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CONTINUATION OF THE LAUDATION TO
Professor Imre Kátai
on his eightieth birthday

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Five years ago in Annales Volume 40 (2013), Professors Jean-Marie De Koninck and Bui Minh Phong wrote the Laudation to Professor Imre Kátai. Recall that Kátai retired in 2008 and was immediately named professor emeritus at Eötvös Loránd University (ELTE). In 2016, he was awarded the title of **Honorary doctor of the Nguyen Tatthanh University in the Ho Chi Minh City**. In the last five years Professor Kátai has remained very active in research as can be seen by examining the 49 papers he published during that period. Here are the highlights of his results regrouped in five main topics.

1. Construction of normal numbers

Recall that, given an integer $q \geq 2$, we say that an irrational number η is a q-*normal number*, or simply a *normal number*, if the q-ary expansion of η is such that any preassigned sequence, of length $k \geq 1$, taken within this expansion, occurs with the expected limiting frequency, namely $1/q^k$. The problem of determining if a given number is normal is unresolved. For instance, fundamental constants such as π, e, $\sqrt{2}$, $\log 2$ as well as the famous Apéry constant $\zeta(3)$, have not yet been proven to be normal numbers, although numerical evidence tends to indicate that they are. This is even more astounding if we recall that in 1909, Émile Borel has shown that almost all numbers are normal in every base.

Since every positive integer n admits a unique q-ary expansion $n = \varepsilon_0(n) + \varepsilon_1(n)q + \cdots + \varepsilon_t(n)q^t$, where each $\varepsilon_i(n) \in \{0, 1, \ldots, q-1\}$ and $\varepsilon_t(n) \neq 0$, we may associate with it the word $n = \varepsilon_0(n)\varepsilon_1(n)\ldots\varepsilon_t(n)$. On the other hand, for a given sequence of integers $a(1), a(2), a(3), \ldots$, we call the concatenation

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of their q-ary expansions $\overline{a(1)}\,\overline{a(2)}\,\overline{a(3)}\ldots$, which we denote by

$$\mathrm{Concat}(a(n) : n \in \mathbb{N}),$$

a normal sequence if the number $0.\overline{a(1)}\,\overline{a(2)}\,\overline{a(3)}\ldots$ is a q-normal number.

In [392], De Koninck and Kátai proved that, given any polynomial $R \in \mathbb{Z}[x]$ such that $R(x) \geq 0$ for $x \geq 0$, if we set $\delta(n) := |\omega(n) - \lfloor \log\log n \rfloor|$, then the sequence $\mathrm{Concat}\Big(\overline{R(\delta(n))} : n = 3, 4, 5, \ldots\Big)$ is normal in base 2. They also obtained that $\mathrm{Concat}\Big(\overline{R(\delta(p+1))} : p \text{ prime}\Big)$ is normal in base q.

Recall that, in a 2012 paper, answering a question raised by Igor Sparlinski, De Koninck and Kátai proved that the number $0.P(2)P(3)P(4)P(5)\ldots$, where $P(n)$ stands for the largest prime factor of n, is a normal number. Now, can one replace $P(n)$ by $p(n)$, the smallest prime factor of n, and maintain the same conclusion? The answer is "yes". Indeed, De Koninck and Kátai [393] proved that the number $0.p(2)p(3)p(4)p(5)\ldots$ is indeed a normal number. The result was also generalised to every base $q \geq 2$.

