Annales Univ. Sci. Budapest., Sect. Comp. 47 (2018) 117–126

ON UNIQUENESS FOR MEROMORPHIC FUNCTIONS AND THEIR nTH DERIVATIVES

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Communicated by Bui Minh Phong

(Received February 23, 2018; accepted July 15, 2018)

Abstract. In this paper, we consider the problem of uniqueness of derivatives of meromorphic functions when they share a set of roots of unity.

1. Introduction

Let $\mathbb{C}$ denote the complex plane. By a *meromorphic function* we mean a meromorphic function in the complex plane $\mathbb{C}$.

In 1926, R. Nevanlinna ([8]) showed that a meromorphic function is uniquely determined by the inverse images, ignoring multiplicities, of 5 distinct values. In 1997 Yang and Hua ([10]) studied the unicity problem for meromorphic functions and differential monomials of the form $f^n f'$, when they share only one value.

S.S. Bhoosnurmath, R.S. Dyavanal ([2]) extend Yang–Hua's result to the case of $(f^n)^{(k)}$.

As a generalization of Nevanlinna's theorem on determining a meromorphic function by its single preimages, one considered the problem of determining a meromorphic function by a finite set of points in $\mathbb{C} \cup \infty$.

Key words and phrases: Meromorphic function, differential polynomial, unique range set.
2010 Mathematics Subject Classification: 30D35.
This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.01-2012.19.
https://doi.org/10.71352/ac.47.117

Inspired by the mentioned above results, in this paper we study possible relations between two meromorphic functions f and g, when $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share a finite set.

We first recall some notations. Let f be a non-constant meromorphic function. For every $a \in \mathbb{C}$, define the function $\nu_f^a : \mathbb{C} \to \mathbb{N}$ by

$$\nu_f^a(z) = \begin{cases} 0 & \text{if } f(z) \neq a \\ m & \text{if } f(z) = a \text{ with multiplicity } m, \end{cases}$$

and set $\nu_f^\infty = \nu_{\frac{1}{f}}^0$. For $f \in \mathcal{M}(\mathbb{C})$ and $S \subset \mathbb{C} \cup \{\infty\}$, we define

$$E_f(S) = \bigcup_{a \in S} \{(z, \nu_f^a(z)) : z \in a \in S\}.$$

In [12] Yang posed the problem: is it true that the equality $f^{-1}(S) = g^{-1}(S)$ with $S = \{-1, 1\}$ for polynomials of the same degree f, g implies that either $f = g$ or $f = -g$? This problem was solved in [9].

Now let $d, n, k \in \mathbb{N}^*$. Concerning the mentioned above problem of Yang, and related topics (see, for example [9]), in this paper, instead of $\{\pm 1\}$ we consider the set of roots of unity of degree d, $S = \{a \in \mathbb{C} : a^d = 1\}$, and the following problem: how we can say about the relations of f, g, if $E_{(f^n)^{(k)}}(S) = E_{(g^n)^{(k)}}(S)$?.

We shall prove the following theorem.

Theorem 1. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, and let n, d, k be positive integers with $n > 2k + \frac{2k+8}{d}$, $d \geq 2$, and $S = \{a \in \mathbb{C} : a^d = 1\}$. If $E_{(f^n)^{(k)}}(S) = E_{(g^n)^{(k)}}(S)$, then one of the following two cases holds:*

1. *$f = c_1 e^{cz}$ and $g = c_2 e^{-cz}$ for three non-zero constants c_1, c_2 and c such that $(-1)^{kd}(c_1 c_2)^{nd}(nc)^{2kd} = 1$;*
2. *$f = tg$ with $t^{nd} = 1$, $t \in \mathbb{C}$.*

2. Lemmas

We assume that the reader is familiar with the notations in the Nevanlinna theory (see [8]).

We first need the following Lemmas.

