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IMPROVED CONVERGENCE CONDITIONS OF A LAVRENTIEV-TYPE METHOD FOR NONLINEAR ILL-POSED EQUATIONS BY USING RESTRICTED CONVERGENCE DOMAINS

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Abstract. In this paper we extend the applicability of an iterative method which converges to the unique solution x_α^δ of the Lavrentiev regularization equation, i.e., $F(x) + \alpha(x - x_0) = y^\delta$, approximating the solution $\hat{x}$ of the ill-posed problem $F(x) = y$ where $F : D(F) \subseteq X \longrightarrow X$ is a nonlinear monotone operator defined on a real Hilbert space X. We use a center-Lipschitz instead of a Lipschitz condition used in [8, 15, 22] as well as our new idea of restricted convergence domains. This idea helps us determine a smaller ball where the iterates lie leading to smaller Lipschitz constants which in turn helps us provide a wider convergence domain, tighter error bounds on the distances involved and an at least as precise information on the location of the solution. These advantages are obtained under the same computational cost, since in practice the computation the old constants requires the computation of the new constants as special cases. The convergence analysis and the stopping rule are based on the majorizing sequence. The choice of the regularization parameter is the crucial issue. We show that the adaptive scheme considered by Perverzev and Schock [19] for choosing the regularization parameter can be effectively used here for obtaining order optimal error estimate.

Key words and phrases: Ill-posed problems, Lavrentiev regularization method, restricted convergence domain, center-Lipschitz condition, adaptive method.

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1. Introduction

This paper extends the analysis of the Lavrentiev regularization for nonlinear ill-posed problems

$$F(x) = y, \tag{1.1}$$

where $F : D(F) \subseteq X \to X$ is a nonlinear monotone operator considered in [7]. Recall that F is a monotone operator if it satisfies the relation

$$\langle F(x_1) - F(x_2), x_1 - x_2 \rangle \geq 0, \qquad \forall x_1, x_2 \in D(F). \tag{1.2}$$

Here X is a real Hilbert space with inner product $\langle ., . \rangle$ and norm $\|.\|$. Let $U(x, R)$ and $\overline{U(x, R)}$, stand respectively, for the open and closed ball in X with center x and radius $R > 0$. Let also $L(X)$ be the space of all bounded linear operators from X into itself.

We assume that (1.1) has a solution, namely $\hat{x}$ and that F possesses a locally uniformly bounded Fréchet derivative $F'(.)$ in a ball around $\hat{x} \in X$.

In application, usually only noisy data y^δ are available, such that

$$\|y - y^\delta\| \leq \delta. \tag{1.3}$$

Then the problem of recovery of $\hat{x}$ from noisy equation $F(x) = y^\delta$ is ill-posed.

In [7], we considered an iterative regularization method;

$$x^\delta_{n+1,\alpha} = x^\delta_{n,\alpha} - (F'(x_0) + \alpha I)^{-1}(F(x^\delta_{n,\alpha}) - y^\delta + \alpha(x^\delta_{n,\alpha} - x_0)), \tag{1.4}$$

and proved that $(x^\delta_{n,\alpha})$ converges linearly to the unique solution x^δ_α of

$$F(x) + \alpha(x - x_0) = y^\delta. \tag{1.5}$$

It is known (cf. [21], Theorem 1.1) that the equation (1.5) has a unique solution x^δ_α for $\alpha > 0$, provided F is Fréchet differentiable and monotone in the ball $B_r(\hat{x}) \subset D(F)$ with radius $r = \|\hat{x} - x_0\| + \delta/\alpha$. However the regularized equation (1.5) remains nonlinear and one may have difficulties in solving them numerically.

Many authors (see [8, 9, 10, 11, 12]) considered iterative regularization methods for obtaining stable approximate solutions for (1.5). Recall ([22]) that, an iterative method with iterations defined by

$$x^\delta_{k+1} = \Phi(x^\delta_0, x^\delta_1, \cdots, x^\delta_k; y^\delta),$$

where $x_0^\delta := x_0 \in D(F)$ is a known initial approximation of $\hat{x}$, for a known function Φ together with a stopping rule which determines a stopping index $k_\delta \in \mathbb{N}$ is called an iterative regularization method if $\|x_{k_\delta}^\delta - \hat{x}\| \to 0$ as $\delta \to 0$.

