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CONTINUATION OF THE LAUDATION TO Professor Bui Minh Phong on his 65-th birthday

by Imre Kátai (Budapest, Hungary)

We congratulated him in this journal (volume 38) five years ago. During the last five years many important things has happened with him.

(1) The number of grandchildren grew up to five. Their names: David (2008), Mia (2014), Daniel (2016), Lili (2016) and Victoria (2018).

(2) He defended his habilitation thesis in 2013 (see [73]).

(3) He has been appointed as a full professor to the Computer Algebra Department of the Eötvös Loránd University.

(4) Eötvös Loránd University honoured his prominent results in number theory.

(5) Bui Minh Phong has been the vice president of Union of Vietnamese living in Hungary during the last twenty years. Appreciating his activity he had been invited to the 9th congress of the Solidarity Union (of Vietnam) which had been held in 2015. Only 15 persons leaving outside of Vietnam had been invited and participated on this event.

He continued his research in number theory. He wrote 34 new papers in number theory, some of them with coauthors.

We enlarge the categories to classify his new results as follows:

I. A characterization of the identity with functional equations ([76], [82], [90], [92], [96], [102]).

II. A characterization of additive and multiplicative functions ([77], [78], [79],[80], [81],[88],[89],[91], [93], [94], [95], [97], [98], [99], [100], [101], [103]).

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III. On additive functions with values in Abelian groups ([74], [104]).

IV. A consequence of the ternary Goldbach theorem ([83]).

V. On the multiplicative group generated by $\{\frac{[\sqrt{2}n]}{n}\}$ ([84], [85], [86], [87], [105]).

I. A characterization of the identity with functional equations

An arithmetic function $g(n) \not\equiv 0$ is said to be multiplicative if $(n, m) = 1$ implies that

$$g(nm) = g(n)g(m)$$

and it is completely multiplicative if this relation holds for all positive integers n and m. Let $\mathcal{M}$ and $\mathcal{M}^*$ denote the classes of complex-valued multiplicative, completely multiplicative functions, respectively.

Let $\mathcal{P}$, $\mathbb{N}$ be the set of primes, positive integers, respectively.

Let $A, B \subseteq \mathbb{N}$. We say that A, B is a pair of additive uniqueness sets (AU-sets) for $\mathcal{M}$ if only one $f \in \mathcal{M}$ exists for which

$$f(a+b) = f(a) + f(b) \quad \text{for all} \quad a \in A \quad \text{and} \quad b \in B,$$

namely $f(n) = n$ for all $n \in \mathbb{N}$.

In 1992, Spiro (*J. Number Theory* 42 (1992), 232–246) showed that $A = B = \mathcal{P}$ are AU-sets for $\mathcal{M}$.

For some generalizations of this result, we refer the works of J.-M. De Koninck, I. Kátai and B. M. Phong [38] , B. M. Phong [60], B. M. Phong [66], K.-H. Indlekofer and B. M. Phong [62], K. K. Chen and Y. G. Chen (*Publ. Math. Debrecen*, **76** (2010), 425–430.), J-H. Fang (*Combinatorica*, 31 (2011), 697-701.), A. Dubickas and P. Šarka (*Aequationes Math.* 86 (2013), no. 1-2, 81–89).

In 1997, J.-M. De Koninck, I. Kátai and B. M. Phong [38] proved the following theorem.

Theorem A. ([38]) *If $f \in \mathcal{M}$ with f(1)=1 satisfies*

$$f(p+m^2) = f(p) + f(m^2) \quad \textit{for all} \quad p \in \mathcal{P},\ m \in \mathbb{N},$$

then $f(n) = n$ for all $n \in \mathbb{N}$.

For two completely multiplicative functions the following theorem had been proved.

