*Annales Univ. Sci. Budapest., Sect. Comp. 46 (2017) 223–233*

# A MULTIPLICATIVE FUNCTION WITH EQUATION $f(p+m^3)=f(p)+f(m^3)$

**Bui Minh Phong** (Budapest, Hungary)

*Dedicated to the memory of Professor Antal Iványi*

Communicated by Imre Kátai
(Received May 10, 2017; accepted June 20, 2017)

**Abstract.** We prove that if a multiplicative function $f$ satisfies the conditions

$$f(p+m^3)=f(p)+f(m^3) \quad \text{and} \quad f(\pi^2)=f(\pi)^2$$

for all primes $p$, $\pi$ and positive integers $m$, then $f(n)=n$ holds for all positive integers $n$.

## 1. Introduction

An arithmetic function $g(n)\not\equiv 0$ is said to be multiplicative if $(n,m)=1$ implies that

$$g(nm)=g(n)g(m)$$

and it is completely multiplicative if this relation holds for all positive integers $n$ and $m$. Let $\mathcal{M}$ and $\mathcal{M}^*$ denote the class of all complex-valued multiplicative, completely multiplicative functions, respectively.

Let $\mathcal{P}$, $\mathbb{N}$ be the set of primes, positive integers, respectively. $n\|m$ denotes that $m$ is a unitary divisor of $n$, i.e. that $m|n$ and $(\frac{n}{m},m)=1$. Let

$$M(n)=\max\{q^\gamma :\ q^\gamma \parallel n,\ q\in\mathcal{P}\}.$$

*Key words and phrases*: Multiplicative function, a Dirichlet character, the identity function, functional equation.
*2010 Mathematics Subject Classification*: 11A07, 11A25, 11N25, 11N64.
https://doi.org/10.71352/ac.46.223

In 1992, Spiro [8] showed that if $f \in \mathcal{M}$ and $f(p_0) \neq 0$ for some prime $p_0$, then

$$f(p+q) = f(p) + f(q) \quad \text{for all} \quad p,\ q \in \mathcal{P}$$

implies that $f(n) = n$ for all $n \in \mathbb{N}$.

In the paper [3] written with J.-M. De Koninck and I. Kátai, we proved that if $f \in \mathcal{M}$ with $f(1) = 1$ and

$$f(p+m^2) = f(p) + f(m^2) \quad \text{for all} \quad p \in \mathcal{P},\ m \in \mathbb{N},$$

then $f(n) = n$ for all $n \in \mathbb{N}$. Recently in [7] we improve this result by proving that if $f, g \in \mathcal{M}$ with f(1)=1 satisfy

$$f(p+m^2) = g(p) + g(m^2) \quad \text{and} \quad g(p^2) = g(p)^2$$

for all primes $p$ and $m \in \mathbb{N}$, then either

$$f(p+m^2) = 0, \quad g(p) = -1 \ \text{ and } \ g(m^2) = 1$$

for all primes $p$ and $m \in \mathbb{N}$ or

$$f(n) = n \ \text{ and } \ g(p) = p, \quad g(m^2) = m^2$$

for all $p \in \mathcal{P},\ n, m \in \mathbb{N}$. The case $f = g \in \mathcal{M}^*$ was investigated in the previous paper [6].

For some generalizations of this topics, we refer to the works mentioned in the references of [7].

In this note, we prove

**Theorem 1.** *If $f \in \mathcal{M}$ satisfies the conditions*

$$f(p+m^3) = f(p) + f(m^3) \quad \textit{for all} \quad p \in \mathcal{P},\ m \in \mathbb{N} \tag{1.1}$$

*and*

$$f(\pi^2) = f(\pi)^2 \quad \textit{for all} \quad \pi \in \mathcal{P}, \tag{1.2}$$

