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ON THE UNIQUENESS PROBLEM OF NON-ARCHIMEDEAN MEROMORPHIC FUNCTIONS AND THEIR DIFFERENTIAL POLYNOMIALS

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Abstract. In this paper, we discuss the uniqueness problem for differential polynomials $(P^n(f))^{(k)}, (Q^n(g))^{(k)}$, sharing the same value, where P, Q are polynomials of Fermat-Waring type, f and g are meromorphic functions on a non-Archimedean field.

1. Introduction

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, complete for a non-Archimedean absolute value. We denote by $\mathcal{A}(\mathbb{K})$ the ring of entire functions in $\mathbb{K}$, by $\mathcal{M}(\mathbb{K})$ the field of meromorphic functions, i.e., the field of fractions of $\mathcal{A}(\mathbb{K})$, and $\widehat{\mathbb{K}} = \mathbb{K} \cup \{\infty\}$. We assume that the reader is familiar with the notations in the non-Archimedean Nevanlinna theory (see [18]). Let f be a non-constant meromorphic function on $\mathbb{K}$. For every $a \in \mathbb{K}$, define the function $\nu_f^a : \mathbb{K} \to \mathbb{N}$ by

$$\nu_f^a(z) = \begin{cases} 0 & \text{if } f(z) \neq a \\ m & \text{if } f(z) = a \text{ with multiplicity } m, \end{cases}$$

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and set $\nu_f^{\infty} = \nu_{\frac{1}{f}}^{0}$. For $f \in \mathcal{M}(\mathbb{K})$ and $S \subset \mathbb{K} \cup \{\infty\}$, we define

$$E_f(S) = \bigcup_{a \in S} \{(z, \nu_f^a(z)) : z \in \mathbb{K}\}.$$

Let $\mathcal{F}$ be a nonempty subset of $\mathcal{M}(\mathbb{K})$. Two functions f, g of $\mathcal{F}$ are said to *share* S, *counting multiplicity*, if $E_f(S) = E_g(S)$. Let a set $S \subset \mathbb{K} \cup \{\infty\}$ and f and g be two non-constant meromorphic (entire) functions. If $E_f(S) = = E_g(S)$ implies $f = g$ for any two non-constant meromorphic (entire) functions f, g, then S is called a unique range set for meromorphic(entire) functions or, in brief, $URSM(URSE)$. Several interesting results on $URSE$ and $URSM$ for non-Archimedean entire and meromorphic functions have been obtained (see[6], [13], [17] and [18]). The smallest unique range set for meromorphic functions has 10 elements and was given by Hu and Yang [17]. Recently, many results were obtained also for differential polynomials, for example, of the form $(f^n)^{(k)}$ (Khoai, An, and Lai [12]; An, Hoa, and Khoai [3]), and of the form $(f)^{(')} P^{'}(f)$, (Boussaf, Escassut and Ojeda [5]). In [12] Khoai, An, and Lai proved the following result.

Theorem A. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions on $\mathbb{K}$, and let n, k be two positive integers with $n \geq 3k+8$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share* 1 *CM, then $f(z) = tg(z)$ for a constant t such that $t^n = 1$.*

In [22] Yang posed the problem: is it true that the equality $f^{-1}(S) = g^{-1}(S)$ with $S = \{-1, 1\}$ for polynomials of the same degree f, g implies that either $f = g$ or $f = -g$? This problem was solved in [19] and [20].

In this paper, instead of functions f and g we consider differential operators of the form $(P^n(f))^{(k)}, (Q^n(g))^{(k)}$, sharing the same value, where P, Q are polynomials of Fermat-Waring type. Then we establish an uniqueness theorem for non-Archimedean meromorphic functions and their differential polynomials.

Concerning the mentioned above problem of Yang, and related topics (see, for example [20]), we consider the following problem. Let S, T be the zero sets of polynomials $P(z), Q(z)$, respectively, then how we can say about the relations of f, g, if $E_f(S) = E_g(T)$?.

Now let us describe main results of the paper.

Let $d, m, n, k \in \mathbb{N}^*$ and $a_1, b_1, c, a_2, b_2 \in \mathbb{K}$; $a_1, b_1, c, a_2, b_2 \neq 0$.