Theorem B. ([66]) *If $f, g \in \mathcal{M}^*$ satisfy the equations*

$$f(p+1) = g(p) + 1 \quad \textit{and} \quad f(p+q^2) = g(p) + g(q^2) \quad \textit{for all} \quad p,\ q \in \mathcal{P},$$

then either

$$f(p+1)=f(p+q^2)=0 \quad \textit{and} \quad g(\pi)=-1 \quad \textit{for all} \quad p,\ q,\ \pi\in\mathcal{P}$$

or

$$f(n)=g(n)=n \quad \textit{for all} \quad n\in\mathbb{N}.$$

Finnaly, B. M. Phong proved in 2016 the following result:

Theorem C. ([92]) *Assume that $f,g\in\mathcal{M}$ with f(1)=1 satisfy*

$$f(p+m^2)=g(p)+g(m^2) \quad \textit{and} \quad g(p^2)=g(p)^2$$

for all primes p and $m\in\mathbb{N}$. Then either

$$f(p+m^2)=0, \quad g(p)=-1 \quad \textit{and} \quad g(m^2)=1$$

for all primes p and $m\in\mathbb{N}$ or

$$f(n)=n \quad \textit{and} \quad g(p)=p, \quad g(m^2)=m^2$$

for all $p\in\mathcal{P},\ n,m\in\mathbb{N}$.

In the other paper in 2016, B. M. Phong gave a complete solution of multiplicative functions which satisfy some equation.

He proved:

Theorem D. ([96]) *Assume that non-negative integers $a,\ b$ with $a+b>0$ and $f,\ g\in\mathcal{M}$ satisfy the condition*

$$f\left(n^2+m^2+a+b\right)=g(n^2+a)+g(m^2+b) \quad \textit{for all} \quad n,m\in\mathbb{N}.$$

Let

$$S_n=g(n^2+a) \quad \textit{and} \quad A=\frac{1}{120}(S_6+4S_5-S_3-S_2-3S_1).$$

Then the following assertions are true:

I. $A\in\{0,1\}$.

II. *If $A=1$, then*

$$g(m^2+a)=m^2+a,\ g(m^2+b)=m^2+b \ \textit{ for all } m\in\mathbb{N}$$

and

$$f(n)=n \quad \textit{for all} \quad n\in\mathbb{N},\ (n,2(a+b))=1.$$

III. *If* $A=0$, *then there is a* $K\in\{1,2,3\}$ *such that* $S_{n+K}=S_n$ *for all* $n\in\mathbb{N}$. *All solutions of* f *and* g *are given.*

Theorem E. ([102]) *If* $f\in\mathcal{M}$ *satisfies the conditions*

$$f(p+m^3)=f(p)+f(m^3)\quad \textit{for all}\quad p\in\mathcal{P},\ m\in\mathbb{N}$$

and

$$f(\pi^2)=f(\pi)^2\quad \textit{for all}\quad \pi\in\mathcal{P},$$

then

$$f(n)=n\quad \textit{for all}\quad n\in\mathbb{N}.$$

II. A characterization of additive and multiplicative functions

We shall list some results among papers [77], [79], [80], [88], [89], [94], [95], [97], [98], [99], [100], [101], [103].

In 1946, among other results on additive functions, P. Erdős (*Ann.Math.*, **47** (1946), 1–20) proved that if $f\in\mathcal{A}$ satisfies

$$f(n+1)-f(n)=o(1)\quad \text{as}\quad n\to\infty,$$

then $f(n)$ is a constant multiple of $\log n$. He stated several conjectures concerning possible improvements and generalizations of his results. In addition P. Erdős conjectured that the last condition could be weakened to

$$\sum_{n\le x}|f(n+1)-f(n)|=o(x)\quad \text{as}\quad x\to\infty.$$

This was later proved by I. Kátai in 1970 and E. Wirsing in 1971.

In 1981 I. Kátai (*Studia Sci. Math. Hungar.*, **16** (1981), 289–295) stated as a conjecture that if $f\in\mathcal{A}$ satisfies the condition

$$\|f(n+1)-f(n)\|=o(1)\quad \text{as}\quad n\to\infty,$$

then there is a suitable real number τ such that

$$\|f(n)-\tau\log n\|=0\quad \text{for all}\quad n\in\mathbb{N}.$$

Improving on earlier results of I. Kátai, E. Wirsing, Tang Yuansheng, Shao Pintsung in 1996 and E. Wirsing, D. Zagier in 2001 showed that this conjecture is true.