In [8], Bakushinskii and Smirnova considered the iteratively regularized Lavrentiev method

$$x_{k+1}^\delta = x_k^\delta - (A_k^\delta + \alpha_k I)^{-1}(F(x_k^\delta) - y^\delta + \alpha_k(x_k^\delta - x_0)), \quad k = 0, 1, 2, \ldots, \tag{1.6}$$

where $A_k^\delta := F'(x_k^\delta)$ and $\{\alpha_k\}$ is a sequence of positive real numbers such that $\lim_{k\to\infty} \alpha_k = 0$. In fact, the stopping index k_δ in [8] was chosen according to the discrepancy principle

$$\|F(x_{k_\delta}^\delta) - y^\delta\| \le \tau\delta < \|F(x_k^\delta) - y^\delta\|, \qquad 0 \le k < k_\delta$$

for some $\tau > 1$ and showed that $x_{k_\delta}^\delta \to \hat{x}$ as $\delta \to 0$ under the following assumptions:

- There exists $L_1 > 0$ such that $\|F'(x) - F'(y)\| \le L_1\|x - y\|$ for all $x, y \in D(F)$.
- There exists $p > 0$ such that
$$\frac{\alpha_k - \alpha_{k+1}}{\alpha_k\alpha_{k+1}} \le p, \quad \forall k \in \mathbb{N}, \tag{1.7}$$
- $\sqrt{(2 + L_1\sigma)\|x_0 - \hat{x}\|td} \le \sigma - 2\|x_0 - \hat{x}\|t \le d\alpha_0$, where $\sigma := (\sqrt{\tau - 1})^2$, $t := p\alpha_0 + 1$ and $d = 2(t\|x_0 - \hat{x}\| + p\sigma)$.

However, no error estimate for $\|x_{k_\delta}^\delta - \hat{x}\|$ was given in [8]. Later in [22], Mahale and Nair considered method (1.6) and obtained an error estimate for $\|x_{k_\delta}^\delta - \hat{x}\|$ under a weaker condition than (1.7). Precisely they choose the stopping index k_δ as the first nonnegative integer such that x_k^δ in (1.7) is defined for each $k \in \{0, 1, 2, \cdots, k_\delta\}$ and

$$\|\alpha_{k_\delta}(A_{k_\delta}^\delta + \alpha_{k_\delta} I)^{-1}(F(x_{k_\delta}) - y^\delta)\| \le c_0 \ \text{ with } \ c_0 > 4.$$

In [22], Mahale and Nair showed that $x_{k_\delta}^\delta \to \hat{x}$ as $\delta \to 0$ and obtained an optimal order error estimate for $\|x_{k_\delta}^\delta - \hat{x}\|$ under the following assumptions:

Assumption 1. *There exists $r > 0$ such that F is Fréchet differentiable for all $x \in U(\hat{x}, r)$.*

Assumption 2. (cf. [20], Assumption 3) *There exists a constant $K > 0$ such that for every $x, u \in U_0$ and $v \in X$ there exists an element $\Phi(x, u, v) \in X$ such that $[F'(x) - F'(u)]v = F'(u)\Phi(x, u, v)$, $\|\Phi(x, u, v)\| \le K\|v\|\,\|x - u\|$.*

Assumption 3. *There exists a continuous, strictly increasing function $\varphi : (0, a] \to (0, \infty)$ with $a \geq \|F'(\hat{x})\|$ satisfying*

1. $\lim_{\lambda \to 0} \varphi(\lambda) = 0$,
2. *for* $\alpha \leq 1, \varphi(\alpha) \geq \alpha$,
3. $\sup_{\lambda \geq 0} \frac{\alpha\varphi(\lambda)}{\lambda+\alpha} \leq c_\varphi \varphi(\alpha), \qquad \forall \lambda \in (0, a]$,
4. *there exists* $w \in X$ *such that*

$$x_0 - \hat{x} = \varphi(F'(\hat{x}))w. \tag{1.8}$$

Assumption 4. *There exists a sequence $\{\alpha_k\}$ of positive real numbers such that* $\lim_{k \to \infty} \alpha_k = 0$ *and there exists $\mu > 1$ such that*

$$1 \leq \frac{\alpha_k}{\alpha_{k+1}} \leq \mu, \quad \forall k \in \mathbb{N}. \tag{1.9}$$

Note that (1.9) is weaker than (1.7).

In [15] motivated by iteratively regularized Lavrentiev method (see [8] and [22]), we showed the quadratic convergence of the method defined by

$$x^\delta_{n+1,\alpha} = x^\delta_{n,\alpha} - (F'(x^\delta_{n,\alpha}) + \alpha I)^{-1}(F(x^\delta_{n,\alpha}) - y^\delta + \alpha(x^\delta_{n,\alpha} - x_0)), \tag{1.10}$$

where $x^\delta_{0,\alpha} := x_0$ is a starting point of the iteration. Let $R_\alpha(x) = F'(x) + \alpha I$ and

$$G(x) = x - R_\alpha(x)^{-1}(F(x^\delta_{n,\alpha}) - y^\delta + \alpha(x^\delta_{n,\alpha} - x_0)). \tag{1.11}$$

Note that with the above notation $x^\delta_{n+1,\alpha} = G(x^\delta_{n,\alpha})$. The assumptions used instead of Assumption 1 and Assumption 2 in [15] are, respectively, as:

Assumption 5. *Let $x_0 \in D(F)$ be fixed. There exists $r > 0$ such that $U_0 := U(x_0, r) \cup U(\hat{x}, r) \subseteq D(F)$ and F is Fréchet differentiable for all $x \in U_0$.*

Assumption 6. *There exists a constant $k_0 > 0$ such that for every $x, u \in \in U(x_0, r) \cup U(\hat{x}, r))$ and $v \in X$ there exists an element $\Phi(x, u, v) \in X$ satisfying $[F'(x) - F'(u)]v = F'(u)\Phi(x, u, v), \|\Phi(x, u, v)\| \leq k_0\|v\|\|x - u\|$.*

The second condition in Assumption 6 is essentially a Lipschitz-type condition. However, it is in general very difficult to verify or may not even be satisfied [1]–[5].

