1. Linear functional equations

In the early paper "Necessary and sufficient conditions for the existence of non-constant solutions of linear functional equations" (in German), one of the first works of Zoltán, published in Acta Sci. Math. Szeged in 1961, the following problem was discussed: let $f : \mathbb{R} \to \mathbb{R}$ be a non-identically zero additive function, and let $a \neq 0$, $p \neq 0$, $a \neq p$ be irrational numbers. What is the necessary and sufficient condition for that

$$f(ax) = pf(x)$$

holds for all x in $\mathbb{R}$? The answer is: either a and p are the roots of the same irreducible polynomial over the rationals, or a and p are transcendental over the rationals. The pre-history of this functional equation goes back to 1960 when Zoltán wrote his work as a student under the same title as above in Hungarian, the first documented scientific publication of him, published in the collection of "Essays in the Student Scientific Circle" (Tankönyvkiadó, Budapest). The problem would be discussed again in his PhD Thesis in 1962. It is worth mentioning that this problem was warmed-up and reconsidered in a more general setting by Adrienn Varga in her PhD thesis in 2012, where the candidate studied the so-called functional equation of Daróczy:

$$a_n A(t) + \sum_{i=1} a_i A(\beta_i t) = 0,$$

which holds for each real t, A is an additive function, and a_i, β_i are real numbers. Adrienn Varga studied this equation using the methods of discrete spectral synthesis and she succeeded to find necessary and sufficient conditions for the existence of nonzero additive solutions A.

2. Information functions

One of his most important achievement is related to information theory; in fact, he was an inventor on this field. His book "On Measures of Information and Their Characterizations" (joint work with János Aczél, published by Academic Press) was very influential and served as a manual for those who were interested in the axiomatization of information theory. Most of his work has become standard subject of college courses on information theory and functional equations.

His approach starts with the following natural question: how much information is contained in a random event? This is different from the classical setup of

Claude Elwood Shannon – Aleksandr Yakovlevich Khinchin and Dmitrii Konstantinovich Faddeev, where the question is about the information content of and experiment with different possible outcomes – in fact, the information content of a probability distribution. Daróczy asks for the information content of a single random event, which is identified with a measurable subset of a probability space. This new approach makes it possible to formulate the basic axioms of information theory in terms of some properties of a single function as soon as we set up the following reasonable postulate: the amount of information contained in a random event depends only on the probability of that event. In other words, there is a real function f such that if A is a random event with probability $P(A)$, then the information $I(A)$ contained in A is $f(P(A))$. Obviously, the function f is required to satisfy some basic properties, which serve as the axioms of information theory. Using these standard properties we arrive at the definition: the function $f : [0,1] \to \mathbb{R}$ is called an information function of β-type, if the following functional equation holds:

$$f(x) + (1-x)^{\beta} f\left(\frac{y}{1-x}\right) = f(y) + (1-y)^{\beta} f\left(\frac{x}{1-y}\right),$$

with the additional conditions $f(0) = f(1)$ and $f(\frac{1}{2}) = 1$, where β is a positive number, and x, y satisfy the conditions $0 \le x < 1$, $0 \le y < 1$, $x + y \le 1$.

In the case $\beta = 1$ f is termed a standard information function, accordingly, the functional equation

$$f(x) + (1-x) f\left(\frac{y}{1-x}\right) = f(y) + (1-y) f\left(\frac{x}{1-y}\right),$$

is called the fundamental equation of information.

The basic novelty and utmost importance of this approach is that the full arsenal of functional equations is available to describe and characterize different types of information functions. We underline that a priori there is no regularity assumption on the information functions. Nevertheless, the following remarkable result of Daróczy shows, that if $\beta = 1$, then all information functions of β-type are a posteriori regular, and they can be described:

Theorem. If $\beta = 1$ and $f : [0,1] \to \mathbb{R}$ is an information function of β-type, then

$$f(x) = \frac{1}{2^{1-\beta} - 1}\left[x^{\beta} + (1-x)^{\beta} - 1\right],$$

and if β tends to 1, then this function tends to the so-called Shannon information function defined by

$$S(x) = -x \log_2 x - (1-x)\log_2(1-x),$$

for each x in the open interval $]0,1[$, and is zero at the endpoints.