*then*

$$f(n) = n \quad \textit{for all} \quad n \in \mathbb{N}.$$

## 2. Auxiliary lemmas

**Lemma 1.** *We have*

$$\mathfrak{S}(3) := \sum_{p \in \mathcal{P}} \frac{1}{p^3} < 0.1747626338.$$

**Proof.** Let
$$\mathfrak{S}(x,3):=\sum_{p\in\mathcal{P},\ p\le x}\frac{1}{p^3}.$$
It is clear that
$$\mathfrak{S}(3)-\mathfrak{S}(x,3)=\sum_{p\in\mathcal{P},\ p>x}\frac{1}{p^3}<\sum_{n=[x]+1}^{\infty}\frac{1}{n^3}<\frac{1}{2[x]^2}<\frac{1}{2(x-1)^2}$$
consequently
$$\mathfrak{S}(3)<\mathfrak{S}(x,3)+\frac{1}{2(x-1)^2}.$$
One can check with Maple that
$$\mathfrak{S}(10^6+1,3)<0.1747626336,$$
which shows that
$$\mathfrak{S}(3)<0.1747626336+\frac{1}{2\cdot 10^{12}}<0.1747626338.$$
Lemma 1 is proved. ■

**Lemma 2.** *If $f\in\mathcal{M}$ satisfies* (1.1) *and* (1.2), *then*

(2.1) $$f(n)=n\quad\textit{for all}\quad n\le 565\cdot 10^{10}.$$

**Proof.** First we prove that (2.1) holds for $n=2$.

Let $f(2):=x$. Then we infer from (1.1) that
$$\begin{aligned}
&f(3)=f(2+1^3)=x+1,\\
&f(4)=f(3+1^3)=f(3)+1=x+2,\\
&f(5)=f(5+1^3)-1=f(2)f(3)-1=x^2+x-1,\\
&f(8)=f(2+2^3)-f(2)=f(2)f(5)-f(2)=x^3+x^2-2x,\\
&f(11)=f(11+1^3)-1=f(3)f(4)-1=x^2+3x+1,\\
&f(19)=f(11+2^3)-1=f(11)+f(8)=x^3+2x^2+x+1,\\
&f(27)=f(19+2^3)=f(19)+f(8)=2x^3+3x^2-x+1,\\
&f(29)=f(2+3^3)=f(2)+f(27)=2x^3+3x^2+1.
\end{aligned}$$
On the other hand, we also get from (1.1)
$$f(19)=f(19+1)-1=f(4)f(5)-1=x^3+3x^2+x-3$$

and

$$f(29) = f(29+1) - 1 = f(2)f(3)f(5) - 1 = x^4 + 2x^3 - x - 1.$$

Thus, we have

$$x^3 + 2x^2 + x + 1 = x^3 + 3x^2 + x - 3, \quad (x-2)(x+2) = 0$$

and

$$2x^3 + 3x^2 + 1 = x^4 + 2x^3 - x - 1, \quad (x-2)(x^3 + 2x^2 + x + 1) = 0,$$

which imply $x = 2$. The assertion (2.1) is proved for $n = 2$.

As we seen above, we have

$$f(n) = n \quad \text{if} \quad n \in \{1, 2, 3, 4, 5, 6, 8, 10, 11, 12, 15, 19, 20, 27\}.$$

These with (1.1) imply

$$\begin{aligned}
&f(7) = f(7 + 1^3) - 1 = f(8) - 1 = 7,\\
&f(17) = \frac{f(34)}{f(2)} = \frac{f(34)}{2} = \frac{f(7) + f(27)}{2} = \frac{7+27}{2} = 17,\\
&f(9) = \frac{f(18)}{f(2)} = \frac{f(17)+1}{2} = \frac{17+1}{2} = 9,\\
&f(13) = f(5 + 2^3) = f(5) + f(8) = 13,\\
&f(16) = \frac{f(48)}{f(3)} = \frac{f(47)+1}{3} = \frac{f(55)-7}{3} = \frac{f(5)f(11)-7}{3} = 16,
\end{aligned}$$

and so

$$f(n) = n \quad \text{for all} \quad n \in \{1, 2, \cdots, 22\}.$$

Now we prove Lemma 2.

Assume by contradiction that there is a number $Q \in \mathbb{N}$, $22 < Q < 565 \cdot 10^{10}$ such that $f(n) = n$ for all $n < Q$ and $f(Q) \neq Q$. Then $Q = \pi^\alpha \geq 23$ is a prime power.