We will let

$$P(z) = z^d + a_1 z^{d-m} + b_1, \; Q(z) = z^d + a_2 z^{d-m} + b_2, \tag{1.1}$$

be polynomials of degree d of Fermat–Waring type in $\mathbb{K}[z]$ without multiple zeros. We shall prove the following theorems.

Theorem 1. *Let f, g be two non-constant meromorphic functions on $\mathbb{K}$ and let $P(z)$, $Q(z)$ be defined in (1.1). Assume that $n \geq 3k+5$, $d \geq 2m+8$ and either $m \geq 3$ or $(d,m) = 1$ and $m \geq 2$. If $(P^n(f))^{(k)}$ and $(Q^n(g))^{(k)}$ share 1 CM, then $g = hf$ for a constant h such that $h^d = \frac{b_2}{b_1}$, $h^{nd} = 1$, $h^m = \frac{a_2}{a_1}$.*

Theorem 2. *Let f, g be two non-constant meromorphic functions on $\mathbb{K}$ and let $P(z)$, $Q(z)$ be defined in (1.1). Assume that $d \geq 2m+8$ and either $m \geq 3$ or $(d,m) = 1$ and $m \geq 2$. If $P(f)$ and $Q(g)$ share 0 CM, then $g = hf$ for a constant h such that $h^d = \frac{b_2}{b_1}$, $h^m = \frac{a_2}{a_1}$.*

As immediate consequences of Theorem 2, we have

Corollary 3. *Let S, T be the zero sets of the above polynomials $P(z), Q(z)$, respectively, and let f, g be two non-constant meromorphic functions on $\mathbb{K}$. Assume that $d \geq 2m+8$ and either $m \geq 3$ or $(d,m) = 1$ and $m \geq 2$. If $E_f(S) = E_g(T)$, then $g = hf$ for a constant h such that $h^d = \frac{b_2}{b_1}$, $h^m = \frac{a_2}{a_1}$.*

Corollary 4. *Let S be the zero sets of the polynomial $P(z)$, and et f, g be two non-constant meromorphic functions on $\mathbb{K}$. Assume that $d \geq 2m+8$ and $(d,m) = 1$ and $m \geq 2$. If $E_f(S) = E_g(S)$, then $f = g$.*

2. Lemmas

We assume that the reader is familiar with the notations in the non-Archimedean Nevanlinna theory (see [4], [8], [9] and [18]).

We first need the following Lemmas.

Lemma 2.1. ([18]) *Let f be a non-constant meromorphic function on $\mathbb{K}$ and let $a_1, a_2, ..., a_q$, be distinct points of $\mathbb{K} \cup \{\infty\}$. Then*

$$(q-2)T(r,f) \leq \sum_{i=1}^{q} N_1(r, \frac{1}{f-a_i}) - \log r + O(1).$$

Lemma 2.2. ([18]) *Let f be a non-constant meromorphic function on $\mathbb{K}$ and let $a_1, a_2, ..., a_q$, be distinct points of $\mathbb{K} \cup \{\infty\}$. Suppose either $f - a_i$ has no zeros, or $f - a_i$ has zeros, in which case all the zeros of the functions $f - a_i$ have multiplicity at least $m_i, i = 1, ..., q$. Then*

$$\sum_{i=1}^{q} (1 - \frac{1}{m_i}) < 2.$$

Lemma 2.3. ([12]) *Let f and g be non-constant meromorphic functions on $\mathbb{K}$. If $E_f(1) = E_g(1)$, then one of the following three cases holds:*

1. $T(r, f) \leq N_2(r, f) + N_2(r, \frac{1}{f}) + N_2(r, g) + N_2(r, \frac{1}{g}) - \log r + O(1)$, *and the same inequality holds for $T(r, g)$;*
2. $fg = 1$;
3. $f = g$.

