ON CONJECTURE CONCERNING THE FUNCTIONAL EQUATION

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Dedicated to the memory of Professor Antal Iványi

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Abstract. We determine all solutions of those $f : \mathbb{N} \to \mathbb{C}$ for which $f(n^2 + Dm^2) = f^2(n) + Df^2(m)$ is satisfied for all positive integers n, m, where D is a given positive integer. This solves a problem of Kátai and Phong.

1. Introduction

Let, as usual, $\mathcal{P}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{C}$ be the set of primes, positive integers, integers and complex numbers, respectively.

In 1992, C. Spiro [15] proved that if a multiplicative function $f : \mathbb{N} \to \mathbb{C}$ satisfies the relations

$$f(p_0) \neq 0 \quad \text{for some} \quad p_0 \in \mathcal{P}$$

and

$$f(p+q) = f(p) + f(q) \quad \text{for every} \quad p, q \in \mathcal{P},$$

then $f(n) = n$ for all $n \in \mathbb{N}$.

In 1997 J.-M. De Koninck, I. Kátai and B. M. Phong [5] proved that if a multiplicative function $f : \mathbb{N} \to \mathbb{C}$ satisfies the relation

$$f(p+n^2) = f(p) + f(n^2) \quad \text{for every} \quad p \in \mathcal{P},\ n \in \mathbb{N},$$

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then f is the identity function. K.-H. Indlekofer and B. M. Phong [6] proved that if $k \in \mathbb{N}$, $f \in \mathcal{M}$ satisfy $f(2)f(5) \neq 0$ and $f\left(n^2+m^2+k+1\right) = f(n^2+1)+f(m^2+k)$ for all $n, m \in \mathbb{N}$, then $f(n) = n$ for all $n \in \mathbb{N}$, $(n, 2) = 1$.

For some generalizations of the above results, we refer the other works of P. V. Chung [2], B. M. Phong [10],[11], [12].

In 2014 B. Bojan determined all solutions of those $f : \mathbb{N} \to \mathbb{C}$ for which

$$f(n^2+m^2) = f^2(n) + f^2(m) \quad \text{for every} \quad n, m \in \mathbb{N}.$$

Our purpose in this paper is to prove a conjecture of Kátai and Phong [3].

Theorem. *Assume that the number $D \in \mathbb{N}$ and the arithmetical function $f : \mathbb{N} \to \mathbb{C}$ satisfy the equation*

$$f(n^2 + Dm^2) = f^2(n) + Df^2(m) \quad \textit{for every} \quad n, m \in \mathbb{N}. \tag{1.1}$$

Then one of the following assertions holds:

$$\begin{aligned} &\text{a)} \quad f(n) = 0 \quad \textit{for every} \quad n \in \mathbb{N}, \\ &\text{b)} \quad f(n) = \frac{\epsilon(n)}{D+1} \quad \textit{for every} \quad n \in \mathbb{N}, \\ &\text{c)} \quad f(n) = \epsilon(n)n \quad \textit{for every} \quad n \in \mathbb{N}, \end{aligned}$$

where $E := \{n^2 + Dm^2 \mid n, m \in \mathbb{N}\}$, $\epsilon(n) = 1$ if $n \in E$ and $\epsilon(n) \in \{-1, 1\}$ if $n \in \mathbb{N} \setminus E$.

It is proved earlier for the case $D = 2, 3$ in [4] and for $D = 4, 5$ in [14] (see also [16]).

Corollary 1. *Assume that the number $D \in \mathbb{N}$ and a multiplicative function $f : \mathbb{N} \to \mathbb{C}$ satisfy the equation* (1.1). *Then*

$$f(n) = \epsilon(n)n \quad \textit{for every} \quad n \in \mathbb{N},$$

where $E := \{n^2 + Dm^2 \mid n, m \in \mathbb{N}\}$, $\epsilon(n) = 1$ if $n \in E$ and $\epsilon(n) \in \{-1, 1\}$ if $n \in \mathbb{N} \setminus E$.