If $\alpha = 1$, then $Q + 1$ is even and $f(Q+1) = f(Q) + 1 \neq Q + 1$. Since $Q + 1$ is an even composite, then either $Q + 1 = 2^e$, $e \geq 5$ or $Q + 1 = uv$ with $1 < u < v < Q + 1$, $(u, v) = 1$. If $Q + 1 = 2^e$, then

$$\begin{aligned}
Q + 27 = 2.(2^{e-1} + 13) = f(2)f(2^{e-1} + 13) =\\
= f(Q + 27) = f(Q) + f(27) = f(Q) + 27,
\end{aligned}$$

because $2^{e-1} + 13 < 2^e - 1 = Q$. The last relation is impossible. In the case $Q + 1 = uv$ with $1 < u < v < Q + 1$, $(u, v) = 1$, we also get a contradiction, because

$$Q + 1 \neq f(Q+1) = f(uv) = f(u)f(v) = uv = Q + 1.$$

If $\alpha=2$, then we obtain from (1.2) that $Q\neq f(Q)=f(\pi^2)=f(\pi)^2= =\pi^2=Q$, which is impossible. If $\alpha=3\beta$, then

$$\pi+\pi^\alpha=\pi(1+\pi^{3\beta-1})=f(\pi)f(1+\pi^{3\beta-1})= \\ =f(\pi+\pi^{3\beta})=f(\pi)+f(\pi^\alpha)=\pi+f(\pi^\alpha),$$

consequently $f(Q)=f(\pi^\alpha)=\pi^\alpha=Q$. This is a a contradiction.

Assume now $\alpha\geq 4,\ 3\nmid\alpha$. Then there are 394 such prime powers $\pi^\alpha\leq \leq 565\cdot 10^{10}$, for which $\alpha\geq 4$ and $3\nmid\alpha$ hold. Since $\pi^4\leq\pi^\alpha\leq 565\cdot 10^{10}$, we have $\pi\leq 1541$. With the help of Maple program, for each prime power $\pi^\alpha\leq 565\cdot 10^{10}$ there is a positive $x_{\pi^\alpha}\in\{1,\cdots,237\}$ (see Table 1 for the smallest value of $x_{\pi^\alpha}$ which is $\geq 30$ ) such that

$$p_{\pi^\alpha}:=\pi^\alpha x_{\pi^\alpha}-1\in\mathcal{P},\quad x_{\pi^\alpha}<\pi^\alpha,\quad (x_{\pi^\alpha},\pi)=1$$

and

$$M(p_{\pi^\alpha}+8)=M(\pi^\alpha x_{\pi^\alpha}+7)<\pi^\alpha.$$

These with the fact that $f(n)=n$ for all $n<Q=\pi^\alpha$ imply

$$x_{\pi^\alpha}f(\pi^\alpha)=f(x_{\pi^\alpha})f(\pi^\alpha)=f(\pi^\alpha x_{\pi^\alpha})=f(p_{\pi^\alpha}+1)=f(p_{\pi^\alpha})+1$$

and

$$p_{\pi^\alpha}+8=f(p_{\pi^\alpha}+8)=f(p_{\pi^\alpha})+8,\quad f(p_{\pi^\alpha})=p_{\pi^\alpha}.$$

Thus $f(Q)=f(\pi^\alpha)=\pi^\alpha=Q$, which is a contradiction.

Lemma 2 is proved. ■

Let $\mathfrak{M}$ be the set of those subset of $\mathcal{L}$ of $\mathbb{N}$ for which

$$n,m\in\mathcal{L},\quad (n,m)=1\quad\Rightarrow\quad nm\in\mathcal{L}. \tag{2.2}$$

**Lemma 3.** *Assume that $\mathcal{L}\in\mathfrak{M}$ and an integer $T\geq 11000$. If*

$$\pi\in\mathcal{L}\quad\textit{for all}\quad \pi\in\mathcal{P},\ \pi<T$$

*and*

$$2^\alpha\in\mathcal{L},\quad 3^\beta\in\mathcal{L},\quad 5^\gamma\in\mathcal{L}\quad (0\leq\alpha\leq 10,\quad 0\leq\beta\leq 8,\quad 0\leq\gamma\leq 5),$$

