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LAUDATION TO Professor Jean-Marie De Koninck on his seventieth birthday

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Jean-Marie De Koninck obtained his Ph.D. in mathematics at Temple University in 1972 under the supervision of Professor Emil Grosswald. He taught mathematics at Université Laval in Quebec City (Canada) from 1972 to 2016. He is now Professor emeritus at Université Laval. Throughout his career, he received numerous awards, including two Honorary Doctorates, one from University of Moncton in 2010 and one from University of Ottawa in 2016; he became a member of the Order of Canada in 1994 and was named Officer of the Order of Canada in 2014. We first provide the highlights of his mathematical results and then describe his math outreach initiatives as well as his contributions to the well being of society.

A. Highlights of mathematical results

1. Reciprocals and quotients of arithmetical functions

Let $\omega(n) := \sum_{p|n} 1$ and $\Omega(n) := \sum_{p^a\|n} a$. In 1974, De Koninck [2] proved that

$$\frac{1}{x}\sum_{2\le n\le x}\frac{\Omega(n)}{\omega(n)} = 1+\frac{c+o(1)}{\log\log x}\qquad (x\to\infty),$$

where $c=\sum_p \frac{1}{p(p-1)}$, with similar estimates holding for $\frac{1}{x}\sum_{2\le n\le x}\frac{f(n)}{g(n)}$, where f and g are various additive functions.

Set $\beta(n) := \sum_{p|n} p$ and $B(n) := \sum_{p^a\|n} ap$. De Koninck and Ivić [12] proved that, given a fixed integer $m\ge 1$, there exist computable constants $a_1, a_2, \ldots, a_m$

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and $b_1, b_2, \dots, b_m$ such that

$$\sum_{2\le n\le x} \frac{\beta(n)}{\omega(n)} = x^2\left(\frac{a_1}{\log x} + \frac{a_2}{\log^2 x} + \cdots + \frac{a_m}{\log^m x} + O\left(\frac{1}{\log^{m+1} x}\right)\right)$$

and

$$\sum_{2\le n\le x} \frac{B(n)}{\Omega(n)} = x^2\left(\frac{b_1}{\log x} + \frac{b_2}{\log^2 x} + \cdots + \frac{b_m}{\log^m x} + O\left(\frac{1}{\log^{m+1} x}\right)\right).$$

Let $P(n)$ stand for the largest prime factor of n. It is known that

$$\sum_{2\le n\le x} \frac{1}{P(n)} = (1+o(1))\frac{x}{L(x)}, \quad \text{where } L(x)^{-1} = \int_2^x \rho\left(\frac{\log x}{\log t}\right)\frac{dt}{t^2},$$

where ρ stands for the Dickman function (Erdős, Ivić and Pomerance, 1986). Given an integer $k \ge 2$, let $P_k(n)$ stand for the k-th largest prime factor of n. De Koninck [28] showed that

$$\sum_{\substack{2^k\le n\le x\\ \Omega(n)\ge k}} \frac{1}{P_k(n)} = \lambda_k \frac{x(\log\log x)^{k-2}}{\log x}\left(1 + O\left(\frac{1}{\log\log x}\right)\right),$$

where $\lambda_k = \dfrac{1}{(k-2)!}\displaystyle\sum_{m\ge 2}\sum_p \frac{1}{p^2}\prod_{q\le p}\left(1-\frac{1}{q}\right)^{-1}$.

Let Q be a set of primes of positive density $\delta < 1$ and denote by $P(n,Q)$ the largest prime factor of n which belongs to Q (setting $P(n,Q) = +\infty$ if no such prime factor exists). Then, letting q_0 be the smallest element of Q, De Koninck [28] showed that there exists a constant $\eta(Q) > 0$ such that

$$\sum_{q_0\le n\le x} \frac{1}{P(n,Q)} = \left(1 + O\left(\frac{1}{\log\log x}\right)\right)\eta(Q)\frac{x}{(\log x)^{\delta}}.$$

In the same paper, De Koninck showed that the median value of $P_k(n)$ among the positive integers $n \le x$ is $n^{\kappa+o(1)}$, as $x \to \infty$, where κ is the unique solution of $\rho_k(1/\kappa) = 1/2$ (here $\rho_k(u)$ is the unique continuous solution of the differential equation $u\rho_k'(u) + \rho_k(u-1) = \rho_{k-1}(u-1)$ with the initial condition $\rho_k(u) = 1$ for $0 \le u \le 1$). De Koninck's estimate thus improved an earlier result of Wunderlich and Selfridge.

2. The most popular largest prime factor

Let $f(x,p) := \#\{n \le x : P(n) = p\}$. De Koninck [29] proved that, for large x, the maximum value of $f(x,p)$ is obtained at some prime $p = p(x)$ satisfying

$$p = \exp\{(1+o(1))\sqrt{(1/2)\log x \log\log x}\} \qquad (x \to \infty),$$

a result improved in 2017 by Nathan McNew, who showed that

$$p = \exp\left\{\sqrt{\nu(x)\log x} + \frac{1}{4}\left(1 - \frac{\nu(x)-3}{2\nu(x)^2 - 3\nu(x) + 1}\right)\right\} \times \\ \times \left(1 + O\left(\left(\frac{\log\log x}{\log x}\right)^{1/4}\right)\right),$$

where $\nu(x)$ is the solution to the equation $e^{\nu(x)} = 1 + \sqrt{\nu(x)\log x} - \nu(x)$ and is given by

$$\nu(x) = \frac{1}{2}\log\log x + \frac{1}{2}\log\log\log x - \frac{1}{2}\log 2 + o(1) \qquad (x \to \infty).$$

De Koninck and Sweeney [38] showed that when the function $f(x,p)$ is considered as a function of p, there exists two functions $v(x)$ and $w(x)$ for which $f(x,p)$ increases in the interval $[2, v(x)]$, oscillates in the interval $(v(x), w(x))$ and decreases in the interval $(w(x), x)$; they also showed that $v(x) \ge \sqrt{\log x}$ whereas $w(x) \le \sqrt{x}$.

