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A GENERALIZATION OF GOLDBACH'S CONJECTURE

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Dedicated to the memory of Professor Antal Iványi

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Abstract. Goldbach's conjecture states that every even number greater than 2 can be expressed as the sum of two primes. The aim of this paper is to propose a generalization – or a set of increasingly generalized forms – of Goldbach's conjecture and to present relevant computational results. The proposed statements also generalize Lemoine's conjecture, according to which every sufficiently large odd integer is the sum of a prime and the double of another (or the same) prime. We present computational results verifying several cases of the statement until certain values and information regarding the resulting decompositions of even and odd integers.

1. Introduction

One of the best-known unsolved problems in number theory is Goldbach's conjecture, which appeared in a correspondence between Christian Goldbach and Leonhard Euler in 1742 [7]. In modern form it reads as follows:

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Conjecture 1.1 (Goldbach's Conjecture). *Every even number larger then* 2 *can be expressed as the sum of two primes.*

The above conjecture is also often referred to as the strong or even Goldbach conjecture. Parallel to this there also appeared the so called week or odd Goldbach conjecture, also known as ternary Goldbach conjecture, which states that every odd integer greater than 5 can be expressed as the sum of three primes. It is easy to see that this is implied by the strong Goldbach conjecture. In [12] Vinogradov showed that the ternary Goldbach conjecture is true for all n above a large constant C. In 2013 H. Helfgott claimed to have proved the ternary Goldbach conjecture [2], [3], [4] and [5].

As the proof of even Goldbach conjecture is still out of reach, there have been several attempts to verify the statement by computation until increasing limits. For example, Nils Pipping in 1938 laboriously verified the conjecture up to $n \leq 10^5$ [9]. With the evolution of computers, the statement has been verified until increasing limits: [1], [11], [10]. The most recent results of this kind were published by T. Oliveira [8]. With his research team they verified the even Goldbach conjecture until $4 \cdot 10^{18}$.

A well-known, stronger variation of the odd Goldbach conjecture is Lemoine's conjecture, published in 1894, which states that every odd number greater than five can be expressed in the form $2p + q$ where p and q are prime [6].

In this paper we would like to introduce some new generalizations of Goldbach's conjecture, which have not been published before, and to present some computational results verifying these generalizations until a certain limit ($\sim 10^9$). The aim of these computations in this current paper is to provide support that these generalizations are worth considering. Our longer term goals include extending the upper limits of our calculations by developing efficient sieving algorithms.

2. Some generalizations of Goldbach's conjecture

In this section increasingly general versions of Goldbach's conjecture are presented. The reasons for including the less general forms as well are twofold. Firstly, we would like to demonstrate the process during which the most general version gradually emerged. Secondly, from a computational point of view, the different versions of the generalization require somewhat different approaches in the design of a verifying algorithm, hence these can be worth tackling separately. The generalizations are divided into two subsections.

2.1. Initial steps of generalization

The versions in this subsection were inspired by Lemoine's conjecture. Goldbach's conjecture can be reformulated as follows: for every even number $n \geq 4$ there is a prime p such that the only number $x > 1$ such that $n \equiv p \pmod{x}$ is some prime q. In Lemoine's conjecture, because n is odd, for any prime $p > 2$, $n \equiv p \pmod{2}$ holds, hence among the possible values of x for which $n \equiv p \pmod{x}$ 2 will always be present and so, in general, $n \equiv p \pmod{x}$ holds at least for $x = 2$, for a prime $x = q$ and thus for $x = 2q$. In Lemoine's conjecture 2 had to be a factor of $n - p$, because n is odd. However, in general, one can ask whether we can use primes (or numbers) other than 2 in a role somewhat similar to that of 2 in Lemoine's conjecture.

Conjecture 2.1.