*then for each $Q<T,\ 6\nmid Q$ we have*

$$\mathcal{A}_Q:=\{p\in\mathcal{P}\ \mid\ p<T,\ p+Q\in\mathcal{L}\}\neq\emptyset. \tag{2.3}$$

**Proof.** This is Lemma 7 in [7]. ■

| $\pi,\alpha,x_{\pi^\alpha}$ | $\pi,\alpha,x_{\pi^\alpha}$ | $\pi,\alpha,x_{\pi^\alpha}$ | $\pi,\alpha,x_{\pi^\alpha}$ | $\pi,\alpha,x_{\pi^\alpha}$ |
|---|---|---|---|---|
| 2, 15, 45 | 41, 5, 84 | 193, 5, 38 | 571, 4, 60 | 1013, 4, 48 |
| 2, 22, 33 | 41, 7, 40 | 223, 4, 32 | 577, 4, 138 | 1021, 4, 54 |
| 2, 24, 39 | 43, 5, 60 | 223, 5, 80 | 601, 4, 84 | 1049, 4, 44 |
| 2, 25, 57 | 47, 5, 42 | 227, 5, 42 | 607, 4, 48 | 1063, 4, 62 |
| 2, 30, 237 | 47, 7, 36 | 229, 5, 62 | 613, 4, 68 | 1109, 4, 60 |
| 2, 33, 49, | 53, 4, 32 | 239, 5, 58 | 619, 4, 38 | 1123, 4, 62 |
| 2, 36, 143 | 53, 5, 90 | 241, 4, 50 | 653, 4, 42 | 1181, 4, 44 |
| 2, 39, 49 | 53, 7, 90 | 251, 5, 42 | 673, 4, 114 | 1187, 4, 62 |
| 2, 40, 107 | 59, 5, 130 | 271, 4, 68 | 701, 4, 122 | 1217, 4, 38 |
| 2, 41, 55 | 61, 5, 108 | 277, 4, 48 | 709, 4, 44 | 1229, 4, 48 |
| 2, 42, 33 | 71, 5, 114 | 277, 5, 146 | 727, 4, 60 | 1259, 4, 98 |
| 3, 16, 38 | 73, 4, 50 | 283, 4, 68 | 751, 4, 30 | 1291, 4, 32 |
| 3, 25, 70 | 73, 5, 128 | 293, 5, 70 | 769, 4, 44 | 1319, 4, 48 |
| 7, 11, 120 | 83, 5, 30 | 307, 5, 60 | 811, 4, 44 | 1321, 4, 90 |
| 11, 5, 40 | 89, 4, 80 | 317, 4, 182 | 823, 4, 74 | 1373, 4, 62 |
| 11, 10, 68 | 97, 5, 42 | 331, 5, 48 | 829, 4, 98 | 1427, 4, 38 |
| 17, 5, 72 | 101, 4, 78 | 337, 4, 98 | 839, 4, 54 | 1429, 4, 30 |
| 17, 8, 38 | 107, 5, 64 | 337, 5, 42 | 857, 4, 68 | 1433, 4, 50 |
| 17, 10, 126 | 109, 4, 32 | 383, 4, 168 | 877, 4, 38 | 1439, 4, 38 |
| 23, 4, 42 | 109, 5, 30 | 389, 4, 38 | 911, 4, 54 | 1447, 4, 78 |
| 23, 8, 32 | 113, 4, 60 | 409, 4, 48 | 941, 4, 62 | 1483, 4, 32 |
| 29, 4, 30 | 127, 5, 30 | 419, 4, 54 | 947, 4, 68 | 1487, 4, 98 |
| 29, 5, 66 | 139, 5, 68 | 421, 4, 32 | 953, 4, 30 | 1489, 4, 128 |
| 31, 7, 72 | 167, 5, 42 | 431, 4, 50 | 983, 4, 32 | 1493, 4, 50 |
| 37, 5, 60 | 179, 5, 48 | 521, 4, 50 | 997, 4, 80 | 1499, 4, 32 |
| 37, 7, 36 | 181, 5, 50 | 557, 4, 42 | 1009, 4, 84 | 1523, 4, 32 |

**Table 1.**

We shall use the following explicit inequality on the distribution of primes.