3. The median value of the k-th prime factor of an integer

Let $p_1(n) < p_2(n) < \cdots < p_{\omega(n)}(n)$ be the distinct prime factors of an integer $n \ge 2$. Given an integer $k \ge 1$, let p_k^* stand for the median value of the k-th prime factor of an integer. Clearly $p_1^* = 2$, and one can show that

$$p_2^* = 37, \qquad p_3^* = 42\,719, \qquad p_4^* = 5\,737\,850\,066\,077.$$

In [41], De Koninck and Tenenbaum proved that

$$p_k^* = \exp\left\{k - b + O\left(\frac{1}{\sqrt{k}}\right)\right\},$$

where $b = 0.59483\ldots$

4. The counting function for the Niven numbers

An integer $n \ge 1$ is called a *Niven number* if it is divisible by the sum of its digits (in base 10). More generally, given an integer $q \ge 2$, a positive integer

n is said to be a *q-Niven number* if it is divisible by the sum of its base q digits. Let $N_q(x)$ stand for number of q-Niven numbers not exceeding x. As of 2003, no asymptotic formula was known for $N_q(x)$, even in the case $q = 10$. In 2003, De Koninck, Doyon and Kátai [45] obtained the long sought asymptotic formula

$$N_q(x) = (\eta_q + o(1))\,\frac{x}{\log x}, \qquad \text{where } \eta_q = \frac{2\log q}{(q-1)^2}\sum_{j=1}^{q-1}\gcd(j, q-1).$$

We say that the positive integers $n, n+1$ are *twin Niven numbers* in base q if they are both q-Niven numbers. More generally, given an integer $r \geq 2$, we say that $(n, n+1, \ldots, n+r-1)$ is a *q-Niven r-tuple* if each number $n+i$, for $i = 0, 1, \ldots, r-1$, is a q-Niven number. In 1994, Grundman showed that, given any $q \geq 2$, no q-Niven r-tuple, with $r > 2q$, exists. Moreover, she conjectured that, for each integer $r \in [2, 2q]$, there exist infinitely many q-Niven r-tuples. So, given fixed integers $q \geq 2$ and $r \in [2, 2q]$, let $N_q^{(r)}(x)$ stand for the number of q-Niven r-tuples whose leading component is $< x$. In 1997, Wilson showed that Grundman's conjecture is true for any given integer $q \geq 2$. In 2008, De Koninck, Doyon and Kátai [79] proved a quantitative version of Wilson's result, namely by establishing that, given fixed integers $q \geq 2$ and $r \in [2, 2q]$, there exists a constant $c = c(q, r)$ such that

$$N_q^{(r)}(x) = (c + o(1))\frac{x}{\log^r x} \qquad (x \to \infty).$$

5. The index of composition of integers

Given an integer $n \geq 2$, let $\gamma(n) := \prod_{p|n} p$ stand for the product of the distinct primes dividing n and let $\lambda(n) := \log n / \log \gamma(n)$ stand for the *index of composition* of n (setting for convenience $\gamma(1) = \lambda(1) = 1$). In particular, if n is a powerful number, then $\lambda(n) \geq 2$. De Koninck and Doyon [46] showed that λ and $1/\lambda$ both have asymptotic mean value 1. Concerning the local behaviour of $\lambda(n)$, they proved that, given any $\varepsilon > 0$ and any integer $k \geq 2$, there exist infinitely many positive integers n such that

$$(*) \qquad \min(\lambda(n), \lambda(n+1), \ldots, \lambda(n+k-1)) > \frac{k}{k-1} - \varepsilon.$$

They also showed that if the *abc* conjecture is true, then, for any fixed $\varepsilon > 0$, there exist only a finite number of integers n such that

$$\min(\lambda(n), \lambda(n+1), \lambda(n+2)) > \frac{3}{2} + \varepsilon,$$

which indicates that the above $(*)$ inequality, in the case $k = 3$, is in a sense best possible. Incidently, letting $n = 85\,016\,574$, one finds that $\lambda(n) \approx$

≈ 1.72085, $\lambda(n+1) \approx 1.80738$ and $\lambda(n+2) \approx 1.97442$, so that $\min(\lambda(n), \lambda(n+1), \lambda(n+2)) > 1.72$. Assuming that 1.72 is indeed the largest possible value of $\min(\lambda(n), \lambda(n+1), \lambda(n+2))$ as n runs through the positive integers, this would be in contradiction with the existence of three consecutive powerful numbers n_1, n_1+1, n_1+2 for which we would clearly have $\min(\lambda(n_1), \lambda(n_1+1), \lambda(n_1+2)) \geq 2$. Concerning the distribution function $F(z,x) := \#\{n \leq x : \lambda(n) > z\}$, they proved that, given a real $z \in (1,2)$ and any small number $\varepsilon > 0$, then, provided x is sufficiently large,

$$\exp\left\{2(1-\varepsilon)\sqrt{\frac{2(1-1/z)\log x}{\log\log x}}\right\} < \frac{F(z,x)}{x^{1/z}} <$$

$$< \exp\left\{2(1+\varepsilon)\sqrt{\frac{2(1-1/z)\log x}{\log\log x}}\right\},$$

this last inequality also holding if $z \geq 2$.