In order for us to expand the applicability of the method, we consider the following even weaker assumptions, respectively.

Assumption 7. *Let $x_0 \in D(F)$ be fixed. There exists $r > \|x_0 - G(x_0)\|$, if $x_0 \neq G(x_0)$ such that $U_1 := U(x_1, r - \|x_0 - G(x_0)\|) \cup U(\hat{x}, r) \subseteq D(F)$ and F is Fréchet differentiable for all $x \in U_1$.*

Assumption 8. *Let $x_0 \in X$ be fixed. There exists a constant $L' > 0$ such that for every $x, u \in U_1 \subseteq D(F)$ and $v \in X$ there exists an element $\Phi(x, u, v) \in X$ satisfying $[F'(x) - F'(u)]v = F'(u)\Phi(x, u, v)$, $\|\Phi(x, u, v)\| \leq L'\|v\|(\|x - x_0\| + +\|u - x_0\|)$.*

Remark 1.1. Note that in view of the estimate

$$\|x - u\| \leq \|x - x_0\| + \|x_0 - u\|,$$

Assumption 6 implies Assumption 8 with $k_0 = L'$ but not necessarily vice versa (see [1–6]). Throughout this paper we assume that the operator F satisfies Assumptions 7 and 8.

Moreover, we have that $U_1 \subseteq U_0 \subseteq D(F)$, so $k_0 \leq K$. Hence, the convergence analysis of method (1.10) is improved. The advantages are obtained under the same computational cost as in the preceding works, since in practice the computation of constant K requires the computation of constants k_0 and L' as special cases. Examples, where $k_0 < K$ can also be found in [1, 2, 3, 4, 5, 6] (see also Section 5).

Remark 1.2. If Assumptions 7 and 8 are fulfilled only for all $x, u \in U(x_1, r - -\|x_0 - G(x_0)\|) \cap Q \neq \emptyset$, where Q is a convex closed a priori set for which $\hat{x} \in Q$, then we can modify the method (1.10) in the following way:

$$x^{\delta}_{n+1,\alpha} = P_Q(G(x^{\delta}_{n,\alpha}))$$

to obtain the same estimates in this paper; here P_Q is the metric projection onto the set Q.

The plan of this paper is as follows. In Section 2, we prove the convergence of the method and in Section 3, we give error bounds under source conditions. Section 4 deals with the starting point and algorithm.

2. Convergence analysis

We use a majorizing sequence for proving our results. Recall (see [1], Definition 1.3.11) that a nonnegative sequence $\{t_n\}$ is said to be a majorizing sequence of a sequence $\{x_n\}$ in X if

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n, \quad \forall n \geq 0.$$

In the convergence analysis we will use the following Lemma on majorization, which is a reformulation of Lemma 1.3.12 in [1].

Lemma 2.1. (cf.[14], Lemma 2.1.) *Let $\{t_n\}$ be a majorizing sequence for $\{x_n\}$ in X. If $\lim_{n\to\infty} t_n = t^*$, then $x^* = \lim_{n\to\infty} x_n$ exists and*

$$\|x^* - x_n\| \le t^* - t_n, \quad \forall n \ge 0. \tag{2.1}$$

We need an auxillary result on majorizing sequences for method (1.10).

Lemma 2.2. *Suppose that there exist non-negative numbers L', η such that*

$$16L'\eta \le 1. \tag{2.2}$$

Let

$$q = \frac{1 - \sqrt{1 - 16L'\eta}}{2}. \tag{2.3}$$

Then, scalar sequence $\{t_n\}$ given by

(2.4)

$$t_0 = 0,\ t_1 = \eta,\ t_{n+1} = t_n + \frac{L'}{2}(5t_n + 3t_{n-1})(t_n - t_{n-1}) \text{ for each } n = 1, 2, \ldots$$

is well defined, nondecreasing, bounded from above by t^{**} given by

$$t^{**} = \frac{\eta}{1-q} \tag{2.5}$$

and converges to its unique least upper bound t^* which satisfies

$$\eta \le t^* \le t^{**}. \tag{2.6}$$

Moreover the following estimates hold for each $n = 1, 2, \ldots$:

$$t_{n+1} - t_n \le q(t_n - t_{n+1}) \tag{2.7}$$

and

$$t^* - t_n \le \frac{q^n}{1-q}\eta. \tag{2.8}$$

Proof. Note that $q \in (0, 1)$. We shall show using mathematical induction that

$$\frac{L'}{2}(5t_m + 3t_{m-1}) \le q. \tag{2.9}$$

Estimate is true for $m = 1$ by the definition of sequence $\{t_n\}$ and (2.2). Then, we have by (2.4) that $t_2 - t_1 \le q(t_1 - t_0)$ and $t_2 \le \eta + q\eta = (1+q)\eta = \frac{1-q^2}{1-q}\eta <$ $< \frac{\eta}{1-q} = t^{**}$. Let us assume (2.9) holds for all integers smaller or equal to m. Hence, we get that