1. *Every even integer $n \geq 14$ which is not divisible by 3 can be expressed in the form $n = p + 3q$ for some primes p and q.*
2. *Every odd integer $n \geq 17$ which is not divisible by 3 can be expressed in the form $n = p + 6q$ for some primes p and q.*

Conjecture 2.2. *Let r be an arbitrary odd prime or 1. Then:*

1. *Every sufficiently large even integer n which is relative prime to r can be expressed in the form $n = p + rq$ for some primes p and q.*
2. *Every sufficiently large odd integer n which is relative prime to r can be expressed in the form $n = p + 2rq$ for some primes p and q.*

In the following two versions we use products of pairwise different primes and powers of primes in the statements, respectively, instead of a single prime r.

Conjecture 2.3. *Let $r_1, r_2, \ldots, r_k$ be arbitrary, pairwise different odd primes for some $k \geq 1$ or let $k = r_1 = 1$. Then:*

1. *Every sufficiently large even integer n which is relative prime to the number(s) $r_1, \ldots, r_k$ can be expressed in the form $n = p + r_1 r_2 \ldots r_k q$ for some primes p and q;*
2. *Every sufficiently large odd integer n which is relative prime to the number(s) $r_1, \ldots, r_k$ can be expressed in the form $n = p + 2r_1 r_2 \ldots r_k q$ for some primes p and q.*

Conjecture 2.4. *Let r be arbitrary odd prime or 1. Then for any natural $k \geq 1$:*

1. *Every sufficiently large even integer n which is relative prime to r can be expressed in the form $n = p + r^k q$ for some primes p and q.*

2. *Every sufficiently large odd integer n which is relative prime to r can be expressed in the form $n = p + 2r^k q$ for some primes p and q.*

Below we replace r in Conjecture 2.2 by an arbitrary positive integer m.

Conjecture 2.5. *Let m be an arbitrary positive integer.*

1. *If m is odd then every sufficiently large even integer n which is relative prime to m can be expressed in the form $n = p + mq$ for some primes p and q;*

2. *If m is even then every sufficiently large odd integer n which is relative prime to m can be expressed in the form $n = p + mq$ for some primes p and q.*

2.2. Final steps of generalization

In the previous conjectures the coefficient of p was always 1. Now we fix arbitrary positive integer coefficients m_1 and m_2 for the primes p and q. In this case clearly we need to require that n satisfies certain further conditions.

Conjecture 2.6. *Let m_1 and m_2 be arbitrary positive integers. Let $d = 2^k e$ be the highest common factor of m_1 and m_2, where e is the highest common odd factor of m_1 and m_2. Then every sufficiently large integer n such that*

1. $\gcd(n, m_1) = \gcd(n, m_2) = d$ *and*

2. $n \equiv m_1 + m_2 \pmod{2^{k+1}}$

can be expressed in the form $n = m_1 p + m_2 q$ for some primes p and q.

Note that Goldbach's conjecture is a special case of Conjecture 2.6 with $m_1 = m_2 = 1$, while Lemoine's conjecture corresponds to the case $m_1 = 1$ and $m_2 = 2$.

Conjecture 2.6 can be reformulated equivalently for the case when m_1 and m_2 are relative primes. This reformulation can be useful for example for the purposes of computational verification:

Proposition 2.7. *Conjecture* 2.6 *is equivalent to the following statement: Let m_1 and m_2 be arbitrary positive relative primes. Then every sufficiently large integer n such that*

1. $\gcd(n, m_1) = \gcd(n, m_2) = 1$ *and*

2. $n \equiv m_1 + m_2 \pmod 2$

can be expressed in the form $n = m_1 p + m_2 q$ *for some primes* p *and* q.

Proof. Conjecture 2.6 clearly implies the statement in Proposition 2.7. For the other direction let m_1, m_2 and n be integers satisfying the conditions of Conjecture 2.6. Then $m_1' = m_1/d$, $m_2' = m_2/d$ and $n' = n/d$ are pairwise relative primes. Also $2^k e n' = n \equiv m_1 + m_2 = 2^k e m_1' + 2^k e m_2' \pmod{2^{k+1}}$ implies $n' \equiv m_1' + m_2' \pmod 2$. Hence $n' = m_1' p + m_2' q$ for some primes p and q and so $n = m_1 p + m_2 q$. ■

In Propositions 2.8 and 2.9 we would like to demonstrate why the criteria in Conjecture 2.6 on n are necessary and might be sufficient.