In [81], De Koninck and Luca showed that the average value of the index of composition of the Euler totient function φ is 1. Then, recently, De Koninck and Kátai [132] showed that, given any integer $k \geq 1$, the index of composition of the k-th iterate of $\varphi(n)$ tends to 1 on a set of density 1.

6. On the distance between smooth numbers

Letting $P(n)$ stand for the largest prime factor of $n \geq 2$ and set $P(1) = 1$. For each integer $n \geq 2$, let $\delta(n)$ be the distance to the nearest $P(n)$-smooth number, that is, to the nearest integer whose largest prime factor is no larger than that of n. The value $\delta(n)$ is called the *index of isolation* of n, and it is said that an integer n is isolated if $\delta(n) \geq 2$ and non-isolated if $\delta(n) = 1$. De Koninck and Doyon [95] provided a heuristic argument showing that $\sum_{n \leq x} 1/\delta(n) = (4\log 2 - 2 + o(1))x$ as $x \to \infty$. Moreover, given an arbitrary real-valued arithmetic function f, they studied the behavior of the more general function $\delta_f(n)$ defined by $\delta_f(1) = 1$ and for $n \geq 2$ by $\delta_f(n) = \min_{1 \leq m \neq n, f(m) \leq f(n)} |n-m|$. In particular, given any positive integers $a < b$, they showed that $\sum_{a \leq n < b} 1/\delta_f(n) \geq 2(b-a)/3$ and that if $f(n) \geq f(a)$ for all $n \in [a, b[$, then one has $\sum_{a \leq n < b} 1/\delta_f(n) \leq (b-a)\log(b-a)/(2\log 2)$. Unconditionally, they proved that, in each of the three cases $f(n) = \omega(n)$, $\Omega(n)$ and $\tau(n) := \sum_{d|n} 1$, we have $\sum_{n \leq x} \delta_f(n) = (1+o(1))2(2\log 2 - 1)x$ as $x \to \infty$.

7. Economical numbers

A positive integer n is said to be *economical* if its prime factorization requires no more digits than the number of decimal digits as n. In 1995, Bernardo

Recamán Santos asked whether there are arbitrarily long sequences of consecutive economical numbers. In 1998, Richard Pinch gave an affirmative answer to this question assuming the prime k-tuple conjecture. He also exhibited one such sequence of length 9 starting with the number 1 034 429 177 995 381 247 and conjectured that such a sequence of arbitrary length always exists. In 2006, De Koninck and Luca [58] provided an unconditional proof of Pinch's conjecture, in fact, in any base $B \geq 2$.

We say that a positive n is *strongly economical* in base $B \geq 2$ if its prime factorization requires less digits than the number of its base B digits. Denote by $N_B(x)$ the number of base B strongly economical integers not exceeding x. In 2007, De Koninck, Doyon and Luca [71] proved that, as $x \to \infty$, $N_B(x) = x/\log^{K(B)+o(1)} x$, where $K(B)$ is defined by

$$K(B) = \min_{\rho>0}\left(\min_{0<\beta<1}(1+\rho+\beta(\log\beta - 1 - \log(R(\rho/\beta\log B))))\right),$$

where $R(\alpha)$ is the continuous function defined by

$$R(\alpha) = \begin{cases} 0 & \text{if } \alpha \leq 0, \\ \min_{t>0} \frac{1-e^{-t}}{t} e^{\alpha t} & \text{if } 0 < \alpha < \frac{1}{2}, \\ 1 & \text{if } \alpha \geq \frac{1}{2}. \end{cases}$$

8. On a question raised by Paul Erdős

During the 1984 Oberwolfach Conference on Analytic Number Theory, Paul Erdős asked De Koninck if he could estimate the sum of the reciprocals of the middle prime factor of an integer. Some 30 years later, De Koninck and Luca answered Erdős' query in part. Indeed, writing an integer $n \geq 2$ as $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ where $p_1 < p_2 < \cdots < p_k$ are its prime factors, let $p^{(1/2)}(n) := p_{\max(1,\lfloor (k+1)/2 \rfloor)}$ stand for the middle prime factor of n, De Koninck and Luca [111] obtained the asymptotic estimate

$$\sum_{2\leq n\leq x} \frac{1}{p^{(1/2)}(n)} = \frac{x}{\log x}\exp\left((1+o(1))\sqrt{2\log_2 x \log_3 x}\right) \qquad (x\to\infty),$$

where $\log_k x$ stands for the k-iterated logarithm of x. Recently, Vincent Ouellet improved their result by establishing that

$$\sum_{2\leq n\leq x} \frac{1}{p^{(1/2)}(n)} = \frac{x}{\log x}\exp\left(\sqrt{2\log_2 x \log_3 x}\left(H(x) + O\left(\frac{1}{(\log_3 x)^2}\right)\right)\right),$$

where

$$H(x) = 1 - \frac{3\log_4 x}{2\log_3 x} + \left(\frac{3}{2}\log 2 - 1\right)\frac{1}{\log_3 x} - \frac{9}{8}\left(\frac{\log_4 x}{\log_3 x}\right)^2 + \\ + \left(\frac{9\log 2}{4} + 1\right)\frac{\log_4 x}{(\log_3 x)^2}.$$