Lemma 2.4. ([12]) *Let f be a non-constant meromorphic function on $\mathbb{K}$ and n, k be positive integers, $n > k$ and a be a pole of f. Then*

1. $(f^n)^{(k)} = \dfrac{\varphi_k}{(z-a)^{np+k}}$, *where* $p = \nu_f^{\infty}(a), \varphi_k(a) \neq 0.$
2. $\dfrac{(f^n)^{(k)}}{f^{n-k}} = \dfrac{h_k}{(z-a)^{pk+k}}$, *where* $p = \nu_f^{\infty}(a), h_k(a) \neq 0.$

Lemma 2.5. *Let f, $(f)^{(k)}$ be non-constant meromorphic functions on $\mathbb{K}$ and k be a positive integer. Then*

$$T(r, (f)^{(k)}) \leq (k+1)T(r, f) + O(1).$$

Proof. By Lemma 2.4, and noting that $m(r, \dfrac{(f)^{(k)}}{f}) = O(1)$ we get

$$\begin{aligned} T(r, (f)^{(k)}) &= m(r, (f)^{(k)}) + N(r, (f)^{(k)}) \leq \\ &\leq m(r, f) + N(r, f) + kN_1(r, f) + O(1) \leq \\ &\leq T(r, f) + kT(r, f) + O(1) = (k+1)T(r, f) + O(1). \end{aligned}$$

Lemma 2.5 is proved. ■

Lemma 2.6. ([12]) *Let f be a non-constant meromorphic function on $\mathbb{K}$ and n, k be positive integers, $n \geq k+1$. Then*

$$T(r, f) \leq T(r, f^n)^{(k)}) + O(1),$$

in particular, $(f^n)^{(k)}$ is not a constant.

Lemma 2.7. ([12]) *Let f be a non-constant meromorphic function on $\mathbb{K}$ and n, k be positive integers, $n > 2k$. Then*

1. $(n-2k)T(r, f) + kN(r, f) + N(r, \dfrac{1}{\frac{(f^n)^{(k)}}{f^{n-k}}}) \leq T(r, (f^n)^{(k)}) + O(1);$
2. $N(r, \dfrac{1}{\frac{(f^n)^{(k)}}{f^{n-k}}}) \leq kT(r, f) + kN_1(r, f) + O(1).$

Lemma 2.8. *Let f be a non-constant meromorphic function on $\mathbb{K}$ and n, k be positive integers, $n > 2k$, and let $P(z)$ be a polynomial of degree $d > 0$. Then*

1. $(n-2k)dT(r,f)+kN(r,P(f))+N(r,\dfrac{1}{\frac{((P(f))^n)^{(k)}}{(P(f))^{n-k}}}) \leq T(r,((P(f))^n)^{(k)})+$
$+O(1) \leq (k+1)ndT(r,f)+O(1).$

2. $N(r,\dfrac{1}{\frac{((P(f))^n)^{(k)}}{(P(f))^{n-k}}}) \leq kdT(r,f)+kN_1(r,P(f))+O(1) =$
$= kdT(r,f)+kN_1(r,f)+O(1) \leq k(d+1)T(r,f)+O(1).$

Proof. 1. Set $A = ((P(f))^n)^{(k)}, C = P(f)$. Then $T(r,C) = T(r,P(f)) = = dT(r,f)+O(1), T(r,P^n(f)) = ndT(r,f)+O(1)$. Therefore, C, C^n are not constants. By Lemma 2.6 we see that $A = (C^n)^{(k)}$ is not a constant. On the other hand, by Lemma 2.7 and Lemma 2.5 we get

$$(n-2k)T(r,C)+kN(r,C)+N(r,\frac{1}{\frac{A}{C^{n-k}}}) \leq T(r,A)+O(1) \leq (k+1)T(r,C^n)+O(1),$$

i.e.

$$(n-2k)dT(r,f)+kN(r,P(f))+N(r,\frac{1}{\frac{((P(f))^n)^{(k)}}{(P(f))^{n-k}}}) \leq$$
$$\leq T(r,((P(f))^n)^{(k)})+O(1) \leq (k+1)ndT(r,f)+O(1).$$