Lemma 2.1. ([8]) *Let f be a non-constant meromorphic function on $\mathbb{C}$ and let $a_1, a_2, ..., a_q$ be distinct points of $\mathbb{C} \cup \{\infty\}$. Then*

$$(q-2)T(r,f) \leq \sum_{i=1}^{q} N_1\Big(r, \frac{1}{f-a_i}\Big) + S(r,f),$$

where $S(r,f) = o(T(r,f))$ for all r, except for a set of finite Lebesgue measure.

Lemma 2.2. ([10]) *Let f and g be non-constant meromorphic functions on $\mathbb{C}$. If $E_f(1) = E_g(1)$, then one of the following three cases holds:*

1. $T(r,f) \leq N_2(r,f) + N_2\Big(r, \frac{1}{f}\Big) + N_2(r,g) + N_2\Big(r, \frac{1}{g}\Big) + S(r,f) + S(r,g),$ *and the same inequality holds for $T(r,g)$;*
2. $fg = 1$;
3. $f = g$.

Lemma 2.3. ([7]) *Let f be a non-constant meromorphic function on $\mathbb{C}$ and n, k be positive integers, $n > k$ and let a be a pole of f. Then we have*

$$(f^n)^{(k)} = \frac{\varphi_k}{(z-a)^{np+k}}, \text{ where } p = \nu_f^{\infty}(a), \varphi_k(a) \neq 0.$$

Lemma 2.4. ([7]) *Let f be a non-constant meromorphic function on $\mathbb{C}$ and n, k be positive integers, $n > k$ and let a be a pole of f. Then we have*

$$\frac{(f^n)^{(k)}}{f^{n-k}} = \frac{h_k}{(z-a)^{pk+k}}, \text{ where } p = \nu_f^{\infty}(a), h_k(a) \neq 0.$$

Lemma 2.5. *Let f be a non-constant meromorphic on $\mathbb{C}$ and k be a positive integer. Then we have*

$$T\left(r, (f)^{(k)}\right) \leq (k+1)T(r,f) + S(r,f).$$

Proof. By Lemma 2.4 and noting that $m\Big(r, \frac{(f)^{(k)}}{f}\Big) = S(r,f)$ we get

$$T\left(r, (f)^{(k)}\right) = m\left(r, (f)^{(k)}\right) + N(r, (f)^{(k)}) \leq m(r,f) + N(r,f) + kN_1(r,f) +$$
$$+ S(r,f) \leq T(r,f) + kT(r,f) + S(r,f) = (k+1)T(r,f) + S(r,f).$$

Lemma 2.5 is proved. ∎

Lemma 2.6. *Let f be a non-constant meromorphic function on $\mathbb{C}$ and n, k be positive integers, $n > 2k$. Then*

1. $(n-2k)T(r,f) + kN(r,f) + N\Big(r, \dfrac{f^{n-k}}{(f^n)^{(k)}}\Big) \le T\left(r, (f^n)^{(k)}\right) + S(r,f)$;
2. $N\Big(r, \dfrac{f^{n-k}}{(f^n)^{(k)}}\Big) \le kT(r,f) + kN_1(r,f) + S(r,f)$.

Proof. 1. By Lemma 2.3 we have

$$N(r,(f^n)^{(k)}) = nN(r,f) + kN_1(r,f). \tag{2.1}$$

From this and noting that $S(r,f) = S(r,f^n)$, $m\Big(r, \dfrac{(f)^{(k)}}{f}\Big) = S(r,f)$ we obtain

$$(n-k)m(r,f) = m(r,f^{n-k}) \le m\big(r,(f^n)^{(k)}\big) + m\Big(r, \frac{f^{n-k}}{(f^n)^{(k)}}\Big) + S(r,f) =$$

$$= m\big(r,(f^n)^{(k)}\big) + T\Big(r, \frac{(f^n)^{(k)}}{f^{n-k}}\Big) - N\Big(r, \frac{f^{n-k}}{(f^n)^{(k)}}\Big) + S(r,f) \le$$

$$\le m\big(r,(f^n)^{(k)}\big) + +kN(r,f) + km(r,f) + kN_1(r,f) - N\Big(r, \frac{f^{n-k}}{(f^n)^{(k)}}\Big) + S(r,f) =$$

$$= m\big(r,(f^n)^{(k)}\big) + kT(r,f) + kN_1(r,f) - N\Big(r, \frac{f^{n-k}}{(f^n)^{(k)}}\Big) + S(r,f). \tag{2.2}$$