Now, in the above result, instead of considering the smallest prime factor, if we considered the middle prime factor, would we still obtain a normal number? More precisely, given an integer $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, where $p_1 < \cdots < p_k$ are its distinct prime factors, let us define its middle prime factor $p_*(n)$ as the prime factor $p_{\lfloor \frac{k+1}{2} \rfloor}$. In [394], De Koninck and Kátai proved that the sequence $\mathrm{Concat}(\overline{p_*(n)} : n \in \mathbb{N})$ is normal. Interestingly, in the same paper they showed that, if $g(x)$ is a function that tends to infinity with x arbitrarily slowly, then

$$\frac{1}{x}\#\left\{n \leq x : \left|\log\log p_m(n) - \frac{1}{2}\log\log x\right| \leq \sqrt{\log\log x}\, g(x)\right\} \to 1 \text{ as } x \to \infty.$$

Given a prime q, set $\xi := e^{2\pi i/q}$. Further set $x_k = 2^k$ and $y_k = x_k^{1/\sqrt{k}}$ for $k = 1, 2, \ldots$ Then, consider the sequence of completely multiplicative functions f_k, $k = 1, 2, \ldots$, defined on the primes p by

$$f_k(p) = \begin{cases} \xi & \text{if } k \leq p \leq y_k, \\ 1 & \text{if } p < k \text{ or } p > y_k. \end{cases}$$

Then, set $\eta_k := f_k(x_k)f_k(x_k+1)f_k(x_k+2)\cdots f_k(x_{k+1}-1)$ for $k = 1, 2, \ldots$ and $\theta := \mathrm{Concat}(\eta_k : k \in \mathbb{N})$. De Koninck and Kátai [395] could show that the number θ is a normal sequence over $C_q := \{\xi \in \mathbb{C} : \xi^q = 1\}$, the group of complex roots of unity of order q.

It is well known that if α is an irrational number, then the sequence $(\alpha n)_{n\geq 1}$ is uniformly distributed modulo 1. In [405], De Koninck and Kátai were successful in constructing, given a prime number $q \geq 3$, an infinite sequence of normal numbers in base $q-1$ which, for any fixed positive integer r, yields an r-dimensional sequence which is uniformly distributed on $[0,1)^r$.

Let $q \geq 2$ be a fixed integer. Given an integer $n \geq n_0 = \max(q,3)$, let N be the unique positive integer satisfying $q^N \leq n < q^{N+1}$ and let $h(n,q)$ stand for the residue modulo q of the number of distinct prime factors of n located in the interval $[\log N, N]$. For each integer $N \geq 1$, set $x_N := e^N$. Then, De Koninck and Kátai [406] showed that $\mathrm{Concat}(h(n,q) : x_{n_0} \leq n \in \mathbb{N})$ is a q-ary normal sequence.

Recall that an irrational number β is said to be a *Liouville number* if for all integers $m \geq 1$, there exist two integers t and $s > 1$ such that $0 < \left|\beta - \frac{t}{s}\right| < \frac{1}{s^m}$. In [412], De Koninck and Kátai proved the following result.

> *Let α be a non Liouville number and let $f(x) = \alpha x^r + a_{r-1}x^{r-1} + + \cdots + a_1 x + a_0 \in \mathbb{R}[x]$ be a polynomial of positive degree r. Consider the sequence $(y_n)_{n\geq 1}$ defined by $y_n = f(h(n))$, where h belongs to a certain family of arithmetic functions. Then, the sequence $(y_n)_{n\geq 1}$ is uniformly distributed modulo 1.*

Let $q \geq 2$ be a fixed integer. Given an integer $n \geq 2$ and writing its prime factorization as $n = p_1 p_2 \cdots p_r$, where $p_1 \leq p_2 \leq \cdots \leq p_r$ stand for all the prime factors of n, let $\ell(n) = \overline{p_1}\,\overline{p_2}\,\ldots\,\overline{p_r}$, that is the concatenation of the respective base q digits of each prime factor p_i, and set $\ell(1) = 1$. In [417], De Koninck and Kátai proved that the real number $0.\ell(1)\ell(2)\ell(3)\ell(4)\ldots$ is a normal number in base q. In fact, they showed more, namely that the same conclusion holds if one replaces each $\overline{p_i}$ by $\overline{S(p_i)}$, where $S(x) \in \mathbb{Z}[x]$ is an arbitrary polynomial of positive degree such that $S(n) > 0$ for all integers $n \geq 1$.