2. By Lemma 2.7 we have

$$N(r,\frac{1}{\frac{A}{C^{n-k}}}) \leq kT(r,C)+kN_1(r,C)+O(1).$$

On the other hand,

$$T(r,C) = dT(r,f)+O(1), N_1(r,C) = N_1(r,f) \leq N(r,f) \leq T(r,f)+O(1).$$

Therefore,

$$N(r,\frac{1}{\frac{((P(f))^n)^{(k)}}{(P(f))^{n-k}}}) \leq kdT(r,f)+kN_1(r,P(f))+O(1) =$$
$$= kdT(r,f)+kN_1(r,f)+O(1) \leq k(d+1)T(r,f)+O(1).$$

Lemma 2.8 is proved. ■

Lemma 2.9. *Let $d \geq 2m+3$ and either $m \geq 3$ or $(d,m)=1$ and $m \geq 2$, $c \neq 0$, and let $P(z), Q(z)$ be defined by (1). Assume that the equation $P(f)=cQ(g)$ has a non-constant meromorphic solution (f,g). Then $g=hf$ for a constant h such that $h^d = \frac{1}{c} = \frac{b_2}{b_1}$, $h^m = \frac{a_2}{a_1}$.*

Proof. Since $P(f)=Q(g)$ we get

$$f^d + a_1 f^{d-m} + b_1 = c(g^d + a_2 g^{d-m} + b_2),\ dT(r,f)+O(1) = dT(r,g),$$

$$T(r,f)+O(1) = T(r,g). \tag{2.1}$$

Equation (2.1) can be rewritten as

$$f_1 + f_2 = cb_2 - b_1,\ \text{where } f_1 = f^{d-m}(f^m + a_1),\ f_2 = -cg^{d-m}(g^m + a_2).$$

If $cb_2 - b_1 \neq 0$, then by Lemma 2.1, we have

$$\begin{aligned}
T(r,f_1) &\leq N_1(r,f_1) + N_1(r,\frac{1}{f_1}) + N_1(r,\frac{1}{f_1-(cb_2-b_1)} - \log r + O(1),\\
dT(r,f) &\leq N_1(r,f) + N_1(r,\frac{1}{f}) + N_1(r,\frac{1}{f^m+a_1}) + N_1(r,\frac{1}{g})+\\
&\qquad + N_1(r,\frac{1}{g^m+a_2}) - \log r + O(1),\\
dT(r,f) &\leq (2m+3)T(r,f) - \log r + O(1),\ (d-2m-3)T(r,f) \leq\\
&\qquad \leq -\log r + O(1),
\end{aligned}$$

which contradicts to $d \geq 2m+3$. Hence $cb_2 - b_1 = 0$. Thus, (2.1) becomes

$$f^d + a_1 f^{d-m} = cg^d + ca_2 g^{d-m}. \tag{2.2}$$

For simplicity, set $h = \frac{g}{f}$, and $\alpha = \frac{1}{c} \neq 0; \beta = \frac{a_1}{ca_2} \neq 0$. Then we obtain

$$f^m(ch^d - 1) = -(ca_2 h^{d-m} - a_1),\ f^m(h^d - \alpha) = -a_2(h^{d-m} - \beta),$$

$$f^m = -a_2 \frac{h^{d-m} - \beta}{h^d - \alpha}. \tag{2.3}$$

Assume that h is not a constant. Consider the following possible cases:

Case 1. $m \geq 2$, $(m,d)=1$. If $h^d - \alpha$ and $h^{d-m} - \beta$ have no common zeros, then all zeros of $h^d - \alpha$ have multipcities $\geq m$. Then

$$N_1(r,\frac{1}{h^d-\alpha}) \leq \frac{1}{m} N(r,\frac{1}{h^d-\alpha}).$$

By Lemma 2.1 we obtain

$$T(r,h^d) \leq N_1(r,h^d) + N_1(r,\frac{1}{h^d}) + N_1(r,\frac{1}{h^d-\alpha}) - \log r + O(1),$$

$$dT(r,h) \leq 2T(r,h) + \frac{1}{m}N(r,\frac{1}{h^d-\alpha}) - \log r + O(1) \leq$$

$$\leq (2+\frac{d}{m})T(r,h) - \log r + O(1),$$

$$(d-2-\frac{d}{m})T(r,h) \leq -\log r + O(1),$$

which leads to $d(m-1) < 2m$, a contradiction to the condition $d \geq 2m+3$.