In 1981 I formulated the conjecture

Conjecture 1. *If $g \in \mathcal{M}$ and*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} |g(n+1) - g(n)| = 0,$$

then either $\frac{1}{x}\sum_{n\le x} |g(n)| \to 0$, or $g(n) = n^s$, $s \in \mathbb{C}, \Re s < 1$.

A consequence of it is the

Conjecture 2. *If $f \in \mathcal{A}$ and*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} \|f(n+1) - f(n)\| = 0,$$

then there is a suitable real number τ for which $f(n) - \tau \log n = h(n), h(n) \in \mathbb{Z}$ for all $n \in \mathbb{N}$.

A. Hildebrand in 1988 proved that there exists a positive constant δ with the following property. If $f \in \mathcal{A}^*$ satisfies $\|f\| \not\equiv 0$ and

$$\|f(p)\| \le \delta \quad \text{for all primes} \quad p \in \mathcal{P},$$

then

$$\liminf_{x\to\infty} \frac{1}{x} \sum_{n\le x} \|g(n+1) - g(n)\| > 0.$$

It follows from this result that the above conjecture is true for additive functions satisfying the condition $\|f(p)\| \le \delta$ for all $p \in \mathcal{P}$.

The following result improves the above theorem of Hildebrand.

Theorem F. ([77]) *Let $k \in \mathbb{N}$ be fixed. Then there exists a suitable $\eta(> 0)$ for which, if $f_j \in \mathcal{A}$ $(j = 0, \dots, k)$, and $\|f_j(p)\| \le \eta$ for every $p \in \mathcal{P}$, $(j = 0, \dots, k)$, then*

$$\liminf_{x\to\infty} \frac{1}{x} \sum_{n\le x} \|f_0(n) + \ldots + f_k(n+k) + \Gamma\| = 0$$

implies that for every $p \in \mathcal{P}$, $p > k+1$, $\alpha \in \mathbb{N}$, we have

$$f_0(p^\alpha) \equiv \ldots \equiv f_k(p^\alpha) \equiv 0 \pmod 1,$$

furthermore

$$f_j(p^\beta) \equiv f_j(p^{\beta+1}) \pmod 1 \quad \textit{if} \quad p^\beta > k+1, j = 0, \dots, k.$$

Consequently, if $f_j \in \mathcal{A}^*$ $(j = 0, \dots, k)$, then

$$\Gamma \equiv 0 \pmod 1 \quad \textit{and } f_0(n) \equiv \ldots f_k(n) \equiv 0 \pmod 1 \quad (n \in \mathbb{N}).$$

Recently, O. Klurman proved Conjecture 1.

From his proof one can prove easily that if $g \in \mathcal{M}^*$ and

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{n\le x} \frac{|g(n+1)-g(n)|}{n} = 0,$$

then either $g(n) = n^s (s \in \mathbb{C}, 0 \le \Re s < 1)$ or

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{n\le x} \frac{|g(n)|}{n} = 0.$$

In a joint paper of Indlekofer, Kátai and Bui [97] we proved: *If* $f \in \mathcal{M}^*$,

$$\sum_{n\le x} \frac{|f(n+1)-f(n)|}{n} = O(\log x),$$

then either $f(n) = n^s$ $(s \in \mathbb{C}, 0 \le \Re s < 1)$ *or* $\sum_{n\le x} \frac{|f(n)|}{n} = O(\log x)$.

They proved furthermore: *Let* $f \in \mathcal{M}$, $f(n) = 0$ *when* $(n, K) > 1$, (K *is a fixed natural number*),

$$\limsup_{x\to\infty} \frac{1}{\log x} \sum_{n\le x} \frac{|f(n)|}{n} = \infty$$

and

$$\limsup_{x\to\infty} \frac{1}{\log x} \sum_{n\le x} \frac{|f(n+K)-f(n)|}{n} < \infty,$$

then $f(n) = n^s\chi(n)$ $(s \in \mathbb{C}, 0 \le \Re s < 1)$, *where* χ *is a Dirichlet character* $(\mathrm{mod}\ K)$.

