UNSOLVED PROBLEMS SECTION

CHARACTERIZATION OF SOME MULTIPLICATIVE FUNCTIONS

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Abstract. We state the following problem: Let $a < b$, $a, b \in \mathbb{Z}$ and f be a completely multiplicative function for which

$$f(n^2 + b) - f(n^2 + a) = b - a \quad (n^2 > a)$$

is satisfied. Determine all f for which the above relation holds.

1. Introduction

Let, as usual, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{C}$ be the set of positive integers, integers and complex numbers, respectively. Let $\mathcal{M}^*$ be the class of complex-valued completely multiplicative functions. Let χ_3, χ_4 be the nonprincipal Dirichlet characters (mod 3) and (mod 4), that is

$$\chi_3(n) = \begin{cases} 1, & n \equiv 1 \pmod 3 \\ -1, & n \equiv -1 \pmod 3 \\ 0, & n \equiv 0 \pmod 3 \end{cases}$$

and

$$\chi_4(n) = \begin{cases} 1, & n \equiv 1 \pmod 4 \\ -1, & n \equiv -1 \pmod 4 \\ 0, & n \equiv 0 \pmod 2. \end{cases}$$

Let I be the identity function, that is $I(n) = n$ for every $n \in \mathbb{N}$.

Theorem 1. *Assume that $f \in \mathcal{M}^*$, $A \in \mathbb{C}$ satisfy the condition*

(1.1) $f(n^2) = f(n^2 - 1) + A$ *for every* $n \in \mathbb{N}, \ n \geq 2.$

Then

(a1) *No solution exists if $A \notin \{0, 1\}$.*

(b1) *If $A = 1$, then $f \in \{I, \ \chi_3, \ \chi_4\}$*

(c1) *If $A = 0$, then either $f(n) = 1$ for every $n \in \mathbb{N}$ or $f(n) = 0$ for every $n \in \mathbb{N}, n \geq 2.$*

We note that the condition $f \in \mathcal{M}^*$ implies that $f(1) = 1$.

Theorem 2. *Assume that $f \in \mathcal{M}^*$, $B \in \mathbb{C}$ satisfy the condition*

(1.2) $f(n^2 + 1) = f(n^2) + B$ *for every* $n \in \mathbb{N}.$

Then

(a2) *No solution exists if $B \notin \{0, 1\}$.*

(b2) *If $B = 1$, then*

$$f(2) = 2, f(p) = p, (p \in \mathcal{P}) \quad if \quad p \equiv 1 \pmod 4$$

and

$$f(q) = \pm q \ (q \in \mathcal{P}) \quad if \quad q \equiv 3 \pmod 4.$$

(c2) *If $B = 0$, then*

$$f(2) = 1, f(p) = 1, (p \in \mathcal{P}) \quad if \quad p \equiv 1 \pmod 4$$

and

$$f(q) = \pm 1 \ (q \in \mathcal{P}) \quad if \quad q \equiv 3 \pmod 4.$$

If these conditions hold, then (1.2) is satisfied.

Unsolved problem. *Let $a < b$, $a, b \in \mathbb{Z}$ and f be a completely multiplicative function for which*

(1.3) $f(n^2 + b) - f(n^2 + a) = b - a$ $(n^2 > a)$

is satisfied. Determine all f for which (1.3) holds.

2.   Proof of Theorem 1

First, by using the fact $f \in \mathcal{M}^*$, we obtain from (1.1) that $f(1) = 1$ and

(2.1) $E_n := f^2(n) - f(n-1)f(n+1) - A = 0$ for every $n \in \mathbb{N}, \ n \geq 2$.

For $n = 2, 3, 4, 5, 9$, we have

$$\begin{cases} E_2 & = f^2(2) - f(3) - A = 0, \\ E_3 & = f^2(3) - f^3(2) - A = 0, \\ E_4 & = f^4(2) - f(3)f(5) - A = 0, \\ E_5 & = f^2(5) - f^3(2)f(3) - A = 0, \\ E_9 & = f^4(3) - f^4(2)f(5) - A = 0. \end{cases}$$

With the help of a Maple program, by solving this system of equations, we obtain that the solutions are:

$$\big(f(2), \ f(3), \ f(5), \ A\big) \in \{(0, 0, 0, 0), \ (1, 1, 1, 0),$$

$$(2, 3, 5, 1), \ (-1, 0, -1, 1), \ (0, -1, 1, 1)\}.$$

Therefore the assertion (a1) is proved.

First we consider the case $A = 1$. Then

$$\big(f(2), \ f(3), \ f(5), \ A\big) \in \{(2, 3, 5, 1), \ (-1, 0, -1, 1), \ (0, -1, 1, 1)\}.$$

We get from (1.1) that

(2.2) $f^2(N-1) = f\big((N-1)^2\big) = f(N-2)f(N) + A = f(N-2)f(N) + 1.$

If $\big(f(2), \ f(3), \ f(5), \ A\big) = (2, 3, 5, 1)$, then $f(n) = n$ for every $n < N$, where $N \geq 6$ is a prime number. Thus, we infer from (2.2) that

$$(N-1)^2 = f^2(N-1) = f(N-2)f(N) + 1 = (N-2)f(N) + 1,$$

and so $f(N) = N$.