Corollary 2. *Assume that the number $k \in \mathbb{N}, k \geq 2$ and an arithmetic function $f : \mathbb{N} \to \mathbb{C}$ satisfy the relation*

$$f(n_1^2 + n_2^2 + \cdots + n_k^2) = f^2(n_1) + f^2(n_2) + \cdots + f^2(n_k)$$

for all $n_1, n_2, \ldots, n_k \in \mathbb{N}$. *Then one of the following assertions holds:*

$$\text{a)}\quad f(n) = 0 \quad \textit{for every} \quad n \in \mathbb{N},$$

$$\text{b)}\quad f(n) = \frac{\epsilon(n)}{k} \quad \textit{for every} \quad n \in \mathbb{N},$$

$$\text{c)}\quad f(n) = \epsilon(n)n \quad \textit{for every} \quad n \in \mathbb{N},$$

where $F := \{n_1^2 + n_2^2 + \cdots + n_k^2 \mid n_1, n_2, \cdots, n_k \in \mathbb{N}\}$, $\epsilon(n) = 1$ *if* $n \in F$ *and* $\epsilon(n) \in \{-1, 1\}$ *if* $n \in \mathbb{N} \setminus F$.

We note that Corollary 2 is proved by Park in [8] and [9] for multiplicative function and recently by Lee in [7] for general arithmetic functions.

2. Lemmas

In this section we assume that the function $F, G : \mathbb{N} \to \mathbb{C}$ and the numbers $D \in \mathbb{N}$, $U \in \mathbb{C}$, $U \neq 0$ satisfy the relation

$$F(n^2 + Dm^2) = G(n) + UG(m) \quad \text{for every} \quad n, m \in \mathbb{N}. \tag{2.1}$$

Lemma 1. *Assume that the function* $F, G : \mathbb{N} \to \mathbb{C}$ *and the numbers* $D \in \mathbb{N}$, $U \in \mathbb{C}$, $U \neq 0$ *satisfy* (2.1). *Then*

$$\begin{aligned} G(\ell + 12m) = {} & G(\ell + 9m) + G(\ell + 8m) + G(\ell + 7m) - \\ & - G(\ell + 5m) - G(\ell + 4m) - G(\ell + 3m) + G(\ell) \end{aligned} \tag{2.2}$$

holds for every $\ell, m \in \mathbb{N}$ *and*

$$\begin{cases} G(7) &= 2G(5) - G(1) \\ G(8) &= 2G(5) + G(4) - 2G(1) \\ G(9) &= G(6) + 2G(5) - G(2) - G(1) \\ G(10) &= G(6) + 3G(5) - G(3) - 2G(1) \\ G(11) &= G(6) + 4G(5) - G(3) - G(2) - 2G(1) \\ G(12) &= G(6) + 4G(5) + G(4) - G(2) - 4G(1) \end{cases} \tag{2.3}$$

Proof. We note from (2.1) that

$$F(x^2 + Dy^2) = G(|x|) + UG(|y|) \quad \text{for every} \quad x, y \in \mathbb{Z} \setminus \{0\}. \tag{2.4}$$

First we prove the following assertion:

$$G(n + 2m) - G(|n - 2m|) = G(2n + m) - G(|2n - m|) \tag{2.5}$$

for every $n, m \in \mathbb{N}, n \neq 2m, m/2$.

Assume that the numbers $n, m \in \mathbb{N}$ satisfy the conditions $n \neq 2m, n \neq m/2$. If $Dn - 2m \neq 0$, then we infer from (2.4) and the next relations

$$(Dn + 2m)^2 + D(n - 2m)^2 = (Dn - 2m)^2 + D(n + 2m)^2$$

and

$$(Dn + 2m)^2 + D(2n - m)^2 = (Dn - 2m)^2 + D(2n + m)^2$$

that

$$G(Dn + 2m) + UG\Big(|n - 2m|\Big) = G(|Dn - 2m|) + UG\Big(n + 2m\Big)$$

and

$$G(Dn + 2m) + UG\Big(|2n - m|\Big) = G(|Dn - 2m|) + UG\Big(2n + m\Big).$$

These prove (2.5) in the case $Dn - 2m \neq 0$.

If $Dn - 2m = 0$, then $2Dn - m \neq 0$. In this case, we infer from (2.4) and the next relations

$$(2Dn + m)^2 + D(n - 2m)^2 = (2Dn - m)^2 + D(n + 2m)^2$$

and

$$(2Dn + m)^2 + D(2n - m)^2 = (2Dn - m)^2 + D(2n + m)^2$$

that

$$G(2Dn + m) + UG\Big(|n - 2m|\Big) = G(|2Dn - m|) + UG\Big(n + 2m\Big)$$

and

$$G(2Dn + m) + UG\Big(|2n - m|\Big) = G(|2Dn - m|) + UG\Big(2n + m\Big).$$

These prove (2.5) in the case $Dn - 2m = 0$, and so (2.5) is proved.