In a paper written jointly with De Koninck and Phong [420], Kátai introduced the concept of *strong normality*. It goes follows. For each positive integer N, let

$$(*) \qquad M = M_N = \lfloor \delta_N \sqrt{N} \rfloor, \text{ where } \delta_N \to 0 \text{ and } \delta_N \log N \to \infty \text{ as } N \to \infty.$$

We shall say that an infinite sequence of real numbers $(x_n)_{n\geq 1}$ is *strongly uniformly distributed* mod 1 if

$$D(x_{N+1}, \ldots, x_{N+M}) \to 0 \quad \text{as} \quad N \to \infty$$

for every choice of δ_N satisfying $(*)$. Now, given a fixed integer $q \geq 2$, we say that an irrational number α is a *strong normal number* in base q (or a strong q-normal number) if the sequence $(x_n)_{n\geq 1}$, defined by $x_n = \{\alpha q^n\}$, is strongly uniformly distributed mod 1. Observe that there exist normal numbers which are not strongly normal. For instance, consider the Champernowne number

$$\theta := 0.1\,10\,11\,100\,101\,110\,111\,1000\,1001\,1010\,1011\,1100\,1101\,1110\,1111\,\ldots$$

that is the number made up of the concatenation of the positive integers written in base 2. It is known since 1933 that θ is normal. However, one can show that θ is not a strongly normal number. After having introduced two simple criteria for strong uniform distribution mod 1 and for strong normality, the authors proved that the Lebesgue measure of the set of all those real numbers $\alpha \in [0,1]$ which are not strongly q-normal is 0.

Finally, in [437], De Koninck and Kátai showed that some sequences of real numbers involving sharp normal numbers (also called strongly normal numbers) or non-Liouville numbers are uniformly distributed modulo 1. In particular, they proved that if $\tau(n)$ stands for the number of divisors of n and α is a binary sharp normal number, then the sequence $(\alpha\tau(n))_{n\geq 1}$ is uniformly distributed modulo 1 and that if $g(x)$ is a real valued polynomial of positive degree whose leading coefficient is a non-Liouville number, then the sequence $(g(\tau(\tau(n))))_{n\geq 1}$ is also uniformly distributed modulo 1.

2. Topics involving arithmetical functions

Let $\wp$ stand for the set of all primes and let $P(n)$ stand for the largest prime factor of n. Given an integer $q \geq 2$, let $s_q(n)$ stand for the sum of the base q digits of n. Finally, given a positive integer N, let $\wp_N := \{p \leq N : p \in \wp\}$. We shall say that the function $\rho_N : \wp_N \longrightarrow [0,1)$ is a *prime weight function* if it satisfies the following four conditions:

(i) $\displaystyle\sum_{p\in\wp_N} \rho_N(p) = 1 + o(1)$ as $N \to \infty$;

(ii) for every non increasing sequence $(\lambda_N)_{N\in\mathbb{N}}$ tending to 0 as $N \to \infty$, the following two assertions hold:

$$\sum_{\substack{p<N^{\lambda_N}\\ p\in\wp_N}} \rho_N(p) \to 0 \quad \text{and} \quad \sum_{\substack{N^{1-\lambda_N}<p<N\\ p\in\wp_N}} \rho_N(p) \to 0 \qquad (N \to \infty);$$

(iii) with $(\lambda_N)_{N\in\mathbb{N}}$ as in (ii),

$$\max_{\substack{N^{\lambda_N}<p_1<p_2<2p_1<N^{1-\lambda_N}\\ p_1,p_2\in\wp_N}} \left|\frac{\rho_N(p_1)}{\rho_N(p_2)} - 1\right| \to 0 \quad \text{as} \quad N \to \infty;$$

