# ON THE EQUATION $f(n^2) = g^2(n)$ FOR $q$-ADDITIVE FUNCTIONS $f$ AND $g$

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*Dedicated to the memory of Professor Antal Iványi*

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**Abstract.** We prove that if the $q$-additive functions $f$ and $g$ satisfy the equation $f(n^2) = g^2(n)$ for every $n \in \mathbb{N}$ and $\{a \in \{0, 1, \ldots, q-1\}, k \in \mathbb{N} \mid g(aq^k) \neq 0\}$ is an infinite set, then there is a non-zero complex number $c$ such that $g(n) = cn$ and $f(n^2) = c^2n^2$ for every $n \in \mathbb{N}$.

## 1. Introduction

Let, as usual, $\mathbb{N}$, $\mathbb{C}$ be the set of positive integers and complex numbers, respectively. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ be the set of non-negative integers. Let $\mathcal{M}^*$ be the class of completely multiplicative functions.

For some integer $q \geq 2$ let $\mathcal{A}_q$ be the set of $q$-additive functions. Let

$$\mathbb{A}_q := \{0, 1, \cdots, q-1\}.$$

Every $n \in \mathbb{N}_0$ can be uniquely represented in the form

$$n = \sum_{r=0}^{\infty} a_r(n) q^r \quad \text{with} \quad a_r(n) \in \mathbb{A}_q$$

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and $a_r(n) = 0$ if $q^r > n$. We say that $f \in \mathcal{A}_q$, if $f : \mathbb{N}_0 \to \mathbb{C}$,

$$f(0) = 0 \quad \text{and} \quad f(n) = \sum_{r=0}^{\infty} f\Big(a_r(n)q^r\Big) \quad \text{holds for every} \quad n \in \mathbb{N}_0.$$

Recently we proved in [2] the following assertion:

**Theorem A.** *Assume that $f \in \mathcal{M}^*$, $g \in \mathcal{A}_q$ satisfy the condition*

$$f(n) = g^2(n) \quad \textit{for every} \quad n \in \mathbb{N}.$$

*Then either*

$$f(q_0) = 0,\ q_0|q, \quad \textit{and} \quad f(n) = \chi_{q_0}(n), g^2(n) = \chi_{q_0}(n),$$

*where $\chi_{q_0}$ is a Dirichlet character* $\pmod{q_0}$, *or $f(n) = n^2$, and either $g(n) = n$, or $g(n) = -n$ for all $n \in \mathbb{N}$.*

Let

$$K(g) := \{(a, \ell) \in (\mathbb{A}_q, \mathbb{N}) \ \mid g(aq^\ell) \neq 0\}.$$

In this paper, we prove the following

**Theorem 1.** *Assume that $f, g \in \mathcal{A}_q$ satisfy*

$$f(n^2) = g^2(n) \quad \textit{for every} \quad n \in \mathbb{N}. \tag{1.1}$$

*If*

$$|K(g)| = \infty, \tag{1.2}$$

*then there is a non-zero complex number $c$ such that $g(n) = cn$ and $f(n^2) = c^2n^2$ for every $n \in \mathbb{N}$.*

*If*

$$|K(g)| < \infty,$$

*then*

$$|K(f)| < \infty.$$

We are unable to give all solutions of (1.1) if $K(g)$ is finite. We prove

**Theorem 2.** *Assume that $f \in \mathcal{A}_q$ satisfies*

$$f(n^2) = f^2(n) \quad \textit{for every} \quad n \in \mathbb{N}.$$

*If $K(f) = \{(a, \ell) \in (\mathbb{A}_q, \mathbb{N}) \mid f(aq^\ell) \neq 0\}$ is a finite set, then there are integers $0 = m_0 < \cdots < m_k$ such that*

$$\mathbb{A}_q = S(m_0) \cup \cdots \cup S(m_k)$$

*and either $f(m_\nu) = 0$ or $f(m_\nu)$ is a root of unity, for which*

$$f(n) = \varepsilon \cdot \Big(f(m_\nu)\Big)^{2^e} \quad \textit{if} \quad n \in S(m_\nu) \quad (\nu \in \{0, \cdots, k\}),$$

*where $e \in \mathbb{N}_0$ and $\varepsilon$ is a root of unity.*

## 2. Proof of Theorem 1

We shall use the following two lemmas.