If $h^d - \alpha$ and $h^{d-m} - \beta$ have common zeros, then there exists z_0 such that $h^d(z_0) = \alpha$, $h^{d-m}(z_0) = \beta$. From (2.3) we get

$$\alpha f^m((\frac{h}{h(z_0)})^d - 1) = -\beta a_2((\frac{h}{h(z_0)})^{d-m} - 1).$$

Since $(m,d) = 1$, the equations $z^d - 1 = 0$ and $z^{d-m} - 1 = 0$ have different roots, except for $z = 1$. Let $r_i, i = 1, ..., 2d-m-2$, be all the roots of them. Then all zeros of $\frac{h}{h(z_0)} - r_i$ have multipcities $\geq m$. Therefore, by Lemma 2.2 we obtain

$$(1-\frac{1}{m})(2d-m-2) < 2, \ \ 2d(m-1) < m^2 + 3m - 2,$$

which contradicts $d \geq 2m+3$, $m \geq 2$. Thus, h is a constant.

Case 2. $m \geq 3$. Note that equation $z^d - \alpha = 0$ has d simple zeros, equation $z^{d-m} - \beta = 0$ has $d-m$ simple zeros. Then $z^d - \alpha = 0$, $z^{d-m} - \beta = 0$ have at most $d-m$ common simple zeros. Therefore, the equation $z^d - \alpha = 0$ has at least m distinct roots, which are not roots of $z^{d-m} - \beta = 0$. Let $r_1, r_2, ..., r_m$ be all these roots. Then all zeros of $h - r_j, j = 1, ..., m$, have multiplicities $\geq m$. By Lemma 2.2 we have $m(1-\frac{1}{m}) < 2$. Therefore, $m < 3$. From $m \geq 3$, we obtain a contradiction. Thus h is a constant. ■

3. Proof of main resutls

3.1. Proof of Theorem 1. We have

$$P(f) = (f-e_1)\cdots(f-e_d), \ e_i \in \mathbb{K}, \ e_i \neq 0,$$

$$(P(f))^n = (f-e_1)^n \cdots (f-e_d)^n,$$

$$Q(g) = (g-k_1)...(g-k_d), \ k_i \in \mathbb{K}, \ k_i \neq 0,$$

$$(Q(g))^n = (g-k_1)^n...(g-k_d)^n.$$

Set

$$A = ((P(f))^n)^{(k)}, \quad B = ((Q(g))^n)^{(k)}, \quad C = P(f),$$
$$D = Q(g), \quad F = \frac{A}{C^{n-k}}, \quad Q = \frac{B}{D^{n-k}}.$$

Then

$$C = (f - e_1)\cdots(f - e_d), \quad D = (g - k_1)\cdots(g - k_d),$$
$$A = (C^n)^{(k)} = FC^{n-k}, \quad B = (D^n)^{(k)} = QD^{n-k}.$$

Applying Lemma 2.3 to $(C^n)^{(k)}, (D^n)^{(k)}$ we have one of the following possibilities:

Case 1.

$$T(r, A) \leq N_2(r, A) + N_2(r, \frac{1}{A}) + N_2(r, B) + N_2(r, \frac{1}{B}) - \log r + O(1),$$

$$T(r, B) \leq N_2(r, A) + N_2(r, \frac{1}{A}) + N_2(r, B) + N_2(r, \frac{1}{B}) - \log r + O(1).$$

We see that, if a is a pole of A, then $C(a) = \infty$ with $\nu_A^\infty(a) \geq n + k \geq 2$. Therefore,