An obvious question arises: what about the case $\beta = 1$, in other words, how to describe all standard information functions? This leads to some further natural questions: what is the general solution of the fundamental equation of information? Are there any solutions different from the Shannon information function? If so, then under what additional – possibly weak, maybe regularity-type – conditions does the fundamental equation of information characterize the Shannon information function?

This couple of questions opened the door to a new research field for functional equationists; a new generation of young scientists started their work in this area and tested their skills in solving related problems on functional equations.

The central problem, the description of the general solution of the fundamental equation of information was solved by Zoltán, using ingenious arguments and some nice algebraic results of Børge Jessen, Jørgen Karpf and Anders Thorup in 1968. The result reads as follows:

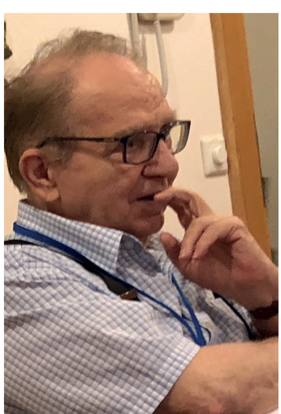
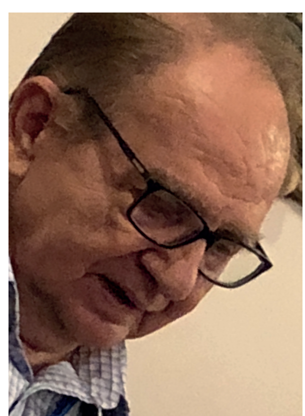
Theorem. The general solution of the fundamental equation of information is

$$f(x) = \begin{cases} xh(x) + (1-x)h(1-x) + ax & \text{for } 0 < x < 1 \\ 0 & \text{for } x = 0 \\ a & \text{for } x = 1, \end{cases}$$

where a is an arbitrary real number and h is an arbitrary solution of

$$h(xy) = h(x) + h(y)$$

on the set of positive real numbers.

Under the side condition $f(0) = f(1)$ we have $a = 0$. The "degree of freedom" in this general solution is the function h which satisfies the functional equation of the logarithm. Choosing a constant multiple of the logarithm function so that the additional condition $f(\frac{1}{2}) = 1$ is also satisfied we obtain the Shannon information function. However, it is well-known from the theory of functional equations that there are highly irregular functions h solving the equation of the logarithm, which means that the fundamental equation of information does not characterize the Shannon information function without additional regularity conditions.

The history of the regularity conditions imposed on the fundamental equation of information is really amazing, it is worth making a short detour in this direction. For obvious reasons, due to the nature of the problem, the most natural regularity condition could be the nonnegativity of the information function. In fact, the Shannon information function is nonnegative. Unfortunately, despite several attempts no proof was found for the conjecture that the only nonnegative standard information function is the Shannon information function. The general form of the solution, provided by Zoltán, is not suitable to extract the nonnegative ones from it. Zoltán Daróczy published several papers with his colleague and friend, Imre Kátai who is also celebrated today, for the same reason as Zoltán. In 1970 they got really close to the target: one of their papers deals with bounded information functions: "Additive number theoretical functions and the measure of information" (in German). In this joint work they proved the result which was the strongest one that time: if a nonnegative information function is bounded above, then it is the Shannon information function. This very nice result is, however, not completely "flawless": in fact, besides nonnegativity they needed boundedness above, which may also be a natural condition though, but the result is still not our heart's desire. So the show had to go on, the hunt for a proof continued. Probably, in those years nobody in the field doubted that a proof would soon be found proving that the Shannon information function is the only nonnegative standard information function. Nobody – but Zoltán's student and colleague, Gyula Maksa, whose birthday is celebrated right now as well... The author of the present laudation had the privilege to be present – for the first time in his life – at the International Symposium on Functional Equations in Graz-Leibnitz, Austria in 1978, exactly forty years ago, where the whole audience was bluffed as Gyula presented their ingenious and simple example for a nonnegative standard information function different from that of Shannon... At the same time they succeeded to prove that, in fact, the Shannon function is the least nonnegative standard information function.