9. On the proximity of additive and multiplicative functions

Given an additive function f and a multiplicative function g, let $E(f, g; x) = \#\{n \le x : f(n) = g(n)\}$. In [123], De Koninck, Doyon and Letendre studied the size of $E(f, g; x)$ for functions f such that $f(n) \ne 0$ for at least one integer $n > 1$. In particular, they showed that for those additive functions f whose values $f(n)$ are concentrated around their mean value $\lambda(n)$, one can find a multiplicative function g such that, given any $\varepsilon > 0$, we have $E(f, g; x) \gg x/\lambda(x)^{1+\varepsilon}$. They also showed that given any additive function satisfying certain regularity conditions, no multiplicative function can coincide with it on a set of positive density. It follows that if $\omega(n)$ stands for the number of distinct prime factors of n, then, given any $\varepsilon > 0$, there exists a multiplicative function g such that $E(\omega, g; x) \gg x/(\log\log x)^{1+\varepsilon}$, whereas for all multiplicative functions g, we have $E(\omega, g; x) = o(x)$ as $x \to \infty$.

In [126], De Koninck and Doyon examined the question of whether, given an additive function f with a limit distribution, one can find a multiplicative function g with the same limit distribution. They showed that if an additive function f has a constant asymptotic mean and constant asymptotic variance, one can construct a multiplicative function g with the same properties. It is known that, when $f = \omega$, both the asymptotic mean and variance of $f(n)$ are of the same order, namely $\log\log n$, but De Koninck and Doyon showed that no multiplicative function $g(n)$ can have the same mean and variance as $\omega(n)$.

In [137], De Koninck, Doyon and Laniel showed that each of the expressions $E(\omega, g; x)$ and $E(\Omega, g; x)$ are $O(x/\sqrt{\log\log x})$ for any integer valued multiplicative function g, thereby improving the above earlier result of De Koninck, Doyon and Letendre.

10. Construction of normal numbers

Given an integer $q \ge 2$, we say that an irrational number η is a *q-normal number*, or simply a *normal number*, if the q-ary expansion of η is such that any preassigned sequence, of length $k \ge 1$, taken within this expansion, occurs with the expected limiting frequency, namely $1/q^k$. The problem of determining if a given number is normal is unresolved. For instance, fundamental constants such as π, e, $\sqrt{2}$, $\log 2$ as well as the famous Apéry constant $\zeta(3)$, have not yet been proven to be normal numbers, although numerical evidence tends to indicate that they are. This is even more astounding if we take into account that in 1909, Borel has shown that almost all numbers are normal in every base.

In 1995, De Koninck and Kátai [31] introduced the notion of a *disjoint classification of primes*, that is a collection of $q+1$ disjoint sets of primes $\mathcal{R}, \wp_0, \wp_1, \ldots, \wp_{q-1}$, whose union is $\wp$, the set of all primes, where $\mathcal{R}$ is a finite set (perhaps empty) and where the other q sets are of positive densities $\delta_0, \delta_1, \ldots, \delta_{q-1}$ (with clearly $\sum_{i=0}^{q-1} \delta_i = 1$); setting $A_q = \{0, 1, \ldots, q-1\}$, letting an expression of the form $i_1 \ldots i_k$, where each $i_j \in A_q$, be a *word* of length k, writing A_q^* for the set of all words regardless of their length, and using the function $H : \mathbb{N} \to A_q^*$ defined by $H(n) = H(p_1^{a_1} \cdots p_r^{a_r}) = \ell_1 \ldots \ell_r$, where each ℓ_j is such that $p_j \in \wp_{\ell_j}$, they investigated the size of the set of positive integers $n \le x$ for which $H(n) = \alpha$ for a given word $\alpha \in A_q^k$. By this approach, they could show [93] the following result: *Let $q \ge 2$ be an integer and let $\mathcal{R}, \wp_0, \wp_1, \ldots, \wp_{q-1}$ be a disjoint classification of primes. Assume that, for a certain constant $c_1 \ge 5$,*

$$\pi([u, u+v] \cap \wp_i) = \frac{1}{q}\pi([u, u+v]) + O\left(\frac{u}{\log^{c_1} u}\right) \tag{1}$$

uniformly for $2 \le v \le u$, $i = 0, 1, \ldots, q-1$, as $u \to \infty$. Furthermore, let $H : \wp \to A_q^*$ be defined by

$$H(p) = \begin{cases} \Lambda & \text{if } p \in \mathcal{R}, \\ \ell & \text{if } p \in \wp_\ell \text{ for some } \ell \in A_q \end{cases} \tag{2}$$

(here Λ stands for the empty word) and further let $T : \mathbb{N} \to A_q^*$ be defined by $T(1) = \Lambda$ and for $n \ge 2$ by

$$T(n) = T(p_1^{a_1} \cdots p_r^{a_r}) = H(p_1) \ldots H(p_r). \tag{3}$$

Then, the number $0.T(1)T(2)T(3)T(4)\ldots$ is a q-normal number.