$$\begin{aligned}
&N_1(r, C) = N_1(r, (f - e_1)...(f - e_d)) = N_1(r, f) \leq T(r, f) + O(1),\\
&N_1(r, \frac{1}{C}) = \sum_{i=1}^{d} N_1(r, \frac{1}{f - e_i}) \leq dT(r, f) + O(1),\\
&N_2(r, A) = 2N_1(r, C) \leq 2T(r, f) + O(1),\\
&N_2(r, \frac{1}{A}) \leq N_2(r, \frac{1}{C^{n-k}}) + N(r, \frac{1}{F}) = 2N_1(r, \frac{1}{C}) + N(r, \frac{1}{F}) \leq\\
&\qquad \leq 2dT(r, f) + N(r, \frac{1}{F}) \leq 2dT(r, f) + kN_1(r, C)+\\
&\qquad + kdT(r, f) + O(1) = d(k + 2)T(r, f) + kN_1(r, C) + O(1).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&N_2(r, B) \leq 2T(r, g) + O(1),\\
&N_2(r, \frac{1}{B}) \leq 2dT(r, g) + N(r, \frac{1}{Q}) \leq d(k + 2)T(r, g) + kN_1(r, D) + O(1).
\end{aligned}$$

Combining the above inequalities, we get

$$T(r, A) \leq (2{+}2d{+}kd)T(r, f){+}(2{+}2d)T(r, g){+}kN_1(r, C){+}N(r, \frac{1}{Q}){-}\log r{+}O(1),$$

$$T(r,B) \le (2+2d+kd)T(r,g)+(2+2d)T(r,f)+kN_1(r,D)+N(r,\frac{1}{F})-\log r+O(1).$$

$$T(r,A)+T(r,B) \le (4+4d+kd)(T(r,f)+T(r,g))+kN_1(r,C)+$$
$$+N(r,\frac{1}{Q})+kN_1(r,D)+N(r,\frac{1}{F})-2\log r+O(1).$$

By Lemma 2.8 we obtain

$$(n-2k)dT(r,f)+kN(r,C)+N(r,\frac{1}{F}) \le T(r,A)+O(1),$$

$$(n-2k)dT(r,g)+kN(r,D)+N(r,\frac{1}{Q}) \le T(r,B)+O(1).$$

Thus,

$$(n-2k)d(T(r,f)+T(r,g))+kN(r,C)+N(r,\frac{1}{F})+kN(r,D)+N(r,\frac{1}{Q}) \le$$
$$\le T(r,A)+T(r,B)+O(1),$$
$$(n-2k)d(T(r,f)+T(r,g))+kN(r,C)+N(r,\frac{1}{F})+kN(r,D)+N(r,\frac{1}{Q}) \le$$
$$\le (4+4d+kd)(T(r,f)+T(r,g))+kN_1(r,C)+N(r,\frac{1}{Q})+$$
$$+kN_1(r,D)+N(r,\frac{1}{F})-2\log r+O(1).$$

Therefore,

$$(n-2k)d(T(r,f)+T(r,g)) \le (4+4d+kd)(T(r,f)+T(r,g))-2\log r+O(1),$$

$$((n-2k)d-4-4d-kd)(T(r,f)+T(r,g)) \le -2\log r+O(1).$$

Since $n \ge 3k+5 > 2k+\frac{4+4d+kd}{d}$, we obtain a contradiction.

Case 2. $(P(f))^n)^{(k)}\ \ ((Q(g))^n)^{(k)}=1$. Then we have

$$C=P(f)=(f-e_1)\cdots(f-e_d),(C^n)^{(k)}=C^{n-k}F,D=Q(g).$$

Therefore

$$(f-e_1)^{n-k}\cdots(f-e_d)^{n-k}.F.(D^n)^{(k)}=(C^n)^{(k)}(D^n)^{(k)}=1.$$

Because $n \ge 3k+5$ we see that, if z_0 is a zero of $f-e_i$ with $1 \le i \le d$, then z_0 is a zero of C, and therefore, z_0 is a zero of $(C^n)^{(k)}$, and then z_0 is a pole of $(D^n)^{(k)}$ and $\nu^{\infty}_{(D^n)^{(k)}}(z_0) = (n-k)\nu^{e_i}_f(z_0)$. Thus, z_0 is a pole of g, and by Lemma 2.4 we get

$$\nu^{\infty}_{(D^n)^{(k)}}(z_0) = nd\nu_g{}^{\infty}(z_0) + k \geq nd + k.$$

So $\nu^{e_i}_f(z_0) = \frac{nd\nu_g{}^{\infty}(z_0)+k}{n-k} \geq \frac{nd+k}{n-k}, i = 1, 2, \ldots, d$. Applying Lemma 2.2, we obtain:

$$\sum_{i=1}^{d}(1 - \frac{n-k}{nd+k}) < 2.$$

From this we have $n(d^2 - 3d) < 2k(1-d)$, and so we obtain a contradiction to $d \geq 12$.