3. Means

One of Zoltán's most frequently discussed topics in functional equations and inequalities is the theory of means. Based on his ideas and initiatives a school was founded at Debrecen University, and a great number of his followers succeeded in developing an independent field related to weighted quasiarithmetic means, deviation means and others. From this fruitful and diversified theory we recall just one result.

He started with the following definition of an n-variable mean: for a continuous and strictly monotonic function $\phi :]0,1[\to \mathbb{R}$ we let

$$^{\phi}M_n(p_1, p_2, \dots, p_n) = \phi^{-1}\left(\frac{\sum_{k=1}^{n} p_k \phi(p_k)}{\sum_{k=1}^{n} p_k}\right)$$

where $p_1, p_2, \dots, p_n$ form a so-called non-complete probability distribution. The characterization problem of such types of means in terms of the generating function ϕ led to various interesting functional equations. A related conjecture of Alfréd Rényi was that such a mean is homogeneous of degree 1 over the non-complete probability distributions if and only if it is generated either by the logarithm, or by the non-constant power function. This conjecture was proved by Zoltán in the following result:

Theorem. If, for all $p_1, p_2, q > 0$ with $p_1 + p_2 \leq 1$,

$$^{\phi}M_2(qp_1, qp_2) = q \cdot {}^{\phi}M_2(p_1, p_2),$$

then either $\phi(t) = \log t$, or $\phi(t) = t^{\alpha}$ holds for all t in the interval $]0,1]$, up to an additive and a nonzero multiplicative constant, where $\alpha = 0$ is an arbitrary constant.

His joint works with Aczél about quasiarithmetic means weighted by weight functions are related to the results of Mahmut Bajraktarević, to the problem of equality of means, and to the characterization of homogeneous means in this class. Zoltán's main contribution on this field was to find new mean values – partly together with László Losonczi and Zsolt Páles –, which could be applied to define new classes of entropies in information theory.

4. Joint works with Imre Kátai

Above we mentioned his joint work with Professor Kátai, our other celebrated scientist today. This work did not restrict to information theoretical problems. A specific part of their joint venture was about additive number theoretical functions. In one of the related papers the following problem was settled: we consider homomorphisms from the additive semigroup of natural numbers into a compact Abelian group. There are two different classes of these functions: the first one is characterized by the property that the limit of their one-step difference is zero:

$$\lim_{n \to \infty} \phi(n+1) - \phi(n) = 0.$$

The second group is characterized in somewhat more complicated way: it consists of all functions ϕ having the property that if for a strictly monotonic increasing sequence $\{n_k\}$ of natural numbers we have

$$\lim_{k \to \infty} \phi(n_k) = g,$$

then also

$$\lim_{k \to \infty} \phi(n_k + 1) = g.$$

The main – and highly nontrivial – theorem in this field is that the two above mentioned classes are the same.

Another very rich area of their cooperation is the theory of interval filling sequences. As in the former laudation, five years ago, we discussed this field in details, here I just want to recall some characteristic results obtained in the cooperation of Daróczy and Kátai.

This problem attracted a number of young colleagues of him, who made their first steps on a scientific road. The basic setting is the following: let a strictly decreasing sequence $\{\lambda_n\}$ of positive numbers, which is summable to L, be given with the property that $\lambda_n \leq \sum_{i=n+1}^{\infty} \lambda_i$. Then a joint result of Zoltán with Kátai and Antal Járai is the discovery of the remarkable property, that exactly in this case, every nonnegative number x, not greater than L, can be represented in the form

$$x = \sum_{n=1}^{\infty} \varepsilon_n \lambda_n,$$

where each ε_n is 0 or 1. Such a sequence will then be called an interval filling sequence.