Applications of (2.5) in the cases $(n, m) \in \{(1,3); (2,3); (1,4); (1,5); (3,4); (2,5)\}$ prove that (2.3) holds for $G(7), G(8), G(9), G(11), G(10)$ and $G(12)$. Thus, (2.3) is proved.

Now we prove (2.2).

By applying (2.5) with $n = \ell + 2m$, we have

$$G(2\ell + 5m) - G(2\ell + 3m) = G(\ell + 4m) - G(\ell) \quad \text{for every} \quad \ell, m \in \mathbb{N}.$$

This shows that

$$G(\ell + 12m) - G(\ell) = G(2\ell + 15m) - G(2\ell + 9m)$$

and

$$\begin{aligned}G(2\ell+15m)-G(2\ell+9m) &= \Big[G\Big(2(\ell+5m)+5m\Big)-G\Big(2(\ell+5m)+3m\Big)\Big]+\\ &+\Big[G\Big(2(\ell+4m)+5m\Big)-G\Big(2(\ell+4m)+3m\Big)\Big]+\\ &+\Big[G\Big(2(\ell+3m)+5m\Big)-G\Big(2(\ell+3m)+3m\Big)\Big]=\\ &=\Big[G\Big(\ell+9m\Big)-G\Big(\ell+5m\Big)\Big]+\\ &+\Big[G\Big(\ell+8m)\Big)-G\Big(\ell+4m\Big)\Big]+\\ &+\Big[G\Big(\ell+7m)\Big)-G\Big(\ell+3m\Big)\Big],\end{aligned}$$

which prove (2.2).

Lemma 1 is proved. ■

In the proof of the next lemma we shall follow a method in part similar to the one used in the proof of Lemma 2 of the paper [13].

Lemma 2. *Assume that the function* $F, G : \mathbb{N} \to \mathbb{C}$ *and the numbers* $D \in \mathbb{N}$, $U \in \mathbb{C}$, $U \neq 0$ *satisfy* (2.1). *Let*

$$\begin{aligned}A :=&\frac{1}{120}\Big(G(6)+4G(5)-G(3)-G(2)-3G(1)\Big),\\ \Gamma_2 :=&\frac{-1}{8}\Big(G(6)-4G(5)+4G(4)-G(3)+3G(2)-3G(1)\Big),\\ \Gamma_3 :=&\frac{-1}{3}\Big(G(6)-2G(5)+2G(3)-G(2)\Big),\\ \Gamma_4 :=&\frac{1}{4}\Big(G(6)-2G(4)-G(3)+G(2)+G(1)\Big),\\ \Gamma_5 :=&\frac{1}{5}\Big(G(6)-G(5)-G(3)-G(2)+2G(1)\Big),\\ \Gamma :=&\frac{1}{4}\Big(G(6)-4G(5)+2G(4)+3G(3)+G(2)+G(1)\Big),\\ B_k :=&\Gamma_2\chi_2(k)+\Gamma_3\chi_3(k)+\Gamma_4\chi_4(k-1)+\Gamma_5\chi_5(k)+\Gamma,\end{aligned}$$

where $\chi_2(k) \pmod 2$, $\chi_3(k) \pmod 3$ *are the principal Dirichlet characters and* $\chi_4(k) \pmod 4$, $\chi_5(k) \pmod 5$ *are the real, non-principal Dirichlet characters, i.e.*

$$\chi_2(0)=0, \chi_2(1)=1, \chi_3(0)=0, \chi_3(1)=\chi_3(2)=1,$$
$$\chi_4(0)=\chi_4(2)=0, \chi_4(1)=1, \chi_4(3)=-1,$$
$$\chi_5(2)=\chi_5(3)=-1, \chi_5(1)=\chi_5(4)=1.$$

Then we have

$$G(\ell)=A\ell^2+B_\ell \quad \textit{for every} \quad \ell\in\mathbb{N}. \tag{2.6}$$

Proof. By computation, we proved that (2.6) holds for $1 \le k \le 12$.