**Lemma 1.** *If $h \in \mathcal{A}_q \cap \mathcal{M}^*$, $h(1) = 1$ and $h(q) \neq 0$, then $h(n) = n$ for every $n \in \mathbb{N}$.*

**Proof.** This lemma is a consequence of Theorem 2 in [1]. ■

**Lemma 2.** *Assume that $f, g \in \mathcal{A}_q$ satisfy* (1.1)*. Then $g(1) \neq 0$ and the function $G(n) := \frac{g(n)}{g(1)}$ is an element of $\mathcal{M}^*$.*

**Proof.** Since $K(g)$ is an infinite set and $g \in \mathcal{A}_q$, then there exists an infinite sequence $k_1 < k_2 < \cdots$ of positive integers and $A \in \mathbb{A}_q$ such that

$$g(Aq^{k_i}) \neq 0 \quad \text{for every} \quad i \in \mathbb{N}.$$

Let $n, m \in \mathbb{N}$. Then the above relation shows that there exists $K \in$ $\in \{k_1, k_2, \cdots\}$ such that

$$q^K > \max(2nm, (nm)^2, (Aq^{k_1})^2) \quad \text{and} \quad g(Aq^K) \neq 0. \tag{2.1}$$

Since $f, g \in \mathcal{A}_q$, we can assume that $f, g \in \mathcal{A}_{q^K}$.

Then, we infer from (2.1) that

$$\begin{aligned} f\Big((Aq^K n + m)^2\Big) &= f\Big(A^2 q^{2K} n^2 + 2Aq^K nm + m^2\Big) = \\ &= f\Big(A^2 q^{2K} n^2\Big) + f\Big(2Aq^K nm\Big) + f(m^2) \end{aligned}$$

and

$$\Big(g(Aq^K n+m)\Big)^2 = \Big(g(Aq^K n)+g(m)\Big)^2 = \\ = g^2(Aq^K n) + 2g(Aq^K n)g(m) + g^2(m).$$

From (1.1), we have

$$f\Big(A^2 q^{2K} n^2\Big) = g^2\big(Aq^K n\big), \quad f(m^2) = g^2(m),$$

consequently

$$f\Big(2Aq^K nm\Big) = 2g(Aq^K n)g(m). \tag{2.2}$$

By taking $n=1$ into (2.2), we have

$$f\Big(2Aq^K m\Big) = 2g(Aq^K)g(m),$$

which gives

$$f\Big(2Aq^K nm\Big) = 2g(Aq^K)g(nm). \tag{2.3}$$

It is clear from (2.2) and (2.3) that

$$g(Aq^K n)g(m) = g(Aq^K)g(nm), \tag{2.4}$$

which with $m=1$ implies

$$g(Aq^K n)g(1) = g(Aq^K)g(n).$$

This relation with $n = Aq^{k_1}$ shows that $g(1) \neq 0$. Therefore

$$g(Aq^K n) = \frac{g(Aq^K)}{g(1)} g(n).$$

Finally, we obtain from (2.4) and the fact $g(Aq^K) \neq 0$ that

$$g(Aq^K)g(nm) = g(Aq^K n)g(m) = \frac{g(Aq^K)}{g(1)} g(n)g(m), \tag{2.5}$$

which implies

$$\frac{g(nm)}{g(1)} = \frac{g(n)}{g(1)} \frac{g(m)}{g(1)}$$

and so

$$G(nm) = G(n)G(m) \quad \text{for every} \quad n, m \in \mathbb{N}.$$

Lemma 2 is thus proved. ■

**Proof of Theorem 1.** Assume that $f, g \in \mathcal{A}_q$ satisfy (1.1) and (1.2). Since $G(n) = \frac{g(n)}{g(1)}$, we have $G \in \mathcal{A}_q$, consequently

$$G \in \mathcal{A}_q \cap \mathcal{M}^*. \tag{2.6}$$

We shall prove that $g(q) \neq 0$. Assume that $g(q) = 0$. Then we obtain from (2.6) that

$$\begin{aligned} g(mq^e) &= g(1)G(mq^e) = g(1)G(m)G(q)^e = \\ &= g(1)\frac{g(m)}{g(1)}\Big(\frac{g(q)}{g(1)}\Big)^e = 0 \quad \text{for every} \quad m, e \in \mathbb{N}, \end{aligned}$$

which contradicts the assumption (1.2).

Assume now that $g(q) \neq 0$. Then $G(1) = 1$, $G(q) \neq 0$ and we infer from Lemma 1 that

$$G(n) = \frac{g(n)}{g(1)} = n, g(n) = g(1)n, \quad \text{and} \quad f(n^2) = g(n)^2 = g(1)^2 n^2,$$

consequently Theorem 1 is proved for $c = g(1) \neq 0$.