It is well-known that for any integers n, m_1 and m_2, n can be expressed in the form $n = m_1 a + m_2 b$ for some integers a and b if and only if $\gcd(m_1, m_2)$ is a divisor of n. Howerer, as in our case a and b are primes, not arbitrary numbers, further conditions on n are necessary. Below we consider the case of odd numbers a and b.

Proposition 2.8. *Let* n, m_1 *and* m_2 *be positive integers and* $d = 2^k e$ *be the highest common factor of* m_1 *and* m_2, *where* e *is the highest common odd factor of* m_1 *and* m_2. *Then there exist integers* a *and* b *such that*

1. $n = m_1 a + m_2 b$ *and*

2. a *and* b *are odd*

if and only if

1. $d | n$ *and*

2. $n \equiv m_1 + m_2$ *(mod* 2^{k+1}*).*

Proof. Suppose $n = m_1 a + m_2 b$ for some odd numbers a and b. Then Condition 1 clearly holds. For the integers $m_1' = m_1/2^k$, $m_2' = m_2/2^k$ and $n' = n/2^k$, $n' = m_1' a + m_2' b$. Hence $n' \equiv m_1' a + m_2' b \equiv m_1' + m_2' \pmod 2$ and so $n = 2^k n' \equiv 2^k m_1' + 2^k m_2' = m_1 + m_2 \pmod{2^{k+1}}$.

For the converse, let n be an integer satisfying Conditions 1 and 2 and let $m_1' = m_1/d$, $m_2' = m_2/d$ and $n' = n/d$. Then by Condition 1, m_1', m_2', n' are all integers and by Condition 2, $n' \equiv m_1' + m_2' \pmod 2$. As $\gcd(m_1', m_2') = 1$, there exists $1 \le a \le m_2'$ such that $a m_1' \equiv n' \pmod{m_2'}$.

Case 1: m_1' and m_2' are odd and n' is even. If a is odd then $n - a m_1'$ is odd and divisible by m_2', hence $n - a m_1' = b m_2'$ for some odd b. Hence $n' = a m_1' + b m_2'$ and so $n = a m_1 + b m_2$, where a and b are odd. If a is even

then $a + m_2'$ is odd and $(a + m_2')m_1' \equiv n' \pmod{m_2'}$. Hence $n' - (a + m_2')m_1'$ is odd and divisible by m_2' and so $n' - (a + m_2')m_1' = bm_2'$, where b is odd. Therefore $n' = (a + m_2')m_1' + bm_2'$ and $n = (a + m_2')m_1 + bm_2$ where $(a + m_2')$ and b are odd.

Case 2: m_1' is even and m_2' and n' are odd. Suppose a is odd. As am_1' is even, $n' - am_1'$ is odd and divisible by m_2' and so $n' - am_1' = bm_2'$, for some odd b. Hence $n' = am_1' + bm_2'$ and $n = am_1 + bm_2$, where a and b are odd. Suppose now that a is even. Then $a + m_2'$ is odd and $(a + m_2')m_1' \equiv n' \pmod{m_2'}$. As $n' - (a + m_2')m_1'$ is odd, $n' - (a + m_2')m_1' = bm_2'$ for some odd b. Therefore $n' = (a + m_2')m_1' + bm_2'$ and $n = (a + m_2')m_1 + bm_2$ where $(a + m_2')$ and b are odd. ■

The following statement is used in Proposition 2.9.

Lemma 2.1. *Let m and r be relative prime nonzero integers. Let $p_1, p_2, \ldots, p_k$ be a list of pairwise distinct primes and $q_1, q_2, \ldots, q_k$ be a list of (not necessarily pairwise different) integers. Assume further that in case m is odd and $p_i = 2$ for some $1 \le i \le k$ then: q_i is even iff r is even (hence q_i is odd iff r is odd). Then there exists an integer x with the properties:*

1. $xm + r \not\equiv 0 \pmod{p_i}$, *for every* $1 \le i \le k$ *and*
2. $x \not\equiv q_i \pmod{p_i}$, *for every* $1 \le i \le k$.