From (2.1) and (2.2) it implies that

$$nN(r,f) + (n-k)m(r,f) = (n-k)\Big(N(r,f) + m(r,f)\Big) + kN(r,f) =$$

$$= (n-k)T(r,f) + kN(r,f) \le N\Big(r,(f^n)^{(k)}\Big) + m\Big(r,(f^n)^{(k)}\Big) - kN_1(r,f) +$$

$$+kT(r,f) + kN_1(r,f) - N\Big(r, \frac{f^{n-k}}{(f^n)^{(k)}}\Big) + S(r,f) =$$

$$= T\Big(r,(f^n)^{(k)}\Big) - N\Big(r, \frac{f^{n-k}}{(f^n)^{(k)}}\Big) + kT(r,f) + S(r,f).$$

Thus

$$(n-2k)T(r,f) + kN(r,f) + N\Big(r, \frac{f^{n-k}}{(f^n)^{(k)}}\Big) \le T\Big(r,(f^n)^{(k)}\Big) + S(r,f).$$

2. By Lemma 2.4 and noting that $m\Big(r, \dfrac{(f)^{(k)}}{f}\Big) = S(r,f)$ we have

$$N\left(r, \frac{1}{\frac{(f^n)^{(k)}}{f^{n-k}}}\right) \le T\Big(r, \frac{(f^n)^{(k)}}{f^{n-k}}\Big) = m\Big(r, \frac{(f^n)^{(k)}}{f^{n-k}}\Big) + N\Big(r, \frac{(f^n)^{(k)}}{f^{n-k}}\Big) \le$$

$$\le km(r,f) + N\Big(r, \frac{(f^n)^{(k)}}{f^{n-k}}\Big) + S(r,f) \le k(T(r,f) - N(r,f)) +$$

$$+kN_1(r,f) + kN(r,f) + S(r,f) = kT(r,f) + kN_1(r,f) + S(r,f).$$

So

$$N\Big(r, \frac{f^{n-k}}{(f^n)^{(k)}}\Big) \le kT(r,f) + kN_1(r,f) + S(r,f).$$

Lemma 2.6 is proved. ■

Lemma 2.7. ([11]) *Let $f(z)$ and $g(z)$ be two non-constant entire functions and n, k be positive integers, $n > k$. If $(f^n)^{(k)}(g^n)^{(k)} = h$, $h \in \mathbb{C}, h \neq 0$, then $f = l_1 e^{lz}$ and $g = l_2 e^{-lz}$ for three non-zero constants l_1, l_2 and l such that $(-1)^k (l_1 l_2)^n (nl)^{2k} = h$.*

Lemma 2.8. *Let f be a non-constant meromorphic function and n, k be positive integers, $n \ge k+3$, $a \in \mathbb{C}$, $a \neq 0$. Then*

$$\frac{n-k-2}{n+k} T_f(r) \le N_1\left(r, \frac{1}{(f^n)^{(k)} - a}\right) + S(r,f).$$

Proof. Since $n \ge k+3$ we have $\frac{n-k-2}{n+k} > 0$. Because $n \ge k+3$ it follows that $(f^n)^{(k)}$ is not constant.