In a subsequent paper [102], they weakened condition (1) to allow for the construction of even larger families of normal numbers. For instance, they showed the following result: *Assume that $\mathcal{R}, \wp_0, \ldots, \wp_{q-1}$ are disjoint sets of primes, whose union is $\wp$, and assume that there exists a positive number $\delta < 1$ and a real number $c_1 \ge 5$ such that*

$$\pi([u, u+v] \cap \wp_i) = \delta\pi([u, u+v]) + O\left(\frac{u}{\log^{c_1} u}\right)$$

holds uniformly for $2 \le v \le u$, $i = 0, 1, \ldots, q-1$, and similarly

$$\pi([u, u+v] \cap \mathcal{R}) = (1 - q\delta)\pi([u, u+v]) + O\left(\frac{u}{\log^{c_1} u}\right).$$

Let H and T be defined as in (2) and (3). Then, the numbers $0.T(1)T(2)T(3)\ldots$ and $0.T(1)T(2)T(4)T(6)T(10)\ldots T(p-1)\ldots$ (where p runs through the sequence of primes) are q-normal numbers.

This type of result motivated Igor Shparlinski to ask if the number

$$0.P(2)P(3)P(4)\ldots,$$

where $P(n)$ stands for the largest prime factor of n, is a normal number in base 10. He further asked if the number

$$0.P(2+1)P(3+1)P(5+1)P(7+1)P(11+1)\ldots P(p+1)\ldots$$

is also a normal number in base 10.

In 2011, De Koninck and Kátai [96] answered both these questions in the affirmative, proving even more. To understand their breakthrough, we must first introduce some notation. Let $q \geq 2$ be a fixed integer and let A_q be as above. Given a positive integer n, write its q-ary expansion as

$$n = \varepsilon_0(n) + \varepsilon_1(n)q + \cdots + \varepsilon_t(n)q^t,$$

where $\varepsilon_i(n) \in A_q$ for $0 \leq i \leq t$ and $\varepsilon_t(n) \neq 0$. To this representation, associate the word

$$\overline{n} = \varepsilon_0(n)\varepsilon_1(n)\ldots\varepsilon_t(n) \in A_q^{t+1}.$$

Let $F \in \mathbb{Z}[x]$ be a polynomial with positive leading coefficient and of positive degree r. In their 2011 paper, De Koninck and Kátai proved that

$$0.\overline{F(P(2))}\,\overline{F(P(3))}\,\overline{F(P(4))}\ldots$$

and

$$0.\overline{F(P(2+1))}\,\overline{F(P(3+1))}\ldots\overline{F(P(p+1))}\ldots$$

are normal numbers in any given base $q \geq 2$.

In [98], they used polynomials to further construct various families of normal numbers. Let $Q_1, Q_2, \ldots, Q_h \in \mathbb{Z}[x]$ be distinct irreducible primitive monic polynomials each of degree no larger than 3. For each $\nu = 0, 1, 2, \ldots, q-1$, let $c_1^{(\nu)}, c_2^{(\nu)}, \ldots, c_h^{(\nu)}$ be distinct integers, $F_\nu(x) = \prod_{j=1}^{h} Q_j(x + c_j^{(\nu)})$, with $F_\nu(0) \neq 0$ for each ν. Moreover, assume that the integers $c_i^{(\nu)}$ are chosen in such a way that $F_\nu(x)$ are squarefree polynomials and $\gcd(F_\nu(x), F_\mu(x)) = 1$ when $\nu \neq \mu$. Let $\wp_0$ be the finite set of prime numbers p for which there exist $\mu \neq \nu$ and $m \in \mathbb{N}$ such that $p | \gcd(F_\nu(m), F_\mu(m))$ and let $\mathcal{N}(\wp_0)$ stand for the semigroup generated by $\wp_0$. Further let

$$U(n) := F_0(n)F_1(n)\cdots F_{D-1}(n) = \vartheta\, p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r},$$

where $\vartheta \in \mathcal{N}(\wp_0)$ and $p_1 < p_2 < \cdots < p_r$ are primes not belonging to $\mathcal{N}(\wp_0)$, while the a_i's are positive integers. Then, let h_n be defined on the prime divisors p^a of $U(n)$ by

$$h_n(p^a) = h_n(p) = \begin{cases} \Lambda & \text{if } p|\vartheta, \\ \ell & \text{if } p|F_\ell(n),\ p \notin \wp_0 \end{cases}$$

and further define $\alpha_n := h_n(p_1^{a_1})h_n(p_2^{a_2})\dots h_n(p_r^{a_r})$, where on the right hand side we omit Λ, the empty word, when $h_n(p_i^{a_i}) = \Lambda$ for some i. They considered the real number η whose q-ary expansion is given by $\eta = 0.\alpha_1\alpha_2\alpha_3\dots$ and showed that η is a q-normal number. Moreover, assuming that $\deg(Q_j) \le 2$ for $j = 1, 2, \dots, h$, they proved that the number $0.\alpha_2\alpha_3\alpha_5\dots\alpha_p\dots$ (where the subscripts run over the primes p) is a normal number.