Now we prove the second assertion of Theorem 1. Assume that $|K(g)| < \infty$. Then there is a number $K \in \mathbb{N}, K \geq 3$ such that $g(mq^k) = 0$ for every $m \in \mathbb{N}$ and $k \geq K$. Then $g(n) = g(\nu)$ if $n \equiv \nu \pmod{q^K}$. Let $\nu, s \in \mathbb{A}_q$, $n = \nu + sq^k$. Then $n^2 = \nu^2 + 2\nu sq^k + s^2q^{2k}$ and in the case $k \geq K$, we have

$$2\nu s < 2q^2 \leq q^3 \leq q^K < q^k \quad \text{and} \quad \nu^2 < q^K \leq q^k,$$

consequently

$$g^2(n) = \Big(g(\nu + sq^k)\Big)^2 = \Big(g(\nu) + g(sq^k)\Big)^2 = g^2(\nu) + 2g(\nu)g(sq^k) + g^2(sq^k)$$

and

$$\begin{aligned} g^2(n) = f\Big((\nu + sq^k)^2\Big) &= f(\nu^2) + f(2\nu sq^k) + f(s^2q^{2k}) = \\ &= g^2(\nu) + f(2\nu sq^k) + g^2(sq^k). \end{aligned}$$

Thus

$$f(2\nu sq^k) = 0 \quad \text{if} \quad k \geq K \quad \text{and} \quad \nu, s \in \mathbb{A}_q. \tag{2.7}$$

Assume first that $q$ is even, $q = 2Q$. Let $s = Q$. Then $2\nu sq^k = \nu q^{k+1}$, and so $f(\nu q^{k+1}) = 0$ for every $\nu \in \mathbb{A}_q$ if $k \geq K$. Therefore, we have $|K(f)| < \infty$.

Assume now that $q$ is odd. If $\nu \in \mathbb{A}_q$, $\nu$ is even, then $\nu/2 \in \mathbb{A}_q$ and we infer from (2.7) with $s = 1$ that $f(\nu q^k) = 0$ if $k \geq K$.

We note from (2.7) that

$$f(q^k)+f(q^{k+1})=f\Big((q+1)q^k\Big)=0 \quad \text{if} \quad k\ge K$$

and

$$f(q^{2t})=g^2(q^t)=0 \quad \text{for every} \quad t\ge K,$$

consequently

$$f(q^\ell)=0 \quad \text{if} \quad \ell\ge 2K. \tag{2.8}$$

Let now $\nu\in\mathbb{A}_q$ and $\nu$ is odd. Then $\frac{q+\nu}{2}\in\mathbb{A}_q$ and we obtain from (2.7) that

$$f\Big((q+\nu)q^k\Big)=f\Big(2\frac{q+\nu}{2}q^k\Big)=0$$

and so we obtain from (2.8) that

$$0=f\Big((q+\nu)q^k\Big)=f\Big(q^{k+1}\Big)+f\Big(\nu q^k\Big)=f\Big(\nu q^k\Big) \quad \text{if} \quad k\ge 2K.$$

Consequently $|K(f)|<\infty$ in the case $q$ is odd.

Theorem 1 is thus proved. ■

## 3. Proof of Theorem 2

Assume that $f\in\mathcal{A}_q$ satisfies $|K(f)|<\infty$ and

$$f(n^2)=f^2(n) \quad \text{for every} \quad n\in\mathbb{N}. \tag{3.1}$$

If

$$f(qm)\ne 0 \quad \text{for some} \quad m\in\mathbb{N},$$

then we infer from (3.1) that

$$f\Big((qm)^{2^\alpha}\Big)=\Big(f(qm)\Big)^{2^\alpha}\ne 0 \quad \text{for every} \quad \alpha\in\mathbb{N},$$

which is impossible. Thus we proved that

$$f(qm)=0 \quad \text{for every} \quad m\in\mathbb{N}, \tag{3.2}$$

and so $f\in\mathcal{A}_q$ implies that

$$f(qm+a)=f(a) \quad \text{for every} \quad a\in\mathbb{A}_q,\ m\in\mathbb{N}.$$

It is clear that $f$ is a solution of (3.1) under the condition (3.2) if and only if

$$f^2(\ell) = f\big(\ell^2 \pmod q\big) \quad \text{for every} \quad \ell \in \mathbb{A}_q. \tag{3.3}$$