Assume that $G(k) = Ak^2 + B_k$ holds for $\ell \le k \le \ell + 11$, where $\ell \ge 1$. By applying (2.2) with $m = 1$, we have

$$\begin{aligned} G(\ell+12) = G(\ell+9) + G(\ell+8) + G(\ell+7) - \\ - G(\ell+5) - G(\ell+4) - G(\ell+3) + G(\ell) = \\ = A\Big[(\ell+9)^2 + (\ell+8)^2 + (\ell+7)^2 - \\ - (\ell+5)^2 - (\ell+4)^2 - (\ell+3)^2 + \ell^2\Big] + \\ + \Big[B_{\ell+9} + B_{\ell+8} + B_{\ell+7} - B_{\ell+5} - B_{\ell+4} - B_{\ell+3} + B_\ell\Big] = \\ = A(\ell+12)^2 + B_{\ell+12}, \end{aligned}$$

which proves that (2.6) holds for $\ell + 12$ and so it is true for all ℓ. In the last relation we have used

$$\begin{aligned} & B_{\ell+9} + B_{\ell+8} + B_{\ell+7} - B_{\ell+5} - B_{\ell+4} - B_{\ell+3} + B_\ell = \\ & = \Gamma_2\Big[\sum_{k=\ell+6}^{\ell+9} \chi_2(k) - \sum_{k=\ell+3}^{\ell+6} \chi_2(k) + \chi_2(\ell)\Big] + \\ & + \Gamma_3\Big[\sum_{k=\ell+7}^{\ell+9} \chi_3(k) - \sum_{k=\ell+3}^{\ell+5} \chi_3(k) + \chi_3(\ell)\Big] + \\ & + \Gamma_4\Big[\sum_{k=\ell+6}^{\ell+9} \chi_4(k-1) - \sum_{k=\ell+3}^{\ell+6} \chi_4(k-1) + \chi_4(\ell-1)\Big] + \\ & + \Gamma_5\Big[\sum_{k=\ell+6}^{\ell+10} \chi_5(k) - \sum_{k=\ell+2}^{\ell+6} \chi_5(k) - \chi_5(\ell+10) + \chi_5(\ell+2) + \chi_5(\ell)\Big] + \Gamma = \\ & = \Gamma_2\chi_2(\ell) + \Gamma_3\chi_3(\ell) + \Gamma_4\chi_4(\ell-1) + \Gamma_5\chi_5(\ell+2) + \Gamma = \\ & = \Gamma_2\chi_2(\ell+12) + \Gamma_3\chi_3(\ell+12) + \Gamma_4\chi_4(\ell+11) + \Gamma_5\chi_5(\ell+12) + \Gamma = B_{\ell+12}. \end{aligned}$$

Lemma 2 is proved. ■

3. Proof of the Theorem

Assume that the numbers $D \in \mathbb{N}$ and the arithmetical function $f : \mathbb{N} \to \mathbb{C}$ satisfy the equation (1.1), that is

$$f(n^2 + Dm^2) = f^2(n) + Df^2(m) \quad \text{for every} \quad n, m \in \mathbb{N}.$$

Let $G(n) := f^2(n)$ for every $n \in \mathbb{N}$ and $U = D$. We shall use the notations of Lemmas 1-2. From (2.6) we have

$$G(\ell) = f^2(\ell) = A\ell^2 + B_\ell \quad \text{for every} \quad \ell \in \mathbb{N},$$

where

$$B_\ell := \Gamma_2\chi_2(\ell) + \Gamma_3\chi_3(\ell) + \Gamma_4\chi_4(\ell-1) + \Gamma_5\chi_5(\ell) + \Gamma.$$

Consequently, we obtain from (1.1) that

$$G(n^2 + Dm^2) = f^2(n^2 + Dm^2) = \Big(G(n) + DG(m)\Big)^2,$$

and so (2.6) implies

$$A(n^2 + Dm^2)^2 + B_{n^2+Dm^2} = \Big(A(n^2 + Dm^2) + B_n + DB_m\Big)^2 \tag{3.1}$$

for every $n, m \in \mathbb{N}$. Since

$$|B_\ell| \le |\Gamma_2| + |\Gamma_3| + |\Gamma_4| + |\Gamma_5| + |\Gamma| \quad \text{for every} \quad \ell \in \mathbb{N}$$

and

$$n^2 + Dm^2 \to \infty \quad \text{as} \quad n, m \to \infty,$$

we infer from (3.1) that

$$\begin{aligned} A &= \lim_{n,m\to\infty} \left[A + \frac{B_{n^2+Dm^2}}{(n^2+Dm^2)^2}\right] = \\ &= \lim_{n,m\to\infty} \left(A + \frac{B_n + DB_m}{n^2 + Dm^2}\right)^2 = A^2. \end{aligned}$$

Therefore, we have $A \in \{0, 1\}$.