In [108], they used the large prime divisors of integers to construct other families of normal numbers. Indeed, let $\eta(x)$ be a slowly varying function, that is a function satisfying $\lim\limits_{x\to\infty} \dfrac{\eta(cx)}{\eta(x)} = 1$ for any fixed constant $c > 0$, and assume also that $\eta(x)$ does not tend to infinity too fast in the sense that it satisfies the additional condition $\dfrac{\log \eta(x)}{\log x} \to 0$ as $x \to \infty$. Then, let $Q(n)$ be the smallest prime divisor of n which is larger than $\eta(n)$, while setting $Q(n) = 1$ if $P(n) > \eta(n)$. Then, they showed that the real number $0.\overline{Q(1)}\,\overline{Q(2)}\,\overline{Q(3)}\dots$ is a q-normal number. With various similar constructions, they created large families of normal numbers in any given base $q \ge 2$. For instance, consider the product function $F(n) = n(n+1)\cdots(n+q-1)$. Observe that if for some positive integer n, we have $p = Q(F(n)) > q$, then $p|n+\ell$ only for one $\ell \in \{0, 1, \dots, q-1\}$, implying that ℓ is uniquely determined for all positive integers n such that $Q(F(n)) > q$. Thus we may define the function

$$\tau(n) = \begin{cases} \ell & \text{if } p = Q(F(n)) > q \text{ and } p|n+\ell, \\ \Lambda & \text{otherwise.} \end{cases}$$

Using this notation, they proved that the number $0.\tau(q+1)\tau(q+2)\tau(q+3)\dots$ is a q-normal number.

In [100], De Koninck and Kátai used a totally different approach to create normal numbers. Their idea is based on the behaviour of the size of the gap between the prime factors of a given integer. It goes as follows. Let $q \ge 2$ be a fixed integer. Given a positive integer $n = p_1^{e_1}\cdots p_{k+1}^{e_{k+1}}$ with primes $p_1 < \cdots < < p_{k+1}$ and positive exponents $e_1, \dots, e_{k+1}$, consider the numbers $c_1(n), \dots, c_k(n)$ defined by

$$c_j(n) := \left\lfloor \frac{q \log p_j}{\log p_{j+1}} \right\rfloor \in A_q \qquad (j = 1, \dots, k)$$

and introduce the arithmetic function

$$H(n) = \begin{cases} c_1(n)\dots c_k(n) & \text{if } \omega(n) \ge 2, \\ \Lambda & \text{if } \omega(n) \le 1. \end{cases}$$

They showed that the number $0.H(1)H(2)H(3)\dots$ is a q-normal number.

Reduced residue classes can also yield a way of producing normal numbers. Indeed, letting φ stand for the Euler function, set $B_{\varphi(q)} = \{\ell_1, \ldots, \ell_{\varphi(q)}\}$ as the set of reduced residues modulo q. Let ε_n be a real function which tends monotonically to 0 as $n \to \infty$ but in such a way that $(\log\log n)\varepsilon_n \to \infty$ as $n \to \infty$. Letting $p(n)$ stand for the smallest prime factor of n, consider the set

$$\mathcal{N}^{(\varepsilon_n)} := \{n \in \mathbb{N} : p(n) > n^{\varepsilon_n}\} = \{n_1, n_2, \ldots\}.$$

De Koninck and Kátai proved in [109] that the infinite word $\mathrm{res}_q(n_1)\mathrm{res}_q(n_2)\ldots$, where $\mathrm{res}_q(n) = \ell$ if $n \equiv \ell \pmod q$, contains every finite word whose digits belong to $B_{\varphi(q)}$ infinitely often.

Is it possible to generate normal numbers using the k-th largest prime factor of an integer? It is! Indeed, given an integer $k \geq 1$, for each integer $n \geq 2$, let $P_k(n)$ stand for the k-th largest prime factor of n if $\omega(n) \geq k$, while setting $P_k(n) = 1$ if $\omega(n) \leq k-1$. Thus, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ stands for the prime factorization of n, where $p_1 < p_2 < \cdots < p_s$, then

$$P_1(n) = P(n) = p_s, \qquad P_2(n) = p_{s-1}, \qquad P_3(n) = p_{s-2}, \ldots$$

Let $F \in \mathbb{Z}[x]$ be a polynomial of positive degree satisfying $F(x) > 0$ for $x > 0$. Also, let $T \in \mathbb{Z}[x]$ be such that $T(x) \to \infty$ as $x \to \infty$ and assume that $\ell_0 = \deg T$. Fix an integer $k \geq \ell_0$. Then, De Koninck and Kátai showed in [107] that the numbers

$$0.\overline{F(P_k(T(2)))}\,\overline{F(P_k(T(3)))}\ldots\overline{F(P_k(T(n)))}\ldots$$

and (assuming that $k \geq \ell_0 + 1$)

$$0.\overline{F(P_k(T(2+1)))}\,\overline{F(P_k(T(3+1)))}\ldots\overline{F(P_k(T(p+1)))}\ldots$$

are q-normal numbers.

Given a sequence of integers $a(1), a(2), a(3), \ldots$, we call the concatenation of their q-ary expansions $\overline{a(1)}\,\overline{a(2)}\,\overline{a(3)}\ldots$, which we denote by $\mathrm{Concat}(a(n) : n \in \in \mathbb{N})$, a normal sequence if the number $0.\overline{a(1)}\,\overline{a(2)}\,\overline{a(3)}\ldots$ is a q-normal number.