$$t_{m+1} - t_m \le q(t_m - t_{m-1}) \tag{2.10}$$

and

$$t_{m+1} \le \frac{1-q^{m+1}}{1-q}\eta. \tag{2.11}$$

We shall show that (2.9) holds for m replaced by $m+1$. Using (2.10) and (2.11), estimates (2.9) shall be true if

$$\frac{L'}{2}[5(\frac{1-q^m}{1-q}) + 3(\frac{1-q^{m-1}}{1-q})]\eta \le q. \tag{2.12}$$

Estimate (2.12) motivates us to define recurrent functions f_m on $[0,1)$ by

$$f_m(t) = L'[5(1 + t + \cdots + t^{m-1}) + 3(1 + t + \cdots + t^{m-2})]\eta - 2t. \tag{2.13}$$

We need a relationship between two consecutive functions f_m. Using (2.13) we get that

$$f_{m+1}(t) = f_m(t) + (5t+3)L'\eta t^{m-1} \ge f_m(t). \tag{2.14}$$

Define function f_∞ on $[0,1)$ by

$$f_\infty(t) = \lim_{m\to\infty} f_m(t). \tag{2.15}$$

Then, using (2.13) we get that

$$f_\infty(t) = \frac{8L'}{1-t}\eta - 2t. \tag{2.16}$$

Evidently, (2.12) is true if

$$f_\infty(t) \le 0, \tag{2.17}$$

since

$$f_m(q) \le f_{m+1}(q) \le \cdots \le f_\infty(q). \tag{2.18}$$

But (2.17) is true by (2.12) and (2.13). The induction for (2.9) is complete. Therefore, sequence $\{t_n\}$ is nondecreasing, bounded from above by t^{**} and it converges to some t^* which satisfies (2.6). ■

Lemma 2.3. ([15], Lemma 2.3.) *For* $u, v \in B_{r_0}(x_0)$

$$F(u) - F(v) - F'(u)(u-v) = F'(u)\int_0^1 \Phi(v + t(u-v), u, u-v)dt.$$

Here, we assume that for $r > 0, L' > 0$, for q given in (2.3) and :

$$\frac{\delta}{\alpha} < \eta \leq min\{\frac{1}{16L'}, r(1-q)\} \tag{2.19}$$

and $\|\hat{x} - x_0\| \leq \rho$, where

$$\rho \leq \frac{1}{L'}(\sqrt{1 + 2L'(\eta - \delta/\alpha)} - 1) \tag{2.20}$$

Remark 2.4. Note that (2.19) and (2.20) imply

$$\frac{L'}{2}\rho^2 + \rho + \frac{\delta}{\alpha} \leq \eta \leq \min\{\frac{1}{16L'}, r(1-q)\}. \tag{2.21}$$

Theorem 2.5. *Suppose Assumption* 8 *holds. Let the assumptions in Lemma* 2.2 *are satisfied with* η *as in* (2.21). *Then the sequence* $\{x^\delta_{n,\alpha}\}$ *defined in* (1.11) *is well defined and* $x^\delta_{n,\alpha} \in U(x_0, t^*)$ *for all* $n \geq 0$. *Further* $\{x^\delta_{n,\alpha}\}$ *is a Cauchy sequence in* $U(x_0, t^*)$ *and hence converges to* $x^\delta_\alpha \in \overline{U(x_0, t^*)} \subset U(x_0, t^{**})$ *and* $F(x^\delta_\alpha) = y^\delta + \alpha(x_0 - x^\delta_\alpha)$.

Moreover, the following estimates hold for all $n \geq 0$:

$$\|x^\delta_{n+1,\alpha} - x^\delta_{n,\alpha}\| \leq t_{n+1} - t_n, \tag{2.22}$$

$$\|x^\delta_{n,\alpha} - x^\delta_\alpha\| \leq t^* - t_n \leq \frac{q^n \eta}{1-q} \tag{2.23}$$

and

$$\|x^\delta_{n+1,\alpha} - x^\delta_{n,\alpha}\| \leq \frac{L'}{2}\left[5\|x^\delta_{n,\alpha} - x^\delta_{0,\alpha}\| + 3\|x^\delta_{n-1,\alpha} - x^\delta_{0,\alpha}\|\right]\|x^\delta_{n,\alpha} - x^\delta_{n-1,\alpha}\|. \tag{2.24}$$

Proof. First we shall prove that

$$\|x^\delta_{n+1,\alpha} - x^\delta_{n,\alpha}\| \leq \frac{L'}{2}[5\|x^\delta_{n,\alpha} - x^\delta_{0,\alpha}\| + 3\|x^\delta_{n-1,\alpha} - x^\delta_{0,\alpha}\|]\|x^\delta_{n,\alpha} - x^\delta_{n-1,\alpha}\|. \tag{2.25}$$