Applying Lemma 2.1 to $(f^n)^{(k)}$ with the values ∞, 0 and a, we obtain

$$T\Big(r, (f^n)^{(k)}\Big) \le$$

$$\le N_1\Big(r, (f^n)^{(k)}\Big) + N_1\left(r, \frac{1}{(f^n)^{(k)}}\right) + N_1\left(r, \frac{1}{(f^n)^{(k)} - a}\right) + S(r,f).$$

By the similar arguments as in the proof of [Lemma 3.4, 7] we obtain

$$N_1\left(r, \frac{1}{(f^n)^{(k)}}\right) \le \frac{k+1}{n} N\left(r, \frac{1}{(f^n)^{(k)}}\right) + \frac{k(n-k-1)}{n} N_1(r,f) + O(1),$$

$$\frac{1}{n+k} N\Big(r, (f^n)^{(k)}\Big) \ge N_1(r,f), \qquad N_1\Big(r, (f^n)^{(k)}\Big) = N_1(r,f).$$

Therefore,

$$T\Big(r,(f^n)^{(k)}\Big) \leq \frac{k+1}{n} N\left(r, \frac{1}{(f^n)^{(k)}}\right) + \Big(1+\frac{k(n-k-1)}{n}\Big) N_1\left(r,(f^n)^{(k)}\right) + \\ + N_1\left(r, \frac{1}{(f^n)^{(k)}-a}\right) + S(r,f).$$

From this and by

$$N\left(r, \frac{1}{(f^n)^{(k)}}\right) \leq T\big(r,(f^n)^{(k)}\big) + S(r,f),$$
$$N_1\big(r,(f^n)^{(k)}\big) \leq T\big(r,(f^n)^{(k)}\big) + S(r,f),$$

we have

$$T\big(r,(f^n)^{(k)}\big) \leq \left(\frac{k+1}{n} + \frac{n+k(n-k-1)}{(n+k)n}\right) T\big(r,(f^n)^{(k)}\big) +$$
$$+ N_1\left(r, \frac{1}{(f^n)^{(k)}-a}\right) + S(r,f),$$
$$\frac{n-k-2}{n+k} T_f(r) \leq N_1\left(r, \frac{1}{(f^n)^{(k)}-a}\right) + S(r,f).$$

Lemma 2.8 is proved. ∎

3. Proof of Theorem 1

Since $n \geq k+3$, from Lemma 2.8, applying to $(f^n)^{(k)}$ with the value 1, it implies that $(f^n)^{(k)} = 1$ has a solution. So $E_{(f^n)^{(k)}}(S) \neq \emptyset$ and $E_{(g^n)^{(k)}}(S) \neq \emptyset$. By $E_{(f^n)^{(k)}}(S) = E_{(g^n)^{(k)}}(S)$ we see that $((f^n)^{(k)})^d$ and $((g^n)^{(k)})^d$ share the value 1 CM. Applying Lemma 2.2 to $((f^n)^{(k)})^d, ((g^n)^{(k)})^d$ we arrive to one of the following cases:

Case 1.

$$T\left(r,((f^n)^{(k)})^d\right) \leq N_2\left(r,((f^n)^{(k)})^d\right) + N_2\left(r, \frac{1}{((f^n)^{(k)})^d}\right) + N_2\left(r,((g^n)^{(k)})^d\right) + \\ + N_2\left(r, \frac{1}{((g^n)^{(k)})^d}\right) + S\Big(r,((f^n)^{(k)})^d\Big) + S\Big(r,((g^n)^{(k)})^d\Big),$$

$$T\left(r,((g^n)^{(k)})^d\right) \le N_2\left(r,((f^n)^{(k)})^d\right) + N_2\left(r,\frac{1}{((f^n)^{(k)})^d}\right) + N_2\left(r,((g^n)^{(k)})^d\right) +$$

$$+N_2\left(r,\frac{1}{((g^n)^{(k)})^d}\right) + S\left(r,((f^n)^{(k)})^d\right) + S\left(r,((g^n)^{(k)})^d\right). \tag{3.1}$$