In [121], De Koninck and Kátai proved that, given any polynomial $R \in \mathbb{Z}[x]$ such that $R(x) \geq 0$ for $x \geq 0$, and setting $\delta(n) := |\omega(n) - \lfloor \log\log n \rfloor|$, then the sequence $\mathrm{Concat}\Big(\overline{R(\delta(n))} : n = 3, 4, 5, \ldots\Big)$ is normal in base 2. They also obtained that $\mathrm{Concat}\Big(\overline{R(\delta(p+1))} : p \text{ prime}\Big)$ is normal in base q.

In their 2012 paper, answering a question raised by Igor Sparlinski, De Koninck and Kátai had proved that the number $0.P(2)P(3)P(4)P(5)\ldots$, where $P(n)$ stands for the largest prime factor of n, is indeed a normal number. Can one replace $P(n)$ by $p(n)$, the smallest prime factor of n, and maintain the same conclusion? The answer is "yes". In [118], De Koninck and Kátai thus proved that the number $0.p(2)p(3)p(4)p(5)\dots$ is a normal number. The result was also generalised to every base $q \ge 2$.

Now, in the above result, instead of considering the smallest prime factor, if one considered the middle prime factor, would we still obtain a normal number? More precisely, given an integer $n = p_1^{\alpha_1}\cdots p_k^{\alpha_k}$, where $p_1 < \cdots < p_k$ are its distinct prime factors, recall that its middle prime factor $p^{(1/2)}(n)$ is defined as $p_{\max(1,\lfloor\frac{k+1}{2}\rfloor)}$. In [114], De Koninck and Kátai proved that the sequence $\mathrm{Concat}(\overline{p^{(1/2)}(n)} : n \in \mathbb{N})$ is normal. Interestingly, in the same paper they showed that, if $g(x)$ is a function that tends to infinity with x arbitrarily slowly, then

$$\frac{1}{x}\#\left\{n \le x : \left|\log\log p^{(1/2)}(n) - \frac{1}{2}\log\log x\right| \le \sqrt{\log\log x}\, g(x)\right\} \to 1 \text{ as } x \to \infty.$$

Now, given a prime q, set $\xi := e^{2\pi i/q}$. Further set $x_k = 2^k$ and $y_k = x_k^{1/\sqrt{k}}$ for $k = 1, 2, \dots$ Then, consider the sequence of completely multiplicative functions f_k, $k = 1, 2, \dots$, defined on the primes p by

$$f_k(p) = \begin{cases} \xi & \text{if } k \le p \le y_k, \\ 1 & \text{if } p < k \text{ or } p > y_k. \end{cases}$$

Then, set $\eta_k := f_k(x_k)f_k(x_k+1)f_k(x_k+2)\cdots f_k(x_{k+1}-1)$ for $k = 1, 2, \dots$ and $\theta := \mathrm{Concat}(\eta_k : k \in \mathbb{N})$. Then, De Koninck and Kátai could show [119] that the number θ is a normal sequence over $C_q := \{\xi \in \mathbb{C} : \xi^q = 1\}$, the group of complex roots of unity of order q.

It is well known that if α is an irrational number, then the sequence $(\alpha n)_{n\ge1}$ is uniformly distributed modulo 1. In [127], De Koninck and Kátai were successful in constructing, given a prime number $q \ge 3$, an infinite sequence of normal numbers in base $q-1$ which, for any fixed positive integer r, yields an r-dimensional sequence which is uniformly distributed on $[0,1)^r$.

Let $q \ge 2$ be a fixed integer. Given an integer $n \ge n_0 = \max(q,3)$, let N be the unique positive integer satisfying $q^N \le n < q^{N+1}$ and let $h(n,q)$ stand for the residue modulo q of the number of distinct prime factors of n located in the interval $[\log N, N]$. For each integer $N \ge 1$, set $x_N := e^N$. Then, De Koninck and Kátai [121] showed that $\mathrm{Concat}(h(n,q) : x_{n_0} \le n \in \mathbb{N})$ is a q-ary normal sequence.

Recall that an irrational number β is said to be a *Liouville number* if for all integers $m \ge 1$, there exist two integers t and $s > 1$ such that $0 < \left|\beta - \frac{t}{s}\right| < \frac{1}{s^m}$. In [125], De Koninck and Kátai proved the following result.

> *Let α be a non Liouville number and let $f(x) = \alpha x^r + a_{r-1}x^{r-1} + + \cdots + a_1 x + a_0 \in \mathbb{R}[x]$ be a polynomial of positive degree r. Consider the sequence $(y_n)_{n\geq 1}$ defined by $y_n = f(h(n))$, where h belongs to a certain family of arithmetic functions. Then, the sequence $(y_n)_{n\geq 1}$ is uniformly distributed modulo 1.*

Let $q \geq 2$ be a fixed integer. Given an integer $n \geq 2$ and writing its prime factorization as $n = p_1 p_2 \cdots p_r$, where $p_1 \leq p_2 \leq \cdots \leq p_r$ stand for all the prime factors of n, let $\ell(n) = \overline{p_1}\,\overline{p_2}\,\ldots\,\overline{p_r}$, that is the concatenation of the respective base q digits of each prime factor p_i, and set $\ell(1) = 1$. In [138], De Koninck and Kátai proved that the real number $0.\ell(1)\ell(2)\ell(3)\ell(4)\ldots$ is a normal number in base q. In fact, they showed more, namely that the same conclusion holds if one replaces each $\overline{p_i}$ by $\overline{S(p_i)}$, where $S(x) \in \mathbb{Z}[x]$ is an arbitrary polynomial of positive degree such that $S(n) > 0$ for all integers $n \geq 1$.