Recently, Bui and I proved the following two theorems:

Theorem G. ([100]) *Let* $P(x) = a_0 + a_1x + \cdots + a_kx^k$, $k \in \{1,2,3\}$, $f \in \mathcal{M}$, $P(E)f(n) = a_0f(n) + a_1f(n+1) + \cdots + a_kf(n+k)$. *Assume that*

$$\sum_{n\le x} \frac{|P(E)f(n)|}{n} = O(\log x)$$

Then either

$$\sum_{n\le x} \frac{|f(n)|}{n} = O(\log x)$$

or

$$f(n) = n^sF(n) \quad (n \in \mathbb{N}), \quad 0 \le \Re s < 1$$

and

$$P(E)F(n) = 0 \quad \textit{for all} \quad n.$$

Theorem H. ([106]) *Let* $f, g \in \mathcal{M}^*$, $f(n)g(n) \neq 0 \quad (n \in \mathbb{N})$, $A \neq 0$, *and that*

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{n\le x} \frac{|g(2n+1) - Af(n)|}{n} = 0,$$

$$\overline{\lim} \frac{1}{\log x} \sum_{n\le x} \frac{|f(n)|}{n} > 0.$$

Then

$$A = f(2), f(n) = n^{\delta+it} \quad (n \in \mathbb{N}), \quad 0 \le \delta < 1, t \in \mathbb{R}$$

and

$$f(n) = g(n) \quad \textit{for all odd} \quad n.$$

For some integer $q \ge 2$ let $\mathcal{A}_q$ be the set of q-additive functions. Every $n \in \mathbb{N}_0$ can be uniquely represented in the form

$$n = \sum_{r=0}^{\infty} a_r(n) q^r \quad \text{with} \quad a_r(n) \in \{0, 1, \cdots, q-1\} (= A_q)$$

and $a_r(n) = 0$ if $q^r > n$. We say that $f \in \mathcal{A}_q$, if $f : \mathbb{N}_0 \to \mathbb{C}$

$$f(0) = 0 \quad \text{and} \quad f(n) = \sum_{r=0}^{\infty} f\Big(a_r(n) q^r\Big) \quad \text{for all} \quad n \in \mathbb{N}.$$

Let $G = \mathbb{Z}[i]$ be the set of Gaussian integers. Kátai and Szabó proved (*Acta Sci. Math Szeged*, **37** (1975), 255–260) that if $\theta = -A \pm i \quad (A \in \mathbb{N})$, then every $\gamma \in G$ has a unique representation of the form

$$\gamma = r_0 + r_1\theta + \cdots + r_k\theta^k,$$

where $r_j \in \mathbb{A}_\theta = \{0, 1, \ldots, N-1\}$, $N = A^2 + 1$.

For some Gaussian integer $\theta = -A \pm i \quad (A \in \mathbb{N})$ let $\mathcal{A}_\theta$ be the set of θ-additive functions, that is $f \in \mathcal{A}_\theta$, if $f : G \to \mathbb{C}$,

$$f(0) = 0 \quad \text{and} \quad f(\gamma) = \sum_{j=0}^{\infty} f\Big(r_j \theta^j\Big).$$

A function $f : G \to \mathbb{C}$ is completely multiplicative ($f \in \mathcal{M}^*(G)$), if

$$f(\alpha\beta) = f(\alpha)f(\beta) \quad \text{for all} \quad \alpha, \beta \in G.$$

Theorem I. ([95]) *Let $G=\mathbb{Z}[i]$ be the set of Gaussian integers. Assume that $\theta=-A+i$, $A\in\mathbb{N}$ and $f\in\mathcal{M}^*(G)\cap\mathcal{A}_\theta$.*

(A) If $f(N)=f(\theta)f(\overline{\theta})\neq 0$, then either $f(\gamma)=\gamma$ or $f(\gamma)=\overline{\gamma}$ for every $\gamma\in G$.