Case I. $A = 1$. From (3.1) we obtain that

$$\begin{aligned} &(n^2 + Dm^2)^2 + B_{n^2+Dm^2} - \Big((n^2 + Dm^2) + B_n + DB_m\Big)^2 = \\ &= (-2B_n - 2DB_m)n^2 + W(n,m) = 0, \end{aligned} \tag{3.2}$$

holds for every $n, m \in \mathbb{N}$, where

$$\begin{aligned} W(n,m) := B_{n^2+Dm^2} - B_n^2 - D^2B_m^2 - \\ - 2D^2m^2B_m - 2Dm^2B_n - 2DB_nB_m. \end{aligned} \tag{3.3}$$

Now let $m \in \mathbb{N}$ be fixed, $n \in \mathbb{N}$, $n \equiv a \pmod{60}$ with some $a \in \mathbb{N}$, $0 \le a < 60$. Then $B_n = B_a$ and

$$|W(n,m)| < \infty$$

and so we obtain from (3.2) that

$$B_a + DB_m = \lim_{\substack{n\to\infty \\ n\equiv a \pmod{60}}} \frac{-W(n,m)}{2n^2} = 0,$$

consequently

$$B_m = \frac{-B_a}{D} = c \quad \text{for every} \quad m \in \mathbb{N}, \tag{3.4}$$

where $c \in \mathbb{C}$ is some fixed constant. This shows that

$$c(D+1) = 0 \quad \text{and} \quad c = 0. \tag{3.5}$$

This proves

$$B_m = c = 0 \quad \text{and} \quad G(m) = m^2 \quad \text{for every} \quad m \in \mathbb{N}$$

and in the case $A = 1$, we proved that

$$f^2(m) = m^2 \quad \text{and} \quad f(n^2 + Dm^2) = f^2(n) + Df^2(m) = n^2 + Dm^2$$

for every $n, m \in \mathbb{N}$.

The part (c) of the theorem is proved.

Case II. $A = 0$. In this case, we have

$$T(n,m) := B_{n^2+Dm^2} - \Big(B_n + DB_m\Big)^2 = 0 \quad \text{for every} \quad n, m \in \mathbb{N}. \tag{3.6}$$

We prove now that

$$\Gamma_4 = 0. \tag{3.7}$$

Assume that $\Gamma_4 \neq 0$. By applying (3.6) with $T(2,2)$ and $T(30,30)$, we obtain that

$$B_{4+4D} = (\Gamma_3 + \Gamma_4 - \Gamma_5 + \Gamma + D(\Gamma_3 + \Gamma_4 - \Gamma_5 + \Gamma))^2$$

and

$$B_0 = (\Gamma_4 + \Gamma + D(\Gamma_4 + \Gamma))^2.$$

Consequently

$$T(8,8) = 4(D+1)^2\Gamma_4(-\Gamma_5 + \Gamma + \Gamma_3) = 0$$

and

$$T(60,60) = 4\Gamma_4\Gamma(D+1)^2 = 0,$$

which with $(D+1)^2\Gamma_4 \neq 0$ imply that

$$\Gamma = 0 \quad \text{and} \quad \Gamma_5 - \Gamma_3 = 0.$$

Finally, we obtain (3.6) that

$$\begin{aligned} 0 = T(2,8) = B_{2^2+D8^2} - \Big(B_2 + DB_8\Big)^2 = B_{4+4D} - \Big(B_2 + DB_8\Big)^2 = \\ = \Big(\Gamma_3 + \Gamma_4 - \Gamma_5 + \Gamma + D(\Gamma_3 + \Gamma_4 - \Gamma_5 + \Gamma)\Big)^2 - \\ - \Big(\Gamma_3 + \Gamma_4 - \Gamma_5 + \Gamma + D(\Gamma_3 - \Gamma_4 - \Gamma_5 + \Gamma)\Big)^2 \\ = \Big(\Gamma_4(D+1)\Big)^2 - \Big(\Gamma_4(D-1)\Big)^2 = 4D\Gamma_4^2. \end{aligned}$$

This is impossible, because $\Gamma_4 \neq 0$. Thus, (3.4) is proved.

