = \mathbb{R}$, $D = [-1, 1]$. Define function F on D by

$$F(x) = \sin x.$$

Then, we have that $\alpha = 0$, $a_0 = a = b = 1$ and $a_1 = a_2 = \frac{1}{2}$. Computed values of the parameters q_1, q_2, q_3 and q are given in Table 2. Theorems 2.1 and 2.3 guarantee the convergence of methods to $\alpha = 0$ provided that $x_0 \in B(\alpha, q)$.

Method (1.4)	Method (1.3)
$q_1 = 0.666667$	$q_1 = 0.666667$
$q_2 = 0.307336$	$q_2 = 0.307336$
$q_3 = 0.178198$	–
$q = 0.178198$	$q = 0.307336$

Table 2. Numerical results

Example 3.3. Let $E_1 = E_2 = \mathbb{R}^3$, $D = [-1, 1]^3$. Define mapping F on D for $v = \{x, y, z\}^T$ by

$$F(v) = \left(e^x - 1, \frac{e-1}{2}y^2 + y, z\right)^T.$$

The Fréchet-derivative of this mapping is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then for $\alpha = (0,0,0)^T$, we deduce that $a_0 = e-1$, $a = e$, $b = 2$ and $a_1 = \frac{e-1}{2}$, $a_2 = \frac{e}{2}$. The parameter values of q_1, q_2, q_3 and q are given in Table 3.

Method (1.4)	Method (1.3)
$q_1 = 0.324992$	$q_1 = 0.324992$
$q_2 = 0.114222$	$q_2 = 0.114222$
$q_3 = 0.047501$	–
$q = 0.047501$	$q = 0.114222$

Table 3. Numerical results

Example 3.4. Let $E_1 = E_2 = C[0,1]$, where $C[0,1]$ stands for the space of continuous functions defined on $[0,1]$. We shall use the maximum norm. Let $D = \bar{B}(0,1)$. Define operator F on D by

$$F(\mu)(x) = \mu(x) - 5\int_0^1 x\tau\mu(\tau)^3 d\tau. \tag{3.1}$$

We obtain that

$$F'(\mu(\lambda))(x) = \lambda(x) - 15\int_0^1 x\tau\mu(\tau)^2\lambda(\tau)d\tau, \text{ for each } \lambda \in D.$$

Then, for $\alpha = 0$, we have $a_0 = 7.5$, $a = 15$, $b = 2$ and $a_1 = a_2 = \frac{1}{2}$. Using the definition of q_1, q_2, q_3 and q, the parameter values are given in Table 4.

Method (1.4)	Method (1.3)
$q_1 = 0.0666667$	$q_1 = 0.0666667$
$q_2 = 0.0238729$	$q_2 = 0.0238729$
$q_3 = 0.0098964$	–
$q = 0.00989641$	$q = 0.0238729$

Table 4. Numerical results

Example 3.5. In last example we verify the results of local convergence along with sixth-order convergence of (1.4) by calculating ACOC employing the formula (2.29). Let us consider the function $F := (f_1, f_2, f_3) : D \to \mathbb{R}^3$ defined

by
(3.2)
$$F(x) = \big(10\,x_1 + \sin(x_1+x_2) - 1,\ 8\,x_2 - \cos^2(x_3-x_2) - 1,\ 12\,x_3 + \sin(x_3) - 1\big)^T,$$

where $x = (x_1, x_2, x_3)^T$.

Fréchet-derivative of $F(x)$ is given by

$$F'(x) = \begin{bmatrix} 10+\cos(x_1+x_2) & \cos(x_1+x_2) & 0 \\ 0 & 8+\sin 2(x_2-x_3) & -2\sin(x_2-x_3) \\ 0 & 0 & 12+\cos(x_3) \end{bmatrix}.$$

Then, we get that $a_0 = a = 0.269812$, $a_1 = a_2 = 1.08139$ and $b = 13.0377$. The parameter values of q_1, q_2, q_3 and q for this example are given in Table 5.

Method (1.4)	Method (1.3)
$q_1 = 2.470856$	$q_1 = 2.470856$
$q_2 = 0.263541$	$q_2 = 0.263541$
$q_3 = 0.0314738$	–
$q = 0.0314738$	$q = 0.263541$

Table 5. Numerical results

With the initial approximation $x_0 = \{0, 0.5, 0.1\}^T$, the solution α of the function (3.2) is given by

$$\alpha = \big\{0.0689783491726665 57...,\ 0.24644241860918295...,\\ 0.07692891198753 6964...\big\}^T.$$

Applying the stopping criterion $||x_{n+1} - x_n|| + ||F(x_n)|| < 10^{-300}$, the number of iterations $n = 5$ is required to converge to the solution given above. Then, using the last three approximations x_{n+1}, x_n, x_{n-1} in (2.29), we obtain ACOC $=$ $= 6.0000$. This verifies the theoretical sixth-order convergence of the method (1.4).

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