With G as in (1.11), we have for $u, v \in B_{t^*}(x_0)$,

$$\begin{aligned}
G(u)-G(v) &= u-v-R_\alpha(u)^{-1}[F(u)-y^\delta+\alpha(u-x_0)]+R_\alpha(v)^{-1}\times \\
&\quad \times[F(v)-y^\delta+\alpha(v-x_0)] = \\
&= u-v-[R_\alpha(u)^{-1}-R_\alpha(v)^{-1}](F(v)-y^\delta+\alpha(v-x_0))- \\
&\quad -R_\alpha(u)^{-1}(F(u)-F(v)+\alpha(u-v)) = \\
&= R_\alpha(u)^{-1}[F'(u)(u-v)-(F(u)-F(v))]- \\
&\quad -R_\alpha(u)^{-1}[F'(v)-F'(u)]R_\alpha(v)^{-1}(F(v)-y^\delta+\alpha(v-x_0)) = \\
&= R_\alpha(u)^{-1}[F'(u)(u-v)-(F(u)-F(v))]- \\
&\quad -R_\alpha(u)^{-1}[F'(v)-F'(u)](v-G(v)) = \\
&= R_\alpha(u)^{-1}[F'(u)(u-v)+\int_0^1(F'(u+t(v-u))(v-u))dt]- \\
&\quad -R_\alpha(u)^{-1}[F'(v)-F'(u)](v-G(v)) = \\
&= \int_0^1 R_\alpha(u)^{-1}[(F'(u+t(v-u))-F'(u))(v-u)dt]- \\
&\quad -R_\alpha(u)^{-1}[F'(v)-F'(u)](v-G(v)).
\end{aligned}$$

The last step follows from the Lemma 2.1. Then, by Assumption 8 and the estimate $\|R_\alpha(u)^{-1}F'(u)\| \leq 1$, we have

$$\begin{aligned}
\|G(u)-G(v)\| &\leq L'\int_0^1[\|u+t(v-u)-x_0\|+\|u-x_0\|]dt\|v-u\|+ \\
&\quad +L'[\|v-x_0\|+\|u-x_0\|]\|v-u\| \leq \\
&\leq \frac{L'}{2}[3\|u-x_0\|+\|v-x_0\|]+L'[\|v-x_0\|+\|u-x_0\|] \leq \\
&\leq \frac{L'}{2}[5\|u-x_0\|+3\|v-x_0\|]\|v-G(u)\|. \qquad (2.26)
\end{aligned}$$

Now by taking $u = x_{n,\alpha}^\delta$ and $v = x_{n-1,\alpha}^\delta$ in (2.26), we obtain (2.25).

Next we shall prove that the sequence (t_n) defined in Lemma 2.1 is a majorizing sequence of the sequence $(x_{n,\alpha}^\delta)$. Note that $F(\hat{x}) = y$, so by Lemma 2.2,

$$\begin{aligned}
\|x_{1,\alpha}^\delta-x_0\| &= \|R_\alpha(x_0)^{-1}(F(x_0)-y^\delta)\| = \\
&= \|R_\alpha(x_0)^{-1}(F(x_0)-y+y-y^\delta)\| = \\
&= \|R_\alpha(x_0)^{-1}(F(x_0)-F(\hat{x})-F'(x_0)(x_0-\hat{x})+ \\
&\quad +F'(x_0)(x_0-\hat{x})+y-y^\delta)\| \leq \\
&\leq \|R_\alpha(x_0)^{-1}(F(x_0)-F(\hat{x})-F'(x_0)(x_0-\hat{x}))\|+ \\
&\quad +\|R_\alpha(x_0)^{-1}F'(x_0)(x_0-\hat{x})\|+\|R_\alpha(x_0)^{-1}(y-y^\delta)\| \leq
\end{aligned}$$

$$
\begin{aligned}
&\leq \|R_\alpha(x_0)^{-1}F'(x_0)\int_0^1 \Phi(\hat{x}+t(x_0-\hat{x}),x_0,(x_0-\hat{x}))dt\| + \\
&\quad +\|R_\alpha(x_0)^{-1}F'(x_0))(x_0-\hat{x})\| + \frac{\delta}{\alpha} \leq \\
&\leq \frac{L'}{2}\|x_0-\hat{x}\|^2 + \|x_0-\hat{x}\| + \frac{\delta}{\alpha} \leq \\
&\leq \frac{L'}{2}\rho^2 + \rho + \frac{\delta}{\alpha} \leq \\
&\leq \eta = t_1 - t_0.
\end{aligned}
$$

Assume that $\|x^\delta_{i+1,\alpha} - x^\delta_{i,\alpha}\| \leq t_{i+1} - t_i$ for all $i \leq k$ for some k. Then

$$
\begin{aligned}
\|x^\delta_{k+1,\alpha} - x_0\| &\leq \|x^\delta_{k+1,\alpha} - x^\delta_{k,\alpha}\| + \|x^\delta_{k,\alpha} - x^\delta_{k-1,\alpha}\| + \cdots + \|x^\delta_{1,\alpha} - x_0\| \leq \\
&\leq t_{k+1} - t_k + t_k - t_{k-1} + \cdots + t_1 - t_0 = \\
&= t_{k+1} \leq t^*.
\end{aligned}
$$