In the next part, we assume that $\Gamma_4 = 0$, and so

(3.8) $$B_k := \Gamma_2\chi_2(k) + \Gamma_3\chi_3(k) + \Gamma_5\chi_5(k) + \Gamma \quad \text{for every} \quad k \in \mathbb{N},$$

furthermore

(3.9) $$B_k = B_{k+30} \quad \text{for every} \quad k \in \mathbb{N}.$$

Since

$$T(30,30) = \Gamma - (\Gamma + D\Gamma)^2 = 0,$$

consequently

(3.10) $$\Gamma \in \{0, \frac{1}{(D+1)^2}\}.$$

Lemma 3. *Assume that* (3.6) *and* (3.8) *hold. If* $\Gamma = 0$*, then* $B_n = 0$ *for every* $n \in \mathbb{N}$*.*

Proof. We deduce from $\Gamma = 0$, (3.6) and (3.8) that

$$B_{30} = \Gamma_2\chi_2(30) + \Gamma_3\chi_3(30) + \Gamma_5\chi_5(30) + \Gamma = 0$$

and

$$B_{n^2} - B_n^2 = B_{n^2+D.30^2} - \Big(B_n + DB_{30}\Big)^2 = T(n,30) = 0 \quad \text{for every} \quad n \in \mathbb{N}.$$

Since

$$B_{n^2} = \Gamma_2\chi_2(n^2) + \Gamma_3\chi_3(n^2) + \Gamma_5\chi_5(n^2) = \Gamma_2\chi_2^2(n) + \Gamma_3\chi_3^2(n) + \Gamma_5\chi_5^2(n),$$

we have

$$\Gamma_2\chi_2^2(n) + \Gamma_3\chi_3^2(n) + \Gamma_5\chi_5^2(n) = \Big(\Gamma_2\chi_2(n) + \Gamma_3\chi_3(n) + \Gamma_5\chi_5(n)\Big)^2$$

holds for every $n \in \mathbb{N}$. This with $n = 2, 3, 5, 6, 10, 15$ gives the following equations

$$\begin{cases} \Gamma_3 + \Gamma_5 = (\Gamma_3 - \Gamma_5)^2 \\ \Gamma_2 + \Gamma_5 = (\Gamma_2 - \Gamma_5)^2 \\ \Gamma_2 + \Gamma_3 = (\Gamma_2 + \Gamma_3)^2 \\ \Gamma_5^2 = \Gamma_5 \\ \Gamma_3^2 = \Gamma_3 \\ \Gamma_2^2 = \Gamma_2. \end{cases}$$

Solve this systems of equations, the solutions $(\Gamma_2, \Gamma_3, \Gamma_5)$ are

$$(\Gamma_2, \Gamma_3, \Gamma_5) \in \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)\}.$$

Now we prove that $(\Gamma_2, \Gamma_3, \Gamma_5) = (0, 0, 0)$. Assume that $(\Gamma_2, \Gamma_3, \Gamma_5) \neq (0, 0, 0)$. Then

$$B_k := \chi_i(k) \quad (i = 2, 3, 5),$$

and by applying (3.6) for the case $n = m = 1$, we have

$$B_{D+1} = (B_1 + DB_1)^2,$$

which implies

$$\chi_i(D + 1) = (D + 1)^2.$$

This is impossible, because $1 \geq |\chi_i(D + 1)| = (D + 1)^2 \geq 4$.

Lemma 3 is proved.

Thus we proved the part (a) of the theorem. ■

Lemma 4. *Assume that* (3.6) *and* (3.8) *hold. If*

$$\Gamma = \frac{1}{(D+1)^2},$$

then $B_n = \Gamma = \frac{1}{(D+1)^2}$ *for every* $n \in \mathbb{N}$.

Proof. We shall prove that $\Gamma_2 = \Gamma_3 = \Gamma_5 = 0$.