(iv) $\displaystyle\sup_{H\leq N} \left|\sum_{\substack{H\leq p<2H\\ p\in\wp_N}} \rho_N(p)\right| \to 0 \quad \text{as} \quad N \to \infty.$

An example of a weight function is given by choosing

$$\rho_N(p) := \frac{c_0}{p} \exp\left\{-\frac{\log N}{\log p}\right\}, \quad \text{where } c_0 = \left(\int_1^\infty e^{-v}\frac{dv}{v}\right)^{-1}.$$

De Koninck and Kátai wrote an extensive paper [418] in which they proved the following seven theorems:

- **Theorem 1.** *Let $f : \wp \to \mathbb{C}$ be a bounded function. Assume that for some constant $\eta \in \mathbb{C}$,*

  $$S(x) := \sum_{p \le x} f(p) = (\eta + o(1))\pi(x) \qquad (x \to \infty).$$

  Then, $\sum_{p \le N} f(p)\rho_N(p) \to \eta$ as $N \to \infty$.

- **Theorem 2.** *Let g be a real valued additive function. Then, the function $g(P(n)+1)$ has a limiting distribution if and only if g satisfies the three-series condition*

  $$\sum_{|g(p)| \ge 1} \frac{1}{p} < \infty, \qquad \sum_{|g(p)|<1} \frac{g(p)}{p} \text{ converge}, \qquad \sum_{|g(p)|<1} \frac{g^2(p)}{p} < \infty.$$

- **Theorem 3.** *Let $a \in \mathbb{Z} \backslash \{0\}$ and let $\tau(n)$ stand for the number of divisors of n. Then,*

  $$\sum_{n \le x} \tau(P(n)+a) = (\kappa D_a + o(1))x \log x \qquad (x \to \infty),$$

  where

  $$\kappa = 1 - \int_1^\infty \frac{\rho(v)}{v^2} dv \quad \text{and} \quad D_a = \frac{\zeta(2)\zeta(3)}{\zeta(6)} \prod_{p|a} \left(1 - \frac{p}{p^2 - p + 1}\right)$$

  (here ρ stands for the Dickman function).

- **Theorem 4.** *Let $a \in \mathbb{Z} \setminus \{0\}$ and let $r(n)$ stand for the number of representations of n as a sum of two squares. Then,*

  $$\sum_{n \le x} r(P(n)+a) = (\kappa R_a + o(1))x \qquad (x \to \infty),$$

  where κ is defined in Theorem 3 and where $R_a := \lim_{x\to\infty} \dfrac{\log x}{x} \sum_{p \le x} r(p+a)$.

- **Theorem 5.** *Let* $y = x^{\frac{7}{12}+\varepsilon}$ *where* $0 < \varepsilon < 5/12$ *is a fixed number. Then, given an arbitrary* $M \in \mathbb{N}$,

$$\frac{1}{xy} \sum_{x \le n \le x+y} P(n) = \sum_{k=0}^{M} \frac{\xi_k}{\log^{k+1} x} + O\left(\frac{1}{\log^{M+2} x}\right),$$

where $\xi_k = \sum_{\nu=1}^{\infty} \frac{\log^k \nu}{\nu^2}$.

- **Theorem 6.** *Let* α *be an arbitrary irrational number. Then,*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n \le x} e(\alpha s_q(P(n))) = 0.$$

- **Theorem 7.** *Given an integer* $n \ge 2$, *write its prime factorisation as*

$$n = P_r(n) P_{r-1}(n) \cdots P_1(n),$$

where $r = \Omega(n)$ *(here* $\Omega(n)$ *stands for the number of prime factors of* n *counting their multiplicity) and* $P_r(n) \le P_{r-1}(n) \le \cdots \le P_1(n)$, *setting for convenience* $P_j(n) = 1$ *if* $j > \Omega(n)$. *Further let* $f_1(p), \ldots, f_k(p)$ *be* k *functions defined on primes* p *and assume that each* $f_i(p)$ *is bounded as* p *runs over* $\wp$ *and is such that there exist constants* $C_1, C_2, \ldots, C_k$ *for which*

$$S_j(x) := \sum_{p \le x} f_j(p) = (C_j + o(1)) \frac{x}{\log x} \qquad (x \to \infty).$$

Then, $\lim_{x\to\infty} \frac{1}{x} \sum_{n \le x} \prod_{j=1}^{k} f_j(P_j(n)) = C_1 C_2 \cdots C_k$.