Proof. Condition 1 is equivalent to the statement that $xm \not\equiv -r \pmod{p_i}$ for every $1 \le i \le k$. If $p_i | m$ then $p_i | xm$ and $p_i \nmid -r$, as m and r are relative primes, hence $xm \not\equiv -r$ holds for every integer x. Let $1 \le i \le k$ such that $p_i \nmid m$. Denote by m_i the remainder of m modulo p_i and by m_i^{-1} the multiplicative inverse of m_i in $\mathbb{Z}/p_i\mathbb{Z}$. Then $xm \not\equiv -r \pmod{p_i}$ is equivalent to $x \not\equiv -rm_i^{-1} \pmod{p_i}$. Suppose first $p_i \neq 2$. Then $|\mathbb{Z}/p_i\mathbb{Z}| = p_i > 2$, hence there exists $s_i \in \mathbb{Z}/p_i\mathbb{Z}$ such that $s_i \not\equiv 0 \pmod{p_i}$ and $s_i \not\equiv q_i \pmod{p_i}$. Suppose now $p_i = 2$. Then m is odd. Assume q_i is even. Then r is even, hence for $p_i = 2$ Condition 1 means that xm is odd, equivalently, that x is odd. Similarly, for $p_i = 2$ Condition 2 means that x is odd. Define $s_i = 1$. Assume now that q_i is odd. Then r is odd, hence for $p_i = 2$ Condition 1 means that xm is even, equivalently, that x is even. Similarly, for $p_i = 2$ Condition 2 means that x is even. Define $s_i = 0$. By the Chinese Reminder Theorem there exists a positive integer x such that $x \equiv s_i \pmod{p_i}$ for every $1 \le i \le k$. Then x satisfies Conditions 1 and 2. ■

Beyond being odd numbers, a and b are also primes. Hence, in general, they are relative primes to n (not necessarily always, but this is the case in general).

Proposition 2.9. *Let n, m_1 and m_2 be positive integers and $d = 2^k e$ be the highest common factor of m_1 and m_2, where e is the highest common odd factor of m_1 and m_2. Then there exist integers a and b such that*

1. $n = m_1 a + m_2 b$,

2. *a and b are odd and*

3. $\gcd(a, n) = \gcd(b, n) = 1$

if and only if:

1. $\gcd(n, m_1) = \gcd(n, m_2) = d$ *and*

2. $n \equiv m_1 + m_2 \pmod{2^{k+1}}$.

Proof. Suppose $n = m_1 a + m_2 b$ for odd numbers a and b such that $n = m_1 a + m_2 b$ and $\gcd(a, n) = \gcd(b, n) = 1$. Let $d' = \gcd(n, m_1)$. Clearly, $d | d'$ and $d' | b m_2$, hence – since $\gcd(n, b) = 1$ –, $d' | m_2$, thus $d' | d$ and so $d' = d$. Similarly we can show $\gcd(n, m_2) = d$, hence Condition 1 holds. Hence $n' = n/2^k$, $m_1' = m_1/2^k$ and $m_2' = m_2/2^k$ are all integers. As $n' = a m_1' + b m_2'$ and a and b are odd, $n' \equiv a m_1' + b m_2' \equiv m_1' + m_2' \pmod 2$, thus $n = 2^k n' \equiv 2^k a m_1' + 2^k b m_2' = a m_1 + b m_2 \pmod{2^{k+1}}$.