If $\big(f(2), \ f(3), \ f(5), \ A\big) = (-1, 0, -1, 1)$, then $f(n) = \chi_3(n)$ for every $n < N$, where $N \geq 6$ is a prime number. Thus, we infer from (2.2) that

$$\chi_3^2(N-1) = f^2(N-1) = f(N-2)f(N) + 1 = \chi_3(N-2)f(N) + 1,$$

from which we have $f(N) = \chi_3(N)$ for $N \not\equiv 2 \pmod 3$. If $N \equiv 2 \pmod 3$, then $2N - 1 \equiv 0 \pmod 3$ and $\frac{2N-1}{3} < N$. Consequently

$$0 = \chi_3^2(2N - 1) = f\left((2N - 1)^2\right) = f(2N - 2)f(2N) + A =$$

$$= \chi_3^2(2)\chi_3(N - 1)f(N) + 1$$

and

$$\chi_3^2(2)\chi_3(N - 1)f(N) + 1 = 0, \quad f(N) = -1.$$

Hence

$$f(n) = \chi_3(n) \quad \text{for every} \quad n \in \mathbb{N}.$$

If $\left(f(2), f(3), f(5), A\right) = (0, -1, 1, 1)$, then $f(n) = \chi_4(n)$ for every $n < N$, where $N \geq 6$ is a prime number. Thus, we infer from (2.2) that

$$\chi_4^2(N - 1) = f^2(N - 1) = f(N - 2)f(N) + 1 = \chi_4(N - 2)f(N) + 1,$$

which shows that $f(N) = \chi_4(N)$, consequently $f(n) = \chi_4(n)$ for every $n \in \mathbb{N}$.

Now we consider the case $A = 0$. Then

$$\left(f(2), f(3), f(5), A\right) \in \left\{(0, 0, 0, 0), (1, 1, 1, 0)\right\}.$$

It is easy to show from (1.1) and from (2.2) that $f(1) = 1$ and

$$f(n) = \begin{cases} 0, & \text{for every} \quad n \in \mathbb{N}, \ n \geq 2 \quad \text{if} \quad \left(f(2), f(3), f(5)\right) = (0, 0, 0) \\ 1, & \text{for every} \quad n \in \mathbb{N}, \ n \geq 2 \quad \text{if} \quad \left(f(2), f(3), f(5)\right) = (1, 1, 1). \end{cases}$$

Theorem 1 thus is proved.   ∎

3. Proof of Theorem 2

By using the fact $f \in \mathcal{M}^*$, we obtain from (1.2) that $f(1) = 1$ and

(3.1) $F_n := f(n^2 + 1) - f^2(n) - B = 0$ for every $n \in \mathbb{N}, \ n \geq 1.$

From $F_1 = f(2) - f(1) - B = 0$, we have $f(2) = B + 1$, and so we obtain from F_2, F_4 that

$$f(5) = B^2 + 3B + 1 \quad \text{and} \quad f(17) = B^4 + 4B^3 + 6B^2 + 5B + 1.$$

Therefore we have

$$F_5 = -(1+B)(B^3 + 5B^2 + 6B - f(13) + 1) = 0.$$

If $B = -1$, then

$$F_8 = f(13) - 1 = 0 \quad \text{and} \quad F_{18} = f(13) + 1 = 0,$$

which are impossible. Then

$$B \neq -1 \quad \text{and} \quad f(13) = B^3 + 5B^2 + 6B + 1,$$

consequently

$$F_8 = -B(B+2)(B-1)(B^3 + 4B^2 + 5B + 1) = 0,$$

$$F_{13} = B(B-1)(1+B)(B+2)^2(B^2 + 3B + 1) = 0.$$

These imply that

$$B(B-1)(B+2) = 0.$$

Since

$$F_3 = B^3 + 4B^2 - f(3)^2 + 3B + 1 = 0$$

and

$$F_{18} = (1+B)^2(B^5 + 9B^4 - f(3)^4 + 28B^3 + 33B^2 + 9B + 1) = 0,$$

we have $B + 2 \neq 0$.

Thus we proved that $B \in \{0, 1\}$.

Case $B = 1$.

It is clear that $f(2) = f(1^2 + 1) = f(1)^2 + 1 = 2$, $f(5) = f(2^2 + 1) = f(2)^2 + 1 = 5$, $10 = f(10) = f(9) + 1$, thus $f(9) = 9$.

Let us assume that $f(p) = p$, $p \equiv 1 \pmod 4$ and $f(q^2) = q^2$ is satisfied if $p < N$, $q < N$, where $N > 9$. If N is a critical number, then either $N = P$ ($P \in \mathcal{P}, P \equiv 1 \pmod 4$), or $N = Q^2$ ($Q \in \mathcal{P}, Q \equiv 3 \pmod 4$).

Assume that $N = P$. Let n be the smallest positive integer for which $n^2 + 1 \equiv 0 \pmod P$. Then $n \leq \frac{P}{2}$, consequently, $n^2 + 1 = kP$ and $k < N$. Since $f(n^2) = n^2$, we have

$$kf(P) = f(k)f(P) = f(kP) = f(n^2 + 1) = f(n^2) + 1 = n^2 + 1 = kP,$$

which implies that $f(P) = P$.

In the second case, $N = Q^2$, we obtain that $2|Q^2 + 1$, and $f\left(\frac{Q^2+1}{2}\right) = \frac{Q^2+1}{2}$, since if $\pi \equiv 3 \pmod 4$, $\pi \in \mathcal{P}$, and $\pi | \frac{Q^2+1}{2}$, then $\pi^\alpha \| \frac{Q^2+1}{2}$ and α is even. The assertion (b2) is proved.

Case $B = 0$.

It is clear that $f(2) = f(1^2 + 1) = f(1)^2 + B = 1$, $f(5) = f\left(2^2 + 1\right) =$
$= f(2)^2 + B = 1$, $f(9) = f(9) + B = f(9^2 + 1) = f(2)f(5) = 1$.

Similar as in the proof of the case $B = 1$, we obtain the assertion (c2) of
Theorem 2.

Theorem 2 thus is proved. ∎

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