**Lemma 4.** *Let $\pi(x)$ be the number of primes $p \le x$. We have*

$$\pi(x) < \frac{x}{\log x} + 1.2762\frac{x}{(\log x)^2} \quad \textit{if} \quad x \ge 1.$$

**Proof.** This statement is proved in [2]. ■

**Lemma 5.** *Assume that $\mathcal{H} \in \mathfrak{M}$ and $U$ is a cube-free, $U > 565\cdot 10^{10}$, $(U,2) = = 1$. If*

(*a*) $\pi \in \mathcal{H}$ *for all* $\pi \in \mathcal{P}$, $\pi < U$,

$$(b) \qquad p^2 \in \mathcal{H} \quad \textit{for all} \quad p \in \mathcal{P},\ p < U$$

*and*

$$(c) \qquad 2^\delta \in \mathcal{H} \quad (1 \le \delta \le 23),$$

*then we have*

$$\mathcal{B}_U := \{n \in \mathbb{N} \mid n < \sqrt[3]{U},\ U + n^3 \in \mathcal{H}\} \ne \emptyset. \tag{2.4}$$

**Proof.** Assume that the conditions of Lemma 5 are satisfied.

We define the function $\kappa \in \mathcal{M}$ as follows:

$$\kappa(p^\alpha) = \begin{cases} 1 & \text{if} \quad p = 2 \\ 1 & \text{if} \quad p \ne 2,\ \alpha \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

Let $h(n) = U + n^3$ and let $H = [U^{1/3}]$.

First we consider the congruence

$$h(x) = x^3 + U \equiv 0 \pmod{\pi^3},\ (2 < \pi \le H,\ \pi \in \mathcal{P}). \tag{2.5}$$

If $(\pi, U) = 1$, then the congruence (2.5) has at most $(3, \pi^2(\pi - 1)) \le 3$ solutions (modulo $\pi^3$). If $(\pi, U) > 1$, then for each solution $n \pmod{\pi^3}$ of the congruence (2.5), we have $\pi \mid n$ and $\pi^3 \mid U$. Since $U$ is cube-free, (2.5) has no solutions in the case $\pi \mid U$. Let

$$\mathcal{U} := \sum_{\substack{n \equiv 1 \pmod 2 \\ n \le H}} \kappa(h(n)).$$

We infer from Lemma 1, Lemma 4 and $U > 565 \cdot 10^{10}$ that

$$\begin{aligned} \mathcal{U} &\ge \frac{H}{2} - \sum_{3 < \pi \le \sqrt[3]{H/2}} 3\Big(\frac{H}{2\pi^3} + 1\Big) - 3 \sum_{\sqrt[3]{H/2} < \pi < \sqrt[3]{2U}} 1 \ge \\ &\ge \Big[\frac{1}{2} - \frac{3}{2}\big(\mathfrak{S}(3) - \frac{1}{8}\big)\Big] H - 3\pi(\sqrt[3]{2U}) + 3 \ge \\ &\ge 0.4253560493H - \frac{3\sqrt[3]{2U}}{\log \sqrt[3]{2U}} - \frac{3c\sqrt[3]{2U}}{(\log \sqrt[3]{2U})^2} + 3 \ge \\ &\ge \Big[0.4253560493 - \frac{3\sqrt[3]{2}}{\log \sqrt[3]{2U}} - \frac{3c\sqrt[3]{2}}{(\log \sqrt[3]{2U})^2}\Big] \sqrt[3]{U} + 2, \end{aligned}$$

where $c := 1.2762$.

Now we give the upper estimate for

$$E_\alpha = \sum_{\substack{n\le H,\ n\equiv 1 \pmod 2\\ h(n)\equiv 0 \pmod{2^\alpha}}} 1.$$

One easily check that if $(U,2)=1$, then the congruence

$$h(n) = U + n^3 \equiv 0 \pmod{2^\alpha} \quad (\alpha \ge 2)$$

has at most two solutions (modulo $2^\alpha$). Thus, we have

$$E_\alpha = \sum_{\substack{n\le H,\ n\equiv 1 \pmod 2\\ h(n)\equiv 0 \pmod{2^\alpha}}} 1 \le 2\Big(\frac{H}{2^\alpha}+1\Big) < \frac{\sqrt[3]{U}}{2^{\alpha-1}} + 2.$$