By Lemma 2.6 we obtain

$$(n-2k)T(r,f) \le T\left(r,(f^n)^{(k)}\right) + S(r,f) \le (k+1)nT(r,f) + S(r,f),$$

$$(n-2k)T(r,g) \le T\left(r,(g^n)^{(k)}\right) + S(r,g) \le (k+1)nT(r,g) + S(r,g).$$

From this and since

$$T\left(r,((f^n)^{(k)})^d\right) = dT\left(r,(f^n)^{(k)}\right) + S\left(r,(f^n)^{(k)}\right),$$
$$T\left(r,((g^n)^{(k)})^d\right) = dT\left(r,(g^n)^{(k)}\right) + S\left(r,(g^n)^{(k)}\right)$$

it is easy to see that

$$\begin{aligned} S\left(r,((f^n)^{(k)})^d\right) &= S\left(r,(f^n)^{(k)}\right) = S(r,f), \\ S\left(r,((g^n)^{(k)})^d\right) &= S\left(r,(g^n)^{(k)}\right) = S(r,g). \end{aligned} \tag{3.2}$$

On the other hand, if a is a pole of $((f^n)^{(k)})^d$, then $f(a) = \infty$ with $\nu^{\infty}_{((f^n)^{(k)})^d}(a) \ge n + k \ge 2$. Moreover, because $d \ge 2$, we see that if a is a zero of $((f^n)^{(k)})^d$, then $(f^n)^{(k)}(a) = 0$ with $\nu^0_{((f^n)^{(k)})^d}(a) \ge 2$. Therefore,

$$N_2\left(r,((f^n)^{(k)})^d\right) = 2N_1(r,f) \le 2T(r,f) + S(r,f),$$

$$N_2\left(r,\frac{1}{((f^n)^{(k)})^d}\right) = 2N_1\left(r,\frac{1}{(f^n)^{(k)}}\right) \le$$

$$\le 2\left(N_1\left(r,\frac{1}{f^{n-k}}\right) + +N\left(r,\frac{f^{n-k}}{(f^n)^{(k)}}\right)\right) =$$

$$= 2\left(N_1\left(r,\frac{1}{f}\right) + N\left(r,\frac{f^{n-k}}{(f^n)^{(k)}}\right)\right) \le$$

$$\le 2T(r,f) + 2N\left(r,\frac{f^{n-k}}{(f^n)^{(k)}}\right) + S(r,f) \le 2T(r,f) + 2kN_1(r,f) +$$

$$+2kT(r,f)) + S(r,f) = (2k+2)T(r,f) + 2kN_1(r,f) + S(r,f).$$

Similarly,

$$N_2\left(r,((g^n)^{(k)})^d\right) \le 2T(r,g)+S(r,g),$$

$$N_2\left(r,\frac{1}{((g^n)^{(k)})^d}\right) \le 2(T(r,g)+N\left(r,\frac{g^{n-k}}{(g^n)^{(k)}})\right) \le$$

$$\le 2(k+1)T(r,g)+2kN_1(r,g)+S(r,f).$$

Set

$$\begin{aligned} T(r) &= T(r,f)+T(r,g),\\ S(r) &= S(r,f)+S(r,g),\\ N(r) &= N(r,f)+N(r,g),\\ N_1(r) &= N_1(r,f)+N_1(r,g). \end{aligned}$$

Combining (3.1) and (3.2) we get

$$T\left(r,((f^n)^{(k)})^d\right) \le (4+2k)T(r,f)+4T(r,g)+2kN_1(r,f)+2N\left(r,\frac{g^{n-k}}{(g^n)^{(k)}}\right)+S(r).$$

$$T\left(r,((g^n)^{(k)})^d\right) \le (4+2k)T(r,g)+4T(r,f)+2kN_1(r,g)+2N\left(r,\frac{f^{n-k}}{(f^n)^{(k)}}\right)+S(r).$$

$$T\left(r,((f^n)^{(k)})^d\right)+T\left(r,((g^n)^{(k)})^d\right) \le (4+2k)T(r)+4T(r)+2kN_1(r)+$$

$$+2N\left(r,\frac{g^{n-k}}{(g^n)^{(k)}}\right)+2N\left(r,\frac{f^{n-k}}{(f^n)^{(k)}}\right)+S(r).$$