Conversely, suppose that the integers n, m_1 and m_2 satisfy Conditions 1 and 2. We show that there exist odd integers a and b such that $n = m_1 a + m_2 b$ and $\gcd(a, n) = \gcd(b, n) = 1$. Let $n' = n/d$, $m_1' = m_1/d$ and $m_2' = m_2/d$. By Condition 2, $2^k e n' = n \equiv m_1 + m_2 = 2^k e m_1' + 2^k e m_2' \pmod{2^{k+1}}$, hence – as e is odd – $n' \equiv m_1' + m_2' \pmod 2$. By Condition 1, m_1', m_2' and n' are pairwise relative primes. Hence there is at most one even number among m_1', m_2' and n' and we have one of the following two possibilities: *Case A:* m_1' and m_2' are odd and n' is even or *Case B:* either of m_1' and m_2' is even, the other one is odd and n' is odd. Without loss of generality, in *Case B* we shall assume that m_1' is odd and m_2' is even. As $\gcd(m_1', m_2') = 1$, there exists $1 \le a' \le m_2'$ such that $a' m_1' \equiv n' \pmod{m_2'}$. Let $a = a' + x m_2'$ and $b = \frac{n' - m_1'(a' + x m_2')}{m_2'}$ for some integer x. Then by $a' m_1' \equiv n' \pmod{m_2'}$, b is an integer. Furthermore, $a m_1' + b m_2' = (a' + x m_2') m_1' + \frac{n' - m_1'(a' + x m_2')}{m_2'} m_2' = n'$, hence $a m_1 + b m_2 = n$.

We show that x can be chosen such that a and b satisfy Conditions 2 and 3 as well. Let $p_1, p_2, \ldots, p_k$ be a list of the pairwise distinct prime factors (i.e. without multiplicity) of n if n is even. If n is odd then let $p_1, p_2, \ldots, p_k$ be a list of the pairwise distinct prime factors of n and 2 included. Then Conditions 2 and 3 are equivalent to the statement: $a \not\equiv 0 \pmod{p_i}$ and $b \not\equiv 0 \pmod{p_i}$ for every $1 \le i \le k$, which in turn is equivalent to:

$$a' + x m_2' \not\equiv 0 \pmod{p_i} \tag{2.1}$$

and

$$\frac{n' - m_1'(a' + x m_2')}{m_2'} \not\equiv 0 \pmod{p_i} \tag{2.2}$$

for every $1 \le i \le k$.

We have $\frac{n'-m'_1(a'+xm'_2)}{m'_2} = \frac{n'-m'_1a'}{m'_2} - m'_1x$, where – as $n' \equiv m'_1a' \pmod{m'_2}$ – $\frac{n'-m'_1a'}{m'_2}$ is an integer, therefore Congruence 2.2 is equivalent to $m'_1x \not\equiv \not\equiv \frac{n'-m'_1a'}{m'_2} \pmod{p_i}$ for every $1 \leq i \leq k$. If some $1 \leq i \leq k$ is such that $p_i | m'_1$ then $p_i \nmid n'$, hence $p_i \nmid (n' - m'_1a')$ and so $p_i \nmid \frac{n'-m'_1a'}{m'_2}$, but $p_i | m'_1x$ for every integer x and thus $m'_1x \not\equiv \frac{n'-m'_1a'}{m'_2} \pmod{p_i}$ for every integer x. Now let $1 \leq i \leq k$ be such that $p_i \nmid m'_1$. Denote by r_i the remainder of m'_1 modulo p_i and by r_i^{-1} the multiplicative inverse of r_i in $\mathbb{Z}/p_i\mathbb{Z}$. Then for p_i Congruence 2.2 is equivalent to $x \not\equiv r_i^{-1}\frac{n'-m'_1a'}{m'_2} \pmod{p_i}$. Denote by P the set of those primes p_i, $1 \leq i \leq k$, such that $p_i | m'_1$ and by R the set of those primes p_j, $1 \leq j \leq k$ such that $p_j \nmid m'_1$. For every $1 \leq i \leq k$ define q_i as follows:

$$q_i \equiv \begin{cases} \text{any integer, if } p_i \in P \\ r_i^{-1}\frac{n'-m'_1a'}{m'_2} \pmod{p_i}, \text{ if } p_i \in R. \end{cases}$$