3. A consequence of the ternary Goldbach theorem

The ternary Goldbach theorem states that every odd integer larger than 5 can be written as the sum of three primes. It was first proved by I.M. Vinogradov for large odd numbers and, in 2013, it was proved unconditionally by H.A. Helfgott. In 2015, Kátai and Phong [409] proved the following.

> *Let* $\wp$ *stand for the set of all primes and let* $\mathcal{M}_k = \{p_1 + p_2 + \cdots + {}$ $+ p_k : p_1, p_2, \ldots, p_k \in \wp\}$. *Given an integer* $k \ge 3$ *and arithmetical functions* f *and* g *which satisfy the functional equation*
>
> $$f(p_1 + p_2 + \cdots + p_k) = g(p_1) + g(p_2) + \cdots + g(p_k)$$

for all primes $p_1, p_2, \ldots, p_k$, then there exist two constants A and B such that $f(n) = An + kB$ for all $n \in \mathcal{M}_k$ and $g(p) = Ap + B$ for all $p \in \wp$.

Interestingly, the solution to this problem allowed the authors to completely solve the functional equation

$$f(p_1 + p_2 + p_3) = f(p_1) + f(p_2) + f(p_3)$$

for all primes p_1, p_2, p_3.

4. On the multiplicative group generated by the sequence $(\lfloor\sqrt{2}n\rfloor/n)_{n\geq 1}$ and related topics

Let $\alpha \neq \beta$ be two real numbers, at least one of which is irrational and let f be a completely additive function. Further let $\mathcal{F}_{\alpha,\beta}$ be the multiplicative group generated by the sequence $(\lfloor\alpha n\rfloor/\lfloor\beta n\rfloor)_{n\geq 1}$. In [405], Kátai and Phong conjectured that $\mathcal{F}_{\alpha,\beta} = \mathbb{Q}_+$, the multiplicative group of the positive rational numbers. They proved this conjecture in the particular case $\alpha = \sqrt{2}$, $\beta = 1$. Moreover, they established that if f is a completely additive function such that, for some real number c, we have $f(\lfloor\sqrt{2}n\rfloor) - f(n) \to c$ as $n \to \infty$, then $f(n) = 2c \log n / \log 2$ for all $n \in N$.

In [410], Kátai and Phong proved that if f and g are completely additive functions such that for some $c \in \mathbb{R}$,

$$\lim_{x\to\infty} \frac{1}{x}\#\{n \leq x : |g(\lfloor\sqrt{2}n\rfloor) - f(n) - c| > \varepsilon\} = 0 \quad \text{for every } \varepsilon > 0,$$

then $f(n) = g(n) = 2c \log n / \log 2$.

In [411], Kátai and Phong proved that if f and g are completely multiplicative functions which satisfy $|f(n)| = |g(n)| = 1$ for all $n \in \mathbb{N}$ and for which there exists a real number $c \neq 0$ such that $\displaystyle\sum_{n=1}^{\infty} \frac{|g(\lfloor\sqrt{2}n\rfloor) - cf(n)|}{n} < \infty$, then $f(n) = g(n)$ for each $n \in \mathbb{N}$ and $f(2) = c^2$.