So, $x^\delta_{i+1,\alpha} \in B_{t^*}(x_0)$ for all $i \leq k$, and hence, by (2.25),

$$
\begin{aligned}
\|x^\delta_{k+2,\alpha} - x^\delta_{k+1,\alpha}\| &\leq \frac{L'}{2}\Big[5\|x^\delta_{n,\alpha} - x^\delta_{0,\alpha}\| + \\
&\quad +3\|x^\delta_{n-1,\alpha} - x^\delta_{0,\alpha}\|\Big]\|x^\delta_{n,\alpha} - x^\delta_{n-1,\alpha}\| \leq \\
&\leq \frac{L'}{2}(5t_{k+1} + 3t_{k-1}) = t_{k+2} - t_{k+1}.
\end{aligned}
$$

Thus by induction $\|x^\delta_{n+1,\alpha} - x^\delta_{n,\alpha}\| \leq t_{n+1} - t_n$ for all $n \geq 0$ and hence $\{t_n\}, n \geq 0$ is a majorizing sequence of the sequence $\{x^\delta_{n,\alpha}\}$. In particular $\|x^\delta_{n,\alpha} - x_0\| \leq t_n \leq t^*$, i.e., $x^\delta_{n,\alpha} \in U(x_0,t^*)$, for all $n \geq 0$. So, $\{x^\delta_{n,\alpha}\}, n \geq 0$ is a Cauchy sequence and converges to some $x^\delta_\alpha \in \overline{U(x_0,t^*)} \subset U(x_0,t^{**})$ and by Lemma 2.3

$$
\|x^\delta_\alpha - x^\delta_{n,\alpha}\| \leq t^* - t_n \leq \frac{q^n\eta}{1-q}.
$$

To prove (2.24), we observe that $G(x^\delta_\alpha) = x^\delta_\alpha$, so (2.24) follows from (2.26), by taking $u = x^\delta_{n,\alpha}$ and $v = x^\delta_\alpha$ in (2.26). Now by letting $n \to \infty$ in (1.10) we obtain $F(x^\delta_\alpha) = y^\delta + \alpha(x_0 - x^\delta_\alpha)$. ∎

Remark 2.6. The convergence order of the method is two [15], under Assumption 6. In Theorem 2.5 the error bounds are too pessimistic. That is why in practice we shall use the computational order of convergence (COC) (see eg. [4]) defined by

$$
\varrho \approx \ln\left(\frac{\|x_{n+1} - x^\delta_\alpha\|}{\|x_n - x^\delta_\alpha\|}\right) / \ln\left(\frac{\|x_n - x^\delta_\alpha\|}{\|x_{n-1} - x^\delta_\alpha\|}\right).
$$

The (COC) ϱ will then be close to 2 which is the order of convergence of the method.

3. Error bounds under source conditions

The objective of this section is to obtain an error estimate for $\|x_{n,\alpha}^{\delta} - \hat{x}\|$ under

Assumption 9. *There exists a continuous, strictly increasing function $\varphi : (0, a] \to (0, \infty)$ with $a \geq \|F'(x_0)\|$ satisfying;*

(i) $\lim_{\lambda \to 0} \varphi(\lambda) = 0$,

(ii) $\sup_{\lambda \geq 0} \frac{\alpha\varphi(\lambda)}{\lambda+\alpha} \leq \varphi(\alpha) \qquad \forall \lambda \in (0, a]$ *and*

(iii) *there exists $v \in X$ with $\|v\| \leq 1$ (cf. [18]) such that*

$$x_0 - \hat{x} = \varphi(F'(x_0))v.$$

Proposition 3.1. *Let $F : D(F) \subseteq X \to X$ be a monotone operator in X. Let x_α^δ be the solution of* (1.5) *and $x_\alpha := x_\alpha^0$. Then*

$$\|x_\alpha^\delta - x_\alpha\| \leq \frac{\delta}{\alpha}.$$

Proof. The result follows from the monotonicity of F and the relation

$$F(x_\alpha^\delta) - F(x_\alpha) + \alpha(x_\alpha^\delta - x_\alpha) = y^\delta - y. \qquad \blacksquare$$

Theorem 3.2. (cf. [20], Proposition 4.1 or [21], Theorem 3.3.) *Suppose that Assumption* 7, 8 *and hypotheses of Proposition* 3.1 *hold. Let $\hat{x} \in D(F)$ be a solution of* (1.1)*. Then, the following assertion holds*

$$\|x_\alpha - \hat{x}\| \leq (L'r + 1)\varphi(\alpha).$$

Theorem 3.3. *Suppose hypotheses of Theorem* 2.5 *and Theorem* 3.2 *hold. Then, the following assertion holds*

$$\|x_{n,\alpha}^\delta - \hat{x}\| \leq \frac{q^n\eta}{1-q} + c_1\big(\varphi(\alpha) + \frac{\delta}{\alpha}\big)$$

where $c_1 = \max\{1, (L_0'r + 1)\}$.