In [131], De Koninck, Kátai and Phong introduced the concept of *strong normality.* It goes follows. For each positive integer N, let

$$(*) \qquad M = M_N = \lfloor \delta_N \sqrt{N} \rfloor, \text{ where } \delta_N \to 0 \text{ and } \delta_N \log N \to \infty \text{ as } N \to \infty.$$

We shall say that an infinite sequence of real numbers $(x_n)_{n\geq 1}$ is *strongly uniformly distributed* mod 1 if

$$D(x_{N+1}, \ldots, x_{N+M}) \to 0 \quad \text{as} \quad N \to \infty$$

for every choice of δ_N satisfying $(*)$. Now, given a fixed integer $q \geq 2$, we say that an irrational number α is a *strongly normal number* in base q (or a strongly q-normal number) if the sequence $(x_n)_{n\geq 1}$, defined by $x_n = \{\alpha q^n\}$, is strongly uniformly distributed mod 1. Observe that there exist normal numbers which are not strongly normal. For instance, consider the Champernowne number

$$\theta := 0.1\,10\,11\,100\,101\,110\,111\,1000\,1001\,1010\,1011\,1100\,1101\,1110\,1111\,\ldots$$

that is the number made up of the concatenation of the positive integers written in base 2. It is known since 1933 that θ is normal. However, one can show that θ is not a strongly normal number. After having introduced two simple criteria for strong uniform distribution mod 1 and for strong normality, the authors proved that the Lebesgue measure of the set of all those real numbers $\alpha \in [0,1]$ which are not strongly q-normal is 0.

Finally, in [144], De Koninck and Kátai showed that some sequences of real numbers involving sharp normal numbers (also called strongly normal numbers) or non-Liouville numbers are uniformly distributed modulo 1. In particular, they proved that if $\tau(n)$ stands for the number of divisors of n and α is a binary sharp normal number, then the sequence $(\alpha\tau(n))_{n\geq 1}$ is uniformly distributed modulo 1 and that if $g(x)$ is a real valued polynomial of positive degree whose leading coefficient is a non-Liouville number, then the sequence $(g(\tau(\tau(n))))_{n\geq 1}$ is also uniformly distributed modulo 1.

B. Reaching out to the student community

De Koninck wrote several books intended for undergraduate and graduate students. Among these, we find "1001 problèmes en théorie classique des nombres" first published in 2004 in France by *Ellipse* and then in 2007 as "1001 Problems in Classical Number Theory" by the American Mathematical Society. Another book of De Koninck is "Ces nombres qui nous fascinent" published by *Ellipse* in 2008 and thereafter in its English version as "Those Fascinating Numbers" in 2009 by the American Mathematical Society. In 2011, as a joint venture with Florian Luca, Professor De Koninck published "Analytic Number Theory: Exploring the Anatomy of Integers" as Volume 134 of Graduate Studies in Mathematics from the American Mathematical Society.

C. Math outreach activities

In 2000 and 2001, Professor De Koninck hosted a weekly television show called *C'est mathématique !* (in English: *Math is everywhere !*). This marked the beginning of a series of outreach activities initiated by Professor De Koninck. These include *Show Math*, a multi-media conference offered in high schools in which math and comedy are center stage. The idea was to show teenagers that mathematics is part of their daily lives and that it can be fun to explore mathematics. Another outreach activity initiated by De Koninck is a free online game called *Math en jeu* (or *MathAmaze*), where each player moves on a virtual board collecting points by answering math questions. Again with the goal of reaching out to kids and teenagers and also to the general public, De Koninck wrote a series of books among which we find *En chair et en math*, *En chair et en math 2*, *Cette science qui ne cesse de nous étonner* and *the Secret Life of Mathematics*. Through his math outreach activities, Professor De Koninck has reached more than 100 000 young students over the past decade. For his promotion of mathematics for the youth, Professor De Koninck has won many awards, including the *Margaret Sinclair Memorial Award* (in 2017, from the Fields Institute) and the PromoScience individual award (in 2018, from NSERC, Canada).

D. A talented organizer

Throughout the course of his academic career, Professor De Koninck managed to organize more than 50 national and international scientific events. Among these, we find the 2002 ACFAS conference held at Université Laval and attended by more than 5 000 scientists, and the *International Number Theory Conference* which he organized jointly with Claude Levesque, held July 5–18, 1987, in Quebec City and to which attended no less than 300 mathematicians from 20 countries.

E. The Renaissance Man

In parallel to his academic career, De Koninck has made important contributions to the well being of society. For instance, in 1984, he initiated *Operation Red Nose*, a national campaign against impaired driving which takes the form of a free private chauffeur service offered by volunteers across Canada during the month of December to those drivers who don't feel safe to drive their own car. Now, each year, some 58 000 volunteers participate in this national campaign. On the other hand, having been a swimming competitor in his 20's and because of his good communication skills, De Koninck was invited by CBC, the Canadian public television network, to join CBC's professional sports commentators to act as a sports analyst for the Olympic Summer Games. Thus since 1976, De Koninck has shared his passion for swimming with millions of viewers for nine Olympic Games.