(B) If $f(N)=f(\theta)f(\overline{\theta})=0$, then

$$f(\gamma)=f(c),\quad \textit{where}\quad c\in\mathbb{A}_\theta,\quad \gamma\equiv c \pmod N.$$

Theorem J. ([99]) *Assume that $f\in\mathcal{M}^*$, $g\in\mathcal{A}_2$, $h\in\mathcal{A}_3$ satisfy the condition*

$$f(n)=g(n)+h(n)\quad \textit{for all}\quad n\in\mathbb{N}.$$

Then

$$f(1)=g(1)+h(1)=1.$$

For every $n\in\mathbb{N}$, one of the following assertions hold:

$$f(n)=\chi_2(n),\quad g(n)=-h(1)n+\chi_2(n),\quad h(n)=h(1)n,$$

where χ_2 is the Dirichlet-character $\pmod 2$, or

$$f(n)=\chi_3(n),\quad g(n)=g(1)n,\quad h(n)=-g(1)n+\chi_3(n),$$

$$f(n)=n,\quad g(n)=g(1)n,\quad h(n)=h(1)n.$$

The solutions of all those pairs $f\in\mathcal{M}^*, g\in\mathcal{A}_\theta$, for which $f(n)=g^2(n)$ for all $n\in\mathbb{N}$ are given as follows:

Theorem K. ([101]) *Assume that $f\in\mathcal{M}^*$, $g\in\mathcal{A}_\theta$ satisfy the condition*

$$f(n)=g^2(n)\quad \textit{for all}\quad n\in\mathbb{N}.$$

Then either

$$f(q_0)=0,\ q_0|q\quad \textit{and}\quad f(n)=g^2(n)=\chi_{q_0}(n),$$

where χ_{q_0} is a Dirichlet character $\pmod{q_0}$, or $f(n)=n^2$ for all $n\in\mathbb{N}$, and either $g(n)=n$ $(\forall n\in\mathbb{N})$, or $g(n)=-n$ $(\forall n\in\mathbb{N})$.

III. On additive functions with values in Abelian groups

I formulated the next conjectures.

Conjecture 3. *If $f_0, f_1, \ldots, f_k\in\mathcal{A}^*$ and*

$$\|f_0(n)+f_1(n+1)+\cdots+f_k(n+k)\|\ \to 0,\quad \textit{as}\quad n\to\infty,$$

then there are $\tau_0, \dots, \tau_k \in \mathbb{R}$ *such that*

$$\tau_0 + \cdots + \tau_k = 0$$

and

$$\|f_0(n) - \tau_0 \log n\| = \cdots = \|f_k(n) - \tau_k \log n\| = 0$$

for all $n \in \mathbb{N}$*.*

Conjecture 4. *If* $f_0,\ f_1, \dots, f_k \in \mathcal{A}^*$ *and*

$$f_0(n) + f_1(n+1) + \cdots + f_k(n+k) \in \mathbb{Z} \quad \textit{for all} \quad n \in \mathbb{N},$$

then

$$f_j(n) \in \mathbb{Z} \quad \textit{for all} \quad n \in \mathbb{N} \quad \textit{and} \quad j = 0, 1, \dots, k.$$

Conjecture 2 is known for $k = 2, 3$

It is proved in 2012 by K. Chakraborty, I. Kátai and B. M. Phong that Conjecture 2 is true for the case $k = 4$ by assuming that the relation

$$f_0(n) + f_1(n+1) + f_2(n+1) + f_3(n+1) + f_4(n+1) \in \mathbb{Z}$$

holds for all $n \in \mathbb{Z}$.

Theorem L. ([74]) *If* $\gg_0 \subseteq \gg$ *are Abelian groups,* $\Gamma \in \gg$*,* $\{f_0, f_1, f_2, f_3, f_4, f_5\} \subseteq \subseteq \mathcal{A}^*_{\gg}$ *and*

$$f_0(n) + f_1(n+1) + f_2(n+2) + f_3(n+3) + f_4(n+4) + f_5(n+5) + \Gamma \in \gg_0$$

is true for all $n \in \mathbb{Z}$*, then*

$$\Gamma \in \gg_0 \quad \textit{and} \quad f_j(n) \in \gg_0 \quad \textit{for all} \quad n \in \mathbb{Z},\ j \in \{0, 1, \cdots, 5\}.$$

IV. A consequence of the ternary Goldbach theorem

H. A. Helfgott proved in 2013 that every odd integer larger than 5 can be written as the sum of three primes. We deduced from this the following theorem.