Case 3. $((P(f))^n)^{(k)} = ((Q(g))^n)^{(k)}$. Then $(P(f))^n - s = (Q(g))^n$, where s is a polynomial of degree $< k$. We prove $s \equiv 0$. If it is not the case, then

$$\frac{((P(f))^n}{s} - 1 = \frac{(g-k_1)^n \cdots (g-k_d)^n}{s},$$

$$\frac{(g-k_1)^n \cdots (g-k_d)^n}{s} + 1 = \frac{(f-e_1)^n \cdots (f-e_d)^n}{s}.$$

Set $H = \frac{C^n}{s}, G = \frac{D^n}{s}$. Since f, g are not constants, and so are C, D, C^n, D^n, H, G. Applying Lemma 2.1 to H with values ∞, $0, 1$, we get

$$T(r, H) \leq N_1(r, H) + N_1(r, \frac{1}{H}) + N_1(r, \frac{1}{H-1}) - \log r + O(1).$$

On the other hand,

$$T(r, C^n) = nT(r, C) + O(1) \leq T(r, H) + T(r, s) \leq T(r, H) + (k-1)\log r + O(1),$$

$$nT(r, C) - (k-1)\log r \leq T(r, H) + O(1), \ ndT(r, f) - (k-1)\log r \leq T(r, H) + O(1).$$

$$N_1(r, H) \leq N_1(r, C^n) + N_1(r, \frac{1}{s}) \leq N_1(r, f) + (k-1)\log r \leq T(r, f) + (k-1)\log r,$$

$$N_1(r, \frac{1}{H}) \leq N_1(r, \frac{1}{C^n}) = N_1(r, \frac{1}{C}) \leq T(r, C) + O(1) = dT(r, f) + O(1),$$

$$N_1(r, \frac{1}{H-1}) = N_1(r, \frac{1}{G}) \leq N_1(r, \frac{1}{D^n}) = N_1(r, \frac{1}{D}) \leq T(r, D) + O(1) = dT(r, g)$$

$$+O(1), ndT(r, f) - (k-1)\log r \leq T(r, f) + (k-1)\log r + d(T(r, f) + T(r, g)) + O(1).$$

From this, and noting that $\log r \leq T(r, f)$, we get

$$(nd - 2(k-1))T(r, f) \leq T(r, f) + d(T(r, f) + T(r, g)) + O(1).$$

Applying Lemma 2.1 to G with values $\infty, 0, -1$, and noting that $\log r \le T(r,g)$ we obtain

$$T(r,G) \le N_1(r,G) + N_1(r,\frac{1}{G}) + N_1(r,\frac{1}{G+1}) - \log r + O(1),$$

$$ndT(r,g)-(k-1)\log r \le T(r,g)+(k-1)\log r+d(T(r,f)+T(r,g))-\log r+O(1),$$

$$(nd-2(k-1))T(r,g) \le T(r,g) + d(T(r,f)+T(r,g)) - \log r + O(1).$$

So

$$(nd-2(k-1))(T(r,f)+T(r,g)) \le T(r,f)+T(r,g)+2d(T(r,f)+T(r,g))-$$

$$-2\log r + O(1),$$

$$(nd-2d-2k+1))(T(r,f)+T(r,g)) + 2\log r \le O(1).$$

We obtain a contradiction to $n \ge 3k+5 > \frac{2d+2k-1}{d}$. So $s=0$. Then $(P(f))^n = (Q(g))^n$. Therefore, $P(f) = cQ(g), c^n = 1$. From this and by Lemma 2.9, we obtain the conclusion of Theorem 1. ■

3.2 Proof of Theorem 2. Set

$$C = P(f) = f^d + a_1 f^{d-m} + b_1, \quad D = Q(g) = g^d + a_2 g^{d-m} + b_2,$$

$$M = -\frac{f^{d-m}(f^m + a_1)}{b_1}, \quad N = -\frac{g^{d-m}(g^m+a_2)}{b_2}.$$