There are several ways to construct the digits ε_n. One of them is the so-called eager, or regular algorithm, which is defined recursively by

$$\varepsilon_n(x) = \begin{cases} 1, & \text{if } \sum_{i=1}^{n-1} \varepsilon_i(x)\lambda_i + \lambda_n \leq x \\ 0, & \text{if } \sum_{i=1}^{n-1} \varepsilon_i(x)\lambda_i + \lambda_n > x, \end{cases}$$

and results in

$$x = \sum_{n=1}^{\infty} \varepsilon_n(x)\lambda_n.$$

The following concept is based on this algorithm. Namely, a function $F : [0, L] \to \mathbb{R}$ is called additive with respect to a sequence $\{\lambda_n\}$ if

$$\sum_{n=1}^{\infty} |F(\lambda_n)| < \infty,$$

and we have for each x in $[0, L]$ that

$$F(x) = F\left(\sum_{n=1}^{\infty} \varepsilon_n(x)\lambda_n\right) = \sum_{n=1}^{\infty} \varepsilon_n(x)F(\lambda_n).$$

The study of additive functions with respect to interval filling sequences attracted great interest in the family of functional equationists. With Kátai they proved that there exists an interval filling sequence $\{\lambda_n\}$ and a continuous function $F : [0, L] \to \mathbb{R}$, additive with respect to $\{\lambda_n\}$, which is nowhere differentiable. This result has a special interest in the light of the result of Daróczy, Kátai and Tamás Szabó, which says that if F is additive with respect to an interval filling sequence for all algorithms, then F is linear. From these two results we infer immediately, that there is an interval filling sequence a function additive with respect to it for some algorithms, however, it fails to be additive for some other algorithms. These theorems form the basis of the theory of interval filling sequences, a treasure island for young scientists to start their hunt for brilliant results.

4. Recent works

In this concluding section we present some of Zoltán's results from the last five years which clearly illustrate that his activity has not been decreasing, his inventions are as brilliant as ever, and he always brings some new surprise for his audience.

In 2016 he published a joint paper with Justyna Jarczyk and Witold Jarczyk. In fact, Daróczy posed a problem at the 52nd International Symposium on Functional Equations held in Innsbruck in 2014, related to the following result of Roman Ger and Tomasz Kochanek:

Theorem. (GK) Let M be a continuous mean on an interval I, strictly increasing with respect to each variable. If there is a non-constant solution $f : I \to \mathbb{R}$ of the equation

$$f(M(x,y)) = \frac{f(x) + f(y)}{2},$$

then the mean M is quasi-arithmetic.

Clearly, the equation above reminds us to the Jensen-equation which is the special case with M is the arithmetic mean. Zoltán's problem is the following: suppose that M is a mean on the interval I, which is not quasi-arithmetic. Does the above equation have any non-constant solution? By the above result, if the mean M is not quasi-arithmetic, then every solution of the above equation is constant, hence the answer to Zoltán's question is negative – assuming that M is strictly increasing with respect to both variables. But what if this strictly increasing property of M is relaxed? From the joint result of Zoltán with the Jarczyk couple it follows that the answer in this case is negative – the strictly increasing property of M in both variables is an important ingredient in the (GK)-theorem above. In their joint paper they also generalize Zoltán's problem by formulating some further open problems.

Another joint paper was published with Vilmos Totik in 2015 related means where a so-called conditional functional equation involving pairs of means showed up. They proved that there are only constant solutions of the equation if for the unknown function continuous differentiability is assumed, and there may be non-constant everywhere differentiable solutions. Various other situations are considered, where less smoothness is assumed on the unknown function.

Obviously, there is no time and space to present here all important results and contributions of Professor Zoltán Daróczy: the examples above clearly show that his work is unquestionably determining in the field of functional equations and inequalities. His students earned scientific respect and reputation all over the world, and the school of science he founded and led for decades in Debrecen is considered as a center of functional equations and inequalities.

I conclude this laudation with my personal short comment: I consider him as my master. It is a gift of my life that I have become a mathematician in his hands and I have become also his colleague and his friend. On this wonderful occasion we all wish you, Zoli, a happy birthday, blessed with joy and love together – with your students, colleagues and friends around you!