Theorem M. ([83]) *Let* $k \geq 3, k$ *fix,* $f : \mathbb{N} \to \mathbb{C}$*,* $g : \mathcal{P} \to \mathbb{C}$*. Assume that*

$$f\,(p_1 + p_2 + \cdots + p_k) = g(p_1) + g(p_2) + \cdots + g(p_k)$$

holds for every $p_1, p_2, \cdots, p_k \in \mathcal{P}$*. Then there exist suitable constants* $A, B \in \mathbb{C}$ *such that*

$$f(n) = An + kB \quad \text{for all} \quad n \in \mathbb{N}, n \geq 2k \tag{2.2}$$

and

$$g(p) = Ap + B \quad \text{for all} \quad p \in \mathcal{P}.)$$

V. On the multiplicative group generated by $\left\{\frac{[\sqrt{2}n]}{n}\right\}$

Let α, β be distinct positive real numbers and let $\mathbb{Q}_+$ be the multiplicative group of the positive rationals. In [84], we started the following conjecture.

Conjecture 1. *Let* $\xi_n = \frac{[\alpha n]}{[\beta n]}$ $(n \in \mathbb{N})$, $\mathcal{F}_{\alpha,\beta}$ *be the multiplicative group generated by* $\{\xi_n \mid n \in \mathbb{N}\}$*. If* α, β *are distinct real numbers at least one of which is irrational , then* $\mathcal{F}_{\alpha,\beta} = \mathbb{Q}_+$.

We proved this conjecture in the special case $\alpha = \sqrt{2}, \beta = 1$ and proved some theorems:

- ([84]) *Assume that* $f \in \mathcal{A}^*$ *and*

$$f\Big([\sqrt{2}n]\Big) - f(n) \ \to C \ (n \to \infty).$$

Then $f(n) = A \log n \ \ (n \in \mathbb{N})$*, where* $A = \frac{2C}{\log 2}$.

- ([85]) *Let* $f, g \in \mathcal{A}^*$, $C \in \mathbb{R}$, $\delta(n) = g([\sqrt{2}n]) - f(n) - C$*. Assume that*

$$\lim_{x \to \infty} \frac{1}{x} \sharp \Big\{ n \leq x \ \Big| \ |\delta(n)| > \epsilon \Big\} = 0$$

holds for every $\epsilon > 0$*. Then* $f(n) = g(n) = A \log n$*, where* $A = \frac{2C}{\log 2}$.

- ([86]) *Let* $\varepsilon(n) \downarrow 0$*. Assume that*

$$\sum_{n=2}^{\infty} \frac{\varepsilon(n) \log\log 2n}{n} < \infty.$$

Let $f, g \in \mathcal{M}^*$, $C \in \mathbb{C}$, $|f(n)| = |g(n)| = 1 \ \ (n \in \mathbb{N})$ *and*

$$|g([\sqrt{2}n]) - Cf(n)| \leq \varepsilon(n) \ \ (n \in \mathbb{N}).$$

Then $f(n) = g(n) = n^{i\tau} \ \ (\tau \in \mathbb{R})$*, where* $C = (\sqrt{2})^{i\tau}$.

- ([86]) *If* $f, g \in \mathcal{M}^*$, $C \in \mathbb{C}, C \neq 0$, $|f(n)| = |g(n)| = 1 \ \ (n \in \mathbb{N})$ *and*

$$\sum_{n=1}^{\infty} \frac{|g([\sqrt{2}n]) - Cf(n)|}{n} < \infty,$$

then $f(n) = g(n) \ \ (n \in \mathbb{N})$*, furthermore* $C^2 = f(2)$.