Let us define the directed graph $\ell \to \ell^2 \pmod q$ over $\mathbb{A}_q$. We shall classify the elements of $\mathbb{A}_q$, saying that $a \sim b$ if there is a path from $a$ to $b$, or from $b$ to $a$. Let $U_0, U_1, \ldots, U_k$ be the classes we obtain. Let

$$m_i := min\{t \in U_i\} \quad \text{and} \quad S(m_i) := U_i.$$

These are the connected components of this graph. Each $S(m_i)$ contains a directed circle (loop is allowed):

$$h_0 \to h_1 \to \cdots \to h_{t-1} (\to h_0).$$

Then

$$h_1 \equiv h_0^2 \pmod q,\ h_2 \equiv h_0^{2^2} \pmod q,\ \ldots,\ h_{t-1} \equiv h_0^{2^{t-1}},\ h_0 \equiv h_0^{2^t} \pmod q,$$

and so, if $f$ is a solution, then

$$f(h_j) = f^{2^j}(h_0) \quad \text{and} \quad f(h_0) = f^{2^t}(h_0),$$

consequently $f(h_0)$ is a root of unity of rank $2^t - 1$, or $f(h_0) = 0$.

The values $f(h_j)$ are determined by $f(h_0)$. Let $m \in S(m_\ell)$ which is not on the circle. Let

$$m \to t_1 \to \cdots \to t_{s-1} \to h_\ell$$

be the path from $m$ to the circle. Then $f(h_\ell) = f(m)^{2^s}$, and $f(m) = f(h_\ell)^{2^{-s}}$.

If we do this for every element of $S(m_j)$ and for every $j$, then we choose a solution of (3.1) satisfying $f(mq) = 0$ for every $m \in \mathbb{N}$.

**Examples. 1. $q = 24$.**

$$S(0) = \{0, 6, 12, 18\}, \quad S(1) = \{1, 5, 7, 11, 13, 17, 19, 23\},$$

$$S(2) = \{2, 4, 8, 10, 14, 16, 20, 22\} \quad \text{and} \quad S(3) = \{3, 9, 15, 21\}.$$

It is clear that $f(1) \in \{0, 1\}$ and

$$f(n) = \begin{cases} 0 & \text{if } n \in S(0), \\ \pm f(1) & \text{if } n \in S(1). \end{cases}$$

We have $f(2)^4 = f(2)^8$, consequently $f(2) \in \{0, \pm 1, \pm i\}$. It is easy to check that $f(4) = f(2)^2$, $f(8) = \pm f(2)^2$, $f(10) = \pm f(2)$, $f(14) = \pm f(2)$, $f(16) = = f(2)^4$, $f(20) = \pm f(2)^2$, $f(22) = \pm f(2)$.

In $S(3)$ it is obvious that $S(3) \in \{0, \pm 1\}$, furthermore $f(9) = f(3)^2$, $f(15) = \pm f(3)$, $f(21) = \pm f(3)$.

**2. $q = 40$.**

$$S(0) = \{0, 10, 20, 30\},$$
$$S(1) = \{1, 3, 7, 9, 11, 13, 17, 19, 21, 23, 27, 29, 31, 33, 37, 39\},$$
$$S(2) = \{2, 4, 6, 8, 12, 14, 16, 18, 22, 24, 26, 28, 32, 34, 36, 38\}$$

and

$$S(5) = \{5, 15, 25, 35\}.$$

It is easy to check that $f(1) \in \{0, 1\}$ and

$$f(n) = \begin{cases} 0 & \text{if } n \in S(0), \\ \pm f(1) & \text{if } n \in \{1, 9, 11, 19, 21, 29, 31, 39\}, \\ \pm f(1), \pm i f(1) & \text{if } n \in \{3, 7, 13, 17, 23, 27, 33, 37\}. \end{cases}$$

In $S(2)$, we have $f(2) \in \{0, \pm 1, \pm i\}$, furthermore $f(4) = f(2)^2$, $f(16) = f^4(2)$ and

$$f(n) = \begin{cases} \pm f(2) & \text{if } n \in \{18, 22, 38\}, \\ \pm f^2(2) & \text{if } n \in \{24, 36\}, \\ \pm f(2), \pm i f(2) & \text{if } n \in \{6, 8, 12, 14, 18, 26, 28, 32, 34\}. \end{cases}$$

Finally, we have $f(5) \in \{0, \pm 1\}$,

$$f(15) = \pm f(5), \ f(25) = f(5)^2 \quad \text{and} \quad f(35) = \pm f(5).$$

## References


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