Then Congruence 2.3 below implies Congruence 2.2:

$$x \not\equiv q_i \pmod{p_i}, \text{ for every } 1 \leq i \leq k \tag{2.3}$$

Next we show that with $m = m'_2$, $r = a'$ and x, the system of Congruences 2.1 and 2.3 satisfies the conditions of Lemma 2.1. As $a'm'_1 \equiv n' \pmod{m'_2}$ and n' and m'_2 are relative primes, $a'm'_1$ and m'_2 are relative primes and so $r = a'$ and $m = m'_2$ are relative primes. Suppose now that $m = m'_2$ is odd. Then *Case A* must hold, i.e m'_1 is odd and n' is even, hence $2 \in R$, where $p_{i_0} = 2$ for some $1 \leq i_0 \leq k$. We need to show that q_{i_0} is even iff a' is even. We have $q_{i_0} \equiv m_{i_0}^{-1}\frac{n'-m'_1a'}{m'_2} = \frac{n'-m'_1a'}{m'_2} \pmod 2$. If a' is even then - as n' is also even - $n' - m'_1a'$ is even and so $q_{i_0} \equiv \frac{n'-m'_1a'}{m'_2} \pmod 2$ is even. If a' is odd then $n' - m'_1a'$ is odd, hence $q_{i_0} \equiv \frac{n'-m'_1a'}{m'_2} \pmod 2$ is odd, as required. Hence the system 2.1 and 2.3 satisfies the conditions of Lemma 2.1, therefore it has an integer solution x_0, which hence also satisfies the system 2.1 and 2.2. Then for $a = a' + x_0$ and $b = \frac{n'-m'_1(a'+x_0m'_2)}{m'_2}$ Conditions 1, 2 and 3 hold. ■

Below we formulate the most general version of the conjecture, containing an arbitrary number of coefficients m_i, $1 \geq i \geq r$, where $r \geq 2$.

Conjecture 2.10. *Let $m_1, m_2, \ldots, m_r$ be arbitrary positive integers for some $r \geq 2$. Let $d = 2^k e$ be the highest common factor of $m_1, m_2, \ldots, m_r$, where e is the highest common odd factor of $m_1, m_2, \ldots, m_r$. Then every sufficiently large integer n such that*

1. $\gcd(n, m_2, m_3, \ldots, m_r) = \gcd(m_1, \ldots m_{i-1}, n, m_{i+1}, \ldots, m_r) =$
   $= \gcd(m_1, m_2, \ldots, m_{r-1}, n) = d$ *for every* $2 \leq i \leq r-1$ *and*

2. $n \equiv m_1 + m_2 + \ldots + m_r \pmod{2^{k+1}}$

can be expressed in the form $n = m_1p_1 + m_2p_2 + \ldots + m_rp_r$ *for some primes* $p_1, p_2, \ldots, p_r$.

The assertion below can be shown similarly to Proposition 2.7.

Proposition 2.11. *Conjecture* 2.10 *is equivalent to the following statement: Let* $m_1, m_2, \ldots, m_r$ *be positive integers* $(r \geq 2)$ *such that* $\gcd(m_1, m_2, m_3, \ldots, m_r) = 1$ *. Then every sufficiently large integer* n *such that*

1. $\gcd(n, m_2, m_3, \ldots, m_r) = \gcd(m_1, \ldots m_{i-1}, n, m_{i+1}, \ldots, m_r) =$
$= \gcd(m_1, m_2, \ldots, m_{r-1}, n) = 1$ *for every* $2 \leq i \leq r-1$ *and*

2. $n \equiv m_1 + m_2 + \ldots + m_r \pmod 2$

can be expressed in the form $n = m_1p_1 + m_2p_2 + \ldots + m_rp_r$ *for some primes* $p_1, p_2, \ldots, p_r$.

Generalized Goldbach partitions. In [8] the number of Goldbach partitions of a positive integer n was denoted by $R(n)$ and by $r(n)$, respectively, depending on whether the order of the two primes in the partition matters. Here we shall introduce analogous terminology and notations.

For a given integer $r \geq 2$, any r-tuple $(m_1, m_2, \ldots, m_r)$ of positive integers and positive integer n, we call a decomposition (if it exists) of n in the form $n = m_1p_1 + m_2p_2 + \ldots + m_rp_r$ where $p_1, p_2, \ldots, p_r$ are primes, a(n) *(r-term) generalized Goldbach partition of* n *with coefficients* $m_1, m_2, \ldots, m_r$.