Consequently, $U > 565\cdot 10^{10}$ gives

$$\mathcal{U} - E_{24} = \sum_{\substack{n\equiv 1 \pmod 2\\ n\le H}} \kappa(h(n)) - \sum_{\substack{n\le H,\ n\equiv 1 \pmod 2\\ h(n)\equiv 0 \pmod{2^{24}}}} 1 \ge$$

$$\ge \Big[0.4253560493 - \frac{3\sqrt[3]{2}}{\log\sqrt[3]{2U}} - \frac{3c\sqrt[3]{2}}{(\log\sqrt[3]{2U})^2} - \frac{1}{2^{23}}\Big]\sqrt[3]{U} > 0.$$

We may now complete the proof of (2.4).

Since $\mathcal{U} - E_{24} > 0$, there is a $n\in\mathbb{N}, n^3 < U,\ 2\nmid n$ such that

$$U + n^3 = 2^\delta\eta,\ 1\le\delta\le 23, \kappa(\eta) = 1, \eta < U.$$

Since $U > 565\cdot 10^{10}$ and $\delta \le 23$, we have $1 < \eta < U$ and so by our assumptions, using the conditions (a)-(c) we get

$$U + n^3 = 2^\delta\eta \in \mathcal{H},$$

which proves Lemma 5. ■

**Remark.** By using the method of C. Hooley [4], [5] one can prove that

$$\mathcal{U} = U\prod_{p>2}\Big(1 - \frac{\rho_U(p^2)}{p^2}\Big) + O(U(\log U)^{-1/2}),$$

where $\rho_U(m)$ is the number of those residues for which $(2n+1)^3 + U \equiv 0 \pmod m$ and the constant implied by error term is absolute.

Consequently, one can prove the following assertion: *There exists a constant $c_0$ such that if $f\in\mathcal{M}$ satisfies the condition* (1.1), *furthermore $f(n) = n$ for $n\le c_0$, then $f(n) = n$ for all $n\in\mathbb{N}$.*

The constant $c_0$ is effective, and perhaps $f(n) = n$ $(n\le c_0)$ holds but too much numerical computation would be necessary.

**Lemma 6.** *Assume that $f\in\mathcal{M}$ satisfy* (1.1) *and* (1.2). *Then*

$$f(p)=p \quad \text{for all} \quad p\in\mathcal{P} \tag{2.6}$$

*and*

$$f(m^3)=m^3 \quad \text{for all} \quad m\in\mathbb{N}. \tag{2.7}$$

**Proof.** We apply Lemma 3 and Lemma 5 with

$$\mathcal{L}=\mathcal{H}:=\{n\in\mathbb{N}\ \mid\ f(n)=n\}.$$

Assume that $f(p)=p$ for all prime $p<R$, where $R\in\mathcal{P}$. We may assume that $R>565\cdot 10^{10}$ (see Lemma 2). From (1.2) we have

$$p,\ p^2\in\mathcal{L} \quad \text{for all} \quad p\in\mathcal{P}, p<R.$$

Let $Q:=\pi^{3\alpha}<R$, where $\pi\in\mathcal{P}$ and $\alpha\in\mathbb{N}$. Since $6\nmid Q$, by applying Lemma 3 with $T=R$, there is $p\in\mathcal{P},\ p<R$ such that $p+\pi^{3\alpha}\in\mathcal{L}$, which gives

$$p+\pi^{3\alpha}=f(p+\pi^{3\alpha})=f(p)+f(\pi^{3\alpha})=p+f(\pi^{3\alpha}).$$

Therefore

$$f(\pi^{3\alpha})=\pi^{3\alpha} \quad \text{for all} \quad \pi^{3\alpha}<R,$$

consequently

$$f(m^3)=m^3 \quad \text{for all} \quad m\in\mathbb{N},\ m^3<R. \tag{2.8}$$

Now we apply Lemma 5 with $U=R$. Then there is a $m\in\mathbb{N},\ m^3<R$ such that $R+m^3\in\mathcal{L}$, which with (1.1) and (2.6) gives

$$R+m^3=f(R+m^3)=f(R)+f(m^3)=f(R)+m^3.$$

Consequently $f(R)=R$, and so $f(p)=p$ is satisfied for all primes $p$. The assertion (2.6) is proved.