Let

$$\bar{c} := max\{\frac{\eta}{1-q} + 1, (L_0' r + 2)\}, \tag{3.1}$$

and let

$$n_\delta := min\{n : q^n \leq \frac{\delta}{\alpha}\}. \tag{3.2}$$

Theorem 3.4. *Let $\bar{c}$ and n_δ be as in* (3.1) *and* (3.2) *respectively. Suppose that hypotheses of Theorem* 3.3 *hold. Then, the following assertion holds*

$$\|x^\delta_{n_\delta,\alpha} - \hat{x}\| \leq \bar{c}\big(\varphi(\alpha) + \frac{\delta}{\alpha}\big). \tag{3.3}$$

Note that the error estimate $\varphi(\alpha)+\frac{\delta}{\alpha}$ in (2.21) is of optimal order if $\alpha := \alpha_\delta$ satisfies $\varphi(\alpha_\delta)\alpha_\delta = \delta$.

Now using the function $\psi(\lambda) := \lambda\varphi^{-1}(\lambda), 0 < \lambda \leq a$ we have $\delta = \alpha_\delta\varphi(\alpha_\delta) = = \psi(\varphi(\alpha_\delta))$, so that $\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta))$. In view of the above observations and (2.21) we have the following.

Theorem 3.5. *Let $\psi(\lambda) := \lambda\varphi^{-1}(\lambda)$ for $0 < \lambda \leq a$, and the assumptions in Theorem* 3.4 *hold. For $\delta > 0$, let $\alpha := \alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta))$ and let n_δ be as in* (3.2)*. Then*

$$\|x^\delta_{n_\delta,\alpha} - \hat{x}\| = O\big(\psi^{-1}(\delta)\big).$$

In this section, we present a parameter choice rule based on the balancing principle studied in [17], [19], [13]. In this method, the regularization parameter α is selected from some finite set

$$D_M(\alpha) := \{\alpha_i = \mu^i\alpha_0, i = 0, 1, \dots, M\}$$

where $\mu > 1$, $\alpha_0 > 0$ and let

$$n_i := min\{n : e^{-\gamma_0 n} \leq \frac{\delta}{\alpha_i}\}.$$

Then for $i = 0, 1, \cdots, M$, we have

$$\|x^\delta_{n_i,\alpha_i} - x^\delta_{\alpha_i}\| \leq c\frac{\delta}{\alpha_i}, \qquad \forall i = 0, 1, \dots M.$$

Let $x_i := x^\delta_{n_i,\alpha_i}$. The parameter choice strategy that we are going to consider in this paper, we select $\alpha = \alpha_i$ from $D_M(\alpha)$ and operate only with corresponding x_i, $\quad i = 0, 1, \cdots, M$. Proof of the following theorem is analogous to the proof of Theorem 3.1 in [20].

Theorem 3.6. (cf. [20], Theorem 3.1.) *Assume that there exists* $i \in \{0, 1, 2, \dots, M\}$ *such that* $\varphi(\alpha_i) \leq \frac{\delta}{\alpha_i}$. *Suppose the hypotheses of Theorem* 3.3 *and Theorem* 3.4 *hold and let*

$$l := \max\{i : \varphi(\alpha_i) \leq \frac{\delta}{\alpha_i}\} < M,$$

$$s := \max\{i : \|x_i - x_j\| \leq 4\bar{c}\frac{\delta}{\alpha_j}, \qquad j = 0, 1, 2, \dots, i-1\}.$$

Then $l \leq s$ *and*

$$\|\hat{x} - x_s\| \leq c\psi^{-1}(\delta)$$

where $c = 6\bar{c}\mu$.

4. Implementation of adaptive choice rule

The main goal of this section is to provide a starting point for the iteration approximating the unique solution x_α^δ of (1.5) and then to provide an algorithm for the determination of a parameter fulfilling the balancing principle. The choice of the starting point involves the following steps:

- For $q = \frac{1-\sqrt{1-16L'\eta}}{2}$ choose $0 < \alpha_0 < 1$ and $\mu > 1$.
- Choose η such that η satisfies (2.2).
- Choose ρ such that ρ satisfies (2.20).
- Choose $x_0 \in D(F)$ such that $\|x_0 - \hat{x}\| \leq \rho$.
- Choose the parameter $\alpha_M = \mu^M\alpha_0$ big enough with $\mu > 1$, not too large.
- Choose n_i such that $n_i = min\{n : q^n \leq \frac{\delta}{\alpha_i}\}$.

Finally the adaptive algorithm associated with the choice of the parameter specified in Theorem 3.6 involves the following steps:

4.1. Algorithm

- Set $i \leftarrow 0$.
- Solve $x_i := x^\delta_{n_i,\alpha_i}$ by using the iteration (1.10).
- If $\|x_i - x_j\| > 4c\frac{\sqrt{\delta}}{\mu^j}, \qquad j \leq i$, then take $s = i - 1$.
- Set $i = i + 1$ and return to second step.