In [404], Kátai and Phong obtained the following result:

Let G be an Abelian topological group with the translation invariant metric ρ. Let $\phi : \mathbb{N} \to G$ be a completely additive function. This function can be extended to be a completely additive function on the set of positive rational numbers by the relation $\phi(m/n) = \phi(m) - -\phi(n)$. If there exists some constant c and a function $\varepsilon(n)$ which *tends to 0 as $n \to \infty$ and such that*

$$\rho(\phi(\lfloor\sqrt{2}n\rfloor), \phi(n) + c) \leq \varepsilon(n) \quad \textit{and} \quad \sum_{n\geq 2} \frac{\varepsilon(n)\log\log(2n)}{n} < \infty,$$

then, ϕ can be extended to a continuous homomorphism $\phi : \mathbb{R}^+ \to$ $\to G$.

As above, let G be an Abelian topological group with the translation invariant metric ρ. Let $\phi : \mathbb{N} \to G$ be a completely additive function. Further let $\mathbb{Q}_x$ and $\mathbb{R}_x$ stand respectively for the multiplicative group of the rational numbers and the real numbers. In [438], Kátai and Phong showed that if

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{n\leq x} \frac{\rho(\phi(n), \phi(n+1))}{n} = 0,$$

then the domain of ϕ can be extended uniquely to $\mathbb{R}_x$ by the relation

$$\phi(\alpha) := \lim_{\substack{r_\nu \to \alpha \\ r_\nu \in \mathbb{Q}_x}} \phi(r_\nu),$$

in which case the extended function $\phi : \mathbb{R}_x \to G$ is a continuous homomorphism.

5. On a long standing conjecture of Kátai

In 1983, Kátai made the following conjecture:

Let $f : \mathbb{N} \to \mathbb{C}$ be a multiplicative function such that $\sum_{n\leq x} |f(n+$ $+1) - f(n)| = o(x)$ as $x \to \infty$. Then, either $\sum_{n\leq x} |f(n)| = o(x)$ as $x \to \infty$ or $f(n) = n^s$ for some $s \in \mathbb{C}$ with $\Re(s) < 1$.

In 2016, Oleksiy Klurman proved that Kátai's conjecture is true.

Along the same lines, Indlekofer, Kátai and Phong [426] went beyond Kátai's conjecture and proved that if $f(n)$ is a completely multiplicative function such that $\sum_{n\leq x} \frac{|f(n+1) - f(n)|}{n} = O(\log x)$, then either $f(n) = n^s$, $s \in \mathbb{C}$, $0 \leq \Re(s) \leq 1$, or (trivially) $\sum_{n\leq x} \frac{|f(n)|}{n} = O(\log x)$. They even proved a more general result, namely by establishing that, given a positive integer K, if f is a complex-valued multiplicative function such that $f(n) = 0$ when $(n, K) > 1$ and such that

$$\limsup_{x\to\infty} \frac{1}{\log x} \sum_{n\leq x} \frac{|f(n)|}{n} = \infty \text{ and } \limsup_{x\to\infty} \frac{1}{\log x} \sum_{n\leq x} \frac{|f(n+K) - f(n)|}{n} < \infty,$$

then $f(n) = n^s\chi(n)$, where $s \in \mathbb{C}$, $0 \le \Re(s) \le 1$ and χ is a Dirichlet character mod K.

Given a polynomial $P(x) = a_0 + a_1x + \cdots + a_kx^k \in \mathbb{C}[x]$, where $k \in \{1, 2, 3\}$, and a multiplicative function f, set $P(E)f(n) := a_0f(n) + a_1f(n+1) + \cdots + a_kf(n+k)$. In [432], Kátai and Phong proved that if

$$\sum_{n \le x} \frac{|P(E)f(n)|}{n} = O(\log x),$$

then, either $\sum_{n \le x} \frac{|f(n)|}{n} = O(\log x)$ or there exist $s \in \mathbb{C}$ with $0 < \Re(s) \le k$ and a multiplicative function f such that $f(n) = n^sF(n)$ and $P(E)F(n) = 0$ for all $n \in \mathbb{N}$.