A list $m_1, m_2, \ldots, m_r$ of coefficients may contain repetitions (identical numbers). Consider for example the coefficients $m_1 = 2, m_2 = m_3 = 4, m_4 = 6$. Then $2p_1+4p_2+4p_3+6p_4$ is a generalized Goldbach partition of n if and only if $2p_1 + 4p_3 + 4p_2 + 6p_4$ is a generalized Goldbach partition of n. As suggested by this example, for a given integer $r \geq 2$, any r-tuple $(m_1, m_2, \ldots, m_r)$ of positive integers and positive integer n, the number of generalized Goldbach partitions of n with coefficients $m_1, m_2, \ldots, m_r$ can be interpreted in two different ways, depending on whether we allow for the permutations of primes across identical coefficients. For a list $m_1, m_2, \ldots, m_r$ $(r \geq 2)$ of coefficients and positive integer n, the number of generalized Goldbach partitions of n with coefficients $m_1, m_2, \ldots, m_r$ shall be denoted by $R_{m_1,m_2,\ldots,m_r}(n)$, if we regard two Goldbach partitions $n = m_1p_1 + m_2p_2 + \ldots + m_rp_r$ and $n = m_1q_1 + m_2q_1 + \ldots + m_rq_r$ identical iff $p_i = q_i$ for every $1 \leq i \leq r$. We denote the number of generalized Goldbach partitions of n with coefficients $m_1, m_2, \ldots, m_r$ by $r_{m_1,m_2,\ldots,m_r}(n)$, if two partitions $n = m_1p_1+m_2p_2+\ldots+m_rp_r$ and $n = m_1q_1+m_2q_2+\ldots+m_rq_r$ are considered identical if and only if there exists a permutation σ on the set $\{1, 2, \ldots, r\}$ such that $p_i = q_{\sigma(i)}$ and $m_i = m_{\sigma(i)}$ for every $1 \leq i \leq r$. Note that

these notations are consistent with $R(n)$ and $r(n)$ for the number of Goldbach partitions of n used in [8] and $R_{1,1}(n) = R(n)$ and $r_{1,1}(n) = r(n)$ for all positive integers n.

Conjecture 2.10 clearly implies Conjecture 2.6. By making a further assumption, the converse also becomes true.

Theorem 2.12. *If Conjecture* 2.6 *is true and* $\lim_{n\to\infty} R_{m_1,m_2}(n) = \infty$ *for all integers* m_1, m_2 *and* n *satisfying the conditions of Conjecture* 2.6 *then Conjecture* 2.10 *also holds.*

Proof. Suppose Conjecture 2.6 is true and $\lim_{n\to\infty} R_{m_1,m_2}(n) = \infty$ for all integers m_1, m_2 and n satisfying the conditions of Conjecture 2.6. By induction on r we show that then Conjecture 2.10 also holds and that for any positive integers $m_1, m_2, \ldots, m_r$ ($r \geq 2$) and n satisfying the conditions of Conjecture 2.10, $\lim_{n\to\infty} R_{m_1,m_2,\ldots,m_r}(n) = \infty$. By Proposition 2.11, it is sufficient to consider the case when $m_1, m_2, \ldots, m_r$ are (not necessarily pairwise) relative primes.

Base Step: $r = 2$. This is true since we are assuming that Conjecture 2.6 holds.

Inductive Step: Suppose the statement of Conjecture 2.10 holds for some $r \geq 2$ and for all integers $m_1, m_2, \ldots, m_r$ and n satisfying the conditions of Conjecture 2.10, $\lim_{n\to\infty} R_{m_1,m_2,\ldots,m_r}(n) = \infty$. Let $m_1, m_2, \ldots, m_r, m_{r+1}$ be positive integers. By Proposition 2.11 we can assume that $m_1, m_2, \ldots, m_r, m_{r+1}$ are (not necessarily pairwise) relative primes. Without loss of generality we shall assume that m_{r+1} is odd. Let k be an arbitrary positive integer. We show that there exists K such that for every $n \geq K$ satisfying the conditions of Proposition 2.11 (or equivalently, the conditions of Conjecture 2.10), $R_{m_1,m_2,\ldots,m_{r+1}}(n) > k$. Denote $\gcd(m_1, m_2, \ldots, m_r)$ by d. Let $M = \Pi_{i=1}^{r} m_i$ and denote by p the largest prime factor of M. By our inductive hypothesis there exists L such that for every positive integer $n \geq L$ satisfying