5. Numerical example

We present a numerical example where Assumption 2 or Assumption 6 is not satisfied but our new Assumption 8 does. The example is presented in the more general setting of a Banach space.

Example 5.1. We consider the integral equation

$$u(s) = f(s) + \tau \int_a^b G(s,t)u(t)^{1+1/n}dt, \quad n \in \mathbb{N}. \tag{5.1}$$

Here, f is a given continuous function satifying $f(s) > 0, s \in [a,b], \tau$ is a real number, and the kernel G is continuous and positive in $[a,b] \times [a,b]$.

For example, when $G(s,t)$ is the Green kernel, the corresponding integral equation is equivalent to the boundary value problem

$$u'' = \tau u^{1+1/n}, \tag{5.2}$$

$$u(a) = f(a), u(b) = f(b). \tag{5.3}$$

These types of problems have been considered in [1, 2, 3, 4, 5].

Equation of the form (5.1) generalizes equations of the form

$$u(s) = \int_a^b G(s,t)u(t)^n dt \tag{5.4}$$

studied in [1, 2, 3, 4, 5]. Instead of (5.1) we can try to solve the equation $F(u) = 0$ where

$$F : \Omega \subseteq C[a,b] \to C[a,b], \Omega = \{u \in C[a,b] : u(s) \geq 0, s \in [a,b]\},$$

and

$$F(u)(s) = u(s) - f(s) - \tau \int_a^b G(s,t)u(t)^{1+1/n}dt.$$

The norm we consider is the max-norm.

The derivative F' is given by

$$F'(u)v(s) = v(s) - \tau(1 + \frac{1}{n}) \int_a^b G(s,t)u(t)^{1/n}v(t)dt, \quad v \in \Omega.$$

First of all, we notice that F' does not satisfy a Lipschitz-type condition in Ω. Let us consider, for instance, $[a,b] = [0,1], G(s,t) = 1$ and $y(t) = 0$. Then $F'(y)v(s) = v(s)$ and

$$\|F'(x) - F'(y)\| = |\tau| \left(1 + \frac{1}{n}\right) \int_a^b x(t)^{1/n}dt.$$

If F' were a Lipschitz function, then

$$\|F'(x) - F'(y)\| \leq L_2 \|x - y\|,$$

or, equivalently, the inequality

$$\int_0^1 x(t)^{1/n} dt \leq L_3 \max_{x \in [0,1]} x(s), \tag{5.5}$$

would hold for all $x \in \Omega$ and for a constant L_3. But this is not true. Consider, for example, the functions

$$x_j(t) = \frac{t}{j}, \quad j \geq 1, \quad t \in [0,1].$$

If these are substituted into (5.5)

$$\frac{1}{j^{1/n}(1+1/n)} \leq \frac{L_3}{j} \iff j^{1-1/n} \leq L_3(1+1/n), \quad \forall j \geq 1.$$

This inequality is not true when $j \to \infty$. Therefore, Assumption 6 or the Lipschitz condition on page 2 used in [8] are not satisfied. However, Assumption 8 holds. To show this, let $x_0(t) = f(t)$ and $\gamma = \min_{s\in[a,b]} f(s), \alpha > 0$ Then for $v \in \Omega$,

$$\begin{aligned} \big\| \big[F'(x) - F'(x_0)\big]v\big\| &= |\tau| \left(1 + \frac{1}{n}\right) \times \\ &\times \max_{s\in[a,b]} \left| \int_a^b G(s,t)\big(x(t)^{1/n} - f(t)^{1/n}\big)v(t)dt \right| \leq \\ &\leq |\tau| \left(1 + \frac{1}{n}\right) \max_{s\in[a,b]} G_n(s,t), \end{aligned}$$

where $G_n(s,t) = \dfrac{G(s,t)|x(t) - f(t)|}{x(t)^{(n-1)/n} + x(t)^{(n-2)/n} f(t)^{1/n} + \cdots + f(t)^{(n-1)/n}} \|v\|.$

Hence, we get that

$$\begin{aligned} \big\| \big[F'(x) - F'(x_0)\big]v\big\| &= \frac{|\tau|(1+1/n)}{\gamma^{(n-1)/n}} \max_{s\in[a,b]} \int_a^b G(s,t)dt\, \|x - x_0\| \leq \\ &\leq L_0 \|x - x_0\|, \end{aligned}$$

where $L' = \frac{|\tau|(1+1/n)}{\gamma^{(n-1)/n}} N$ with $N = \max_{s\in[a,b]} \int_a^b G(s,t)dt$. Then, by the following inequality

$$\begin{aligned} \big\| \big[F'(x) - F'(u)\big]v\big\| &\leq \big\| \big[F'(x) - F'(x_0\big]v\big\| + \big\| \big[F'(x_0) - F'(u)\big]v\big\| \leq \\ &\leq L' \|v\| \big(\|x - x_0\| + \|u - x_0\|\big), \end{aligned}$$

Assumption 8 holds for sufficiently small λ.

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