Finally, the assertion (2.7) follows directly from (2.6) and (3.1).

Lemma 6 is proved. ■

## 3. Proof of Theorem 1

From Lemma 6, we obtain that $f(p)=p$ and $g(m^3)=m^3$ for all $p\in\mathcal{P}$, $m\in\mathbb{N}$. Thus

$$f(p+m^3)=p+m^3 \quad \text{for all} \quad p\in\mathcal{P},\ m\in\mathbb{N}. \tag{3.1}$$

Let $E_k(x)$ be the number of those $n \le x$ which can not be written as a sum of a prime and a $k$-th power of an integer, and $n \ne m^k$. Here $k \ge 2$, $k \in \mathbb{N}$.

Devenport and Heilbronn proved in [1] that for each $k \ge 2$ there is a constant $c = c(k) > 0$ such that

$$E_k(x) = O\Big(\frac{x}{(\log x)^c}\Big).$$

For us it is enough to know that $E_3(x)/x \to 0 \quad (x \to \infty)$.

Let $\mathcal{B}$ be the set of those $n$ which can be written as $n = p + m^3 \quad (p \in \mathcal{P}$, $m \in \mathbb{N})$. It follows from (3.1) that $f(n) = n$ if $n \in \mathcal{B}$.

Let $\pi^\alpha$ be an arbitrary prime power. Consider the set of integers $\pi^\alpha \nu \le x$, $(\nu, \pi) = 1$. The size of this set is $\ge \frac{x}{\pi^\alpha}(1 - \frac{1}{\pi}) - 1$. The number of those $\nu$ for which $\nu \notin \mathcal{B}$, or $\pi^\alpha \nu \notin \mathcal{B}$ is $\le E(x) + E(\frac{x}{\pi^\alpha})$. Thus, if $x$ is large enough, then we can find such a $\nu \in \mathcal{B}$ for which $\pi^\alpha \nu \in \mathcal{B}$ and $(\pi, \nu) = 1$. Consequently

$$\pi^\alpha \nu = f(\pi^\alpha \nu) = f(\pi^\alpha) f(\nu) = f(\pi^\alpha)\nu,$$

which proves

$$f(\pi^\alpha) = \pi^\alpha.$$

Thus

$$f(n) = n \quad \text{for all} \quad n \in \mathbb{N}$$

holds, and so our theorem is proved. ■

## References


[1] **Davenport, H. and H. Heilbronn,** Note on a result in the additive theory of numbers, *Proc. London Math. Soc.*, (2) **43** (1937), 142–151.

[2] **Dusart, P.,** Estimates of some functions over primes without R.H., 2002, 20 pages. `http://arxiv.org/pdf/1002.0442v1.pdf`.

[3] **De Koninck, J.-M., I. Kátai and B. M. Phong,** A new characteristic of the identity function, *Journal of Number Theory*, **63** (1997), 325–338.

[4] **Hooley, C.,** On the square-free values of cubic polynomials, *J. Reine Angew. Math.*, **229** 1968, 147–154.

[5] **Hooley, C.,** *Applications of Sieve Methods to the Theory of Numbers*, Cambridge Tracts in Mathematics, No. 70., Cambridge University Press, Cambridge-New York-Melbourne, 1976.

[6] **Phong, B.M.,** A characterization of the identity function with functional equations, *Annales Univ. Sci. Budapest., Sect. Comp.*, **32** (2010), 247–252.

[7] **Phong, B.M.,** A characterization of the identity with functional equations II., *Acta Math. Hungar.*, **148(2)** (2016), 450–465.

[8] **Spiro, C.,** Additive uniqueness sets for arithmetic functions, *J. Number Theory*, **42** (1992), 232–246.

**Bui Minh Phong**
Department of Computer Algebra
Faculty of Informatics
Eötvös Loránd University
H-1117 Budapest, Pázmány Péter sétány 1/C
Hungary
bui@compalg.inf.elte.hu