Since $P(f)$ and $Q(g)$ share 0 CM, we get $E_M(1) = E_N(1)$. Applying Lemma 2.3 to M, N, we have one of the following possibilities:

Case 1.

$$T(r,M) \le N_2(r,M) + N_2(r,\frac{1}{M}) + N_2(r,N) + N_2(r,\frac{1}{N}) - \log r + O(1),$$

$$T(r,N) \le N_2(r,M) + N_2(r,\frac{1}{M}) + N_2(r,N) + N_2(r,\frac{1}{N}) - \log r + O(1).$$

Moreover,

$$T(r,M) = dT(r,f) + O(1) \; N_1(r,M) = N_1(r,f) \le T(r,f) + O(1),$$
$$N_2(r,M) = 2N_1(r,f) \le 2T(r,f) + O(1),$$
$$N_2(r,\frac{1}{M}) \le 2N_1(r,\frac{1}{f}) + N_2(r,\frac{1}{f^m+a_1}) \le 2T(r,f) + mT(r,f) + O(1).$$

Similarly

$$N_2(r,N) \le 2T(r,g) + O(1), \; N_2(r,\frac{1}{N}) \le 2T(r,g) + mT(r,g) + O(1).$$

Therefore,

$$T(r,M) = dT(r,f)+O(1) \leq 4(T(r,f)+T(r,g))+m(T(r,f)+T(r,g))-\log r+O(1).$$

Similarly

$$T(r,N) = dT(r,g)+O(1) \leq 4(T(r,f)+T(r,g))+m(T(r,f)+T(r,g))-\log r+O(1).$$

Combining the above inequalities we get

$$d(T(r,f)+T(r,g)) \leq 8(T(r,f)+T(r,g))+2m(T(r,f)+T(r,g))-2\log r+O(1),$$

$$(d-2m-8)(T(r,f)+T(r,g))+2\log r \leq O(1).$$

We obtain a contradiction to $d \geq 2m+8$.

Case 2. $M.N = 1$, i.e. $f^{d-m}(f^m+a_1)g^{d-m}(g^m+a_2) = \frac{b_1}{b_2}$.

Note that equation $z^m + a_1 = 0$ has m simple zeros. Let $r_1, r_2, ..., r_m$ be all these roots.Therefore

$$f^{d-m}(f-r_1)...(f-r_1)g^{d-m}(g^m+a_2) = \frac{b_1}{b_2}. \tag{3.1}$$

From (3.1) it follows that all zeros of $f - r_j, j = 1, ..., m$, have multiplicities $\geq d$, and all zeros of f have multiplicities $\geq \frac{d}{d-m}$. By Lemma 2.2 we have $1-\dfrac{d-m}{d}+m(1-\dfrac{1}{d}) < 2$. Then $m < 2$. Since $m \geq 2$, we obtain a contradiction.

Case 3. $M = N$, i.e. $\frac{f^{d-m}(f^m+a_1)}{b_1} = \frac{g^{d-m}(g^m+a_2)}{b_2}$. Then

$$f^d + a_1 f^{d-m} + b_1 = \frac{b_1}{b_2}(g^d + a_2 g^{d-m} + b_2). \tag{3.2}$$

Applying Lemma 2.9 to (3.2), we obtain we obtain the conclusion of Theorem 2. ■

3.3 Proof of Corollary 3. Since $P(z)$, $Q(z)$ have no multiple zeros, we see that $E_f(S) = E_g(T)$ if and only if $P(f)$ and $Q(g)$ share 0 CM. From this and Theorem 2, we obtain the conclusion of Corollary 3. ■

3.4 Proof of Corollary 4. By $E_f(S) = E_g(S)$ and Corollary 3, we obtain $g = hf$ for a constant h, such that $h^d = \dfrac{b_2}{b_1}$, $h^m = \dfrac{a_2}{a_1}$ with $b_1 = b_2$, $a_1 = a_2$. Therefore, $h^d = 1$ and $h^m = 1$. Because $(d,m) = 1$ we have $h = 1$. So $f = g$. ■

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