On the other hand, by Lemma 2.6 we have

$$d((n-2k)T(r,f)+kN(r,f)+N\left(r,\frac{f^{n-k}}{(f^n)^{(k)}})\right) \le T\left(r,((f^n)^{(k)})^d\right)+S(r,f),$$

$$d((n-2k)T(r,g)+kN(r,g)+N\left(r,\frac{g^{n-k}}{(g^n)^{(k)}})\right) \le T\left(r,((g^n)^{(k)})^d\right)+S(r,g),$$

Thus,

$$d(n-2k)T(r)+dkN(r)+dN\left(r,\frac{f^{n-k}}{(f^n)^{(k)}}\right)+dN\left(r,\frac{g^{n-k}}{(g^n)^{(k)}}\right) \le$$

$$\le (4+2k)T(r)+4T(r)+2kN_1(r)+2N\left(r,\frac{g^{n-k}}{(g^n)^{(k)}}\right)+2N\left(r,\frac{f^{n-k}}{(f^n)^{(k)}}\right)+S(r).$$

Moreover, because $d \geq 2$, we give

$$dN\left(r, \frac{f^{n-k}}{(f^n)^{(k)}}\right) \geq 2N\left(r, \frac{f^{n-k}}{(f^n)^{(k)}}\right),$$
$$dN\left(r, \frac{g^{n-k}}{(g^n)^{(k)}}\right) \geq 2N\left(r, \frac{g^{n-k}}{(g^n)^{(k)}}\right),$$
$$dkN(r) \geq 2kN_1(r).$$

Therefore,

$$d(n-2k)T(r) \leq (2k+8)T(r) + S(r),\ d(n-2k) \leq 2k+8.$$

From this we obtain a contradiction to $n > 2k + \frac{2k+8}{d}$.

Case 2. $((f^n)^{(k)})^d((g^n)^{(k)})^d = 1$. From this we have $(f^n)^{(k)}(g^n)^{(k)} = h$ with $h^d = 1$. We are going to prove $f(z) \neq 0$, $f(z) \neq \infty$, $g(z) \neq 0$, $g(z) \neq \infty$ for all $z \in \mathbb{C}$. Assume f has a zero a, and $\nu_f^0(a) = \alpha$, $\alpha \geq 1$. Then a is a pole of g with $\nu_g^\infty(a) = \beta$, $\beta \geq 1$ such that $n\alpha - k = n\beta + k$ and $n(\alpha - \beta) = 2k$. From this and by $n \geq 2k + \frac{2k+8}{d} > 2k$ we obtain a contradiction. By similar arguments we have $g(z) \neq 0$, $f(z) \neq \infty$, $g(z) \neq \infty$ for all $z \in \mathbb{C}$. So $f(z)$ and $g(z)$ are two non-constant entire functions. Applying Lemma 2.7 to f and g we obtain $f = c_1e^{cz}$ and $g = c_2e^{-cz}$ for three non-zero constants c_1, c_2 and c such that $(-1)^k(c_1c_2)^n(nc)^{2k} = h$. Because $h^d = 1$ we give $(-1)^{kd}(c_1c_2)^{nd}(nc)^{2kd} = 1$.

Case 3. $((f^n)^{(k)})^d = ((g^n)^{(k)})^d$. Then $(f^n)^{(k)} = h(g^n)^{(k)}$ with $h^d = 1$. Set $e^n = h$ we have $(f^n)^{(k)} = ((eg)^n)^{(k)}$. By the similar arguments as in the proof of [Theorem 1.1, 1] we obtain $f = seg$ with $s^n = 1$. Set $t = se$. Then we get $t^{nd} = s^{nd}e^{nd} = 1$.

Theorem 1 is proved. ∎

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