We infer from (3.6) that

$$T(6, 30) = B_{6^2+D.30^2} - \Big(B_6 + D.B_{30}\Big)^2 = B_6 - \Big(B_6 + \frac{D}{(D+1)^2}\Big)^2 = 0$$

and

$$T(12,30) = B_{12^2+D.30^2} - \Big(B_{12} + D.B_{30}\Big)^2 = B_{24} - \Big(B_{12} + \frac{D}{(D+1)^2}\Big)^2 = 0.$$

Since

$$B_6 = B_{24},$$

$$B_6 = \Gamma_5 + \frac{1}{(D+1)^2}, B_6 + \frac{D}{(D+1)^2} = \Gamma_5 + \frac{1}{D+1}$$

and

$$B_{12} = -\Gamma_5 + \frac{1}{(D+1)^2}, B_{12} + \frac{D}{(D+1)^2} = -\Gamma_5 + \frac{1}{D+1},$$

we obtain that

$$(D\Gamma_5 - D + \Gamma_5 + 1)\Gamma_5 = 0, \quad (D\Gamma_5 - D + \Gamma_5 - 3)\Gamma_5 = 0.$$

These relations show that $\Gamma_5 = 0$. Thus, we have

$$B_k = \Gamma_2\chi_2(k) + \Gamma_3\chi_3(k) + \frac{1}{(D+1)^2}$$

and

$$B_{k+6} = B_k$$

hold for $k \in \mathbb{N}$. By using (3.6) for $(n,m) = (1,6),(2,6),(3,6)$, we have

$$T(1,6) = B_{1^2+D.6^2} - \Big(B_1 + DB_6\Big)^2 = B_1 - \Big(B_1 + \frac{D}{(D+1)^2}\Big)^2 =$$
$$= -\frac{(\Gamma_2+\Gamma_3)(D\Gamma_2 + D\Gamma_3 - D + \Gamma_2 + \Gamma_3 + 1)}{D+1} = 0,$$

$$T(2,6) = B_{2^2+D.6^2} - \Big(B_2 + DB_6\Big)^2 = B_4 - \Big(B_2 + \frac{D}{(D+1)^2}\Big)^2 =$$
$$= -\frac{(D\Gamma_3 - D + \Gamma_3 + 1)\Gamma_3}{D+1} = 0$$

and

$$T(3,6) = B_{3^2+D.6^2} - \Big(B_3 + DB_6\Big)^2 = B_3 - \Big(B_1 + \frac{D}{(D+1)^2}\Big)^2 =$$
$$= -\frac{(D\Gamma_2 - D + \Gamma_2 + 1)\Gamma_2}{D+1} = 0.$$

Solving the above system of equations, we obtain

$$(\Gamma_2,\Gamma_3) \in \Big\{(0,0), \Big(\frac{D-1}{D+1}, 0\Big), \Big(0, \frac{D-1}{D+1}\Big)\Big\}.$$

Let $H := \frac{D-1}{D+1}$. If $(\Gamma_2, \Gamma_3) \neq (0,0)$, then $B_k = H\chi_i(k) + \frac{1}{(D+1)^2} \quad (i = 2,3)$. Therefore

$$\begin{aligned} T(1,1) &= B_{1^2+D1^2} - \Big(B_1 + DB_1\Big)^2 = \\ &= H\chi_i(D+1) + \frac{1}{(D+1)^2} - (D+1)^2(H + \frac{1}{(D+1)^2})^2 = \\ &= H\chi_i(D+1) + \frac{1}{(D+1)^2} - \frac{D^4}{(D+1)^2} = \\ &= H\Big(\chi_i(D+1) - (D^2+1)\Big) = 0. \end{aligned}$$

This is impossible, because $H \neq 0$ and $|\chi_i(D+1)| \leq 1 < D^2 + 1$.

Lemma 4 is proved. ■

Thus we proved the part (b) of the theorem, and so the proof of the theorem is completes.

4. Proof of Corollaries

Corollary 1 follows from the theorem, because if f is multiplicative, then $f(n) \neq 0$ for some $n \in \mathbb{N}$ and $f^2(m) \neq \frac{1}{(D+1)^2}$ for some $m \in \mathbb{N}$.

Corollary 2 is a consequence of the theorem by applying $x_2 = \cdots = x_k$ and $D = k - 1$.

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