(1) $\gcd(n, m_2, m_3, \ldots, m_r) = \gcd(m_1, \ldots m_{i-1}, n, m_{i+1}, \ldots, m_r) = \gcd(m_1, m_2, \ldots, m_{r-1}, n) = d$ for every $2 \leq i \leq r-1$ and

(2) $n \equiv m_1 + m_2 + \ldots + m_r \pmod{2^{k+1}}$,

we have $R_{m_1,\ldots,m_r}(n) > \max\{k\}$.

Let d' an arbitrary divisor of m_{r+1}.

Case 1: $(m_1 + m_2 + \ldots + m_r)/d \equiv 1 \pmod 2$. Define $a = dd'$.

Case 2: $(m_1 + m_2 + \ldots + m_r)/d \equiv 0 \pmod 2$. Define $a = 2dd'$.

As $\gcd(d, m_{r+1}) = 1$ and m_{r+1} is odd, in both of the above cases we have $\gcd(a, m_{r+1}) = d'$. Hence, by our inductive hypothesis, here exists a $K_{d'}$ such that for every $n \geq K_{d'}$ satisfying

(1) $\gcd(n, m_{r+1}) = \gcd(a, n) = \gcd(a, m_{r+1}) = d'$ and

(2) $a + m_{r+1} \equiv n \pmod 2$,

we have $R_{a,m_{r+1}}(n) > \max\{p, L\}$.

Now let $K = \max\{K_{d'} : d' | m_{r+1}\}$ and $n \geq K$ be an integer satisfying

(1) $\gcd(n, m_2, m_3, \ldots, m_{r+1}) = \gcd(m_1, \ldots m_{i-1}, n, m_{i+1}, \ldots, m_{r+1}) =$
$= \gcd(m_1, m_2, \ldots, m_r, n) = 1$ for every $2 \leq i \leq r$ and

(2) $n \equiv m_1 + m_2 + \ldots + m_{r+1} \pmod 2$.

We shall show $R_{m_1,m_2,\ldots,m_{r+1}}(n) > k$. Denote $\gcd(n, m_{r+1})$ by d' and let $a = dd'$ in Case 1 and let $a = 2dd'$ in Case 2.

Case 1: If d is even then $m_1 + m_2 + \ldots + m_r$ and a are even and as m_{r+1} is odd, n is odd. Hence $a + m_{r+1} \equiv 1 \equiv n \pmod 2$. If d is odd then $m_1 + m_2 + \ldots + m_r$ and a are odd and since m_{r+1} is odd, n is even. Hence $a + m_{r+1} \equiv 0 \equiv n \pmod 2$.

Case 2: In this case $m_1 + m_2 + \ldots + m_r$ is even and a is even. Since m_{r+1} is odd, n is odd and so $a + m_{r+1} \equiv 1 \equiv n \pmod 2$.

As noted earlier, in both of the above cases we have $\gcd(a, m_{r+1}) = d'$. Since $d | m_i$ for every $1 \leq i \leq r$ and $\gcd(m_1, \ldots, m_r, n) = 1$, we have $\gcd(d, n) = 1$. Hence in both cases $\gcd(a, n) = d'$. Therefore $\gcd(a, m_{r+1}) = \gcd(a, n) =$
$= \gcd(m_{r+1}, n) = d'$ and as $n \geq K \geq K_{d'}$, $R_{a,m_{r+1}}(n) > max\{p, L\}$. Therefore we have $n = aq_1 + m_{r+1}q_2$ for some primes q_1 and q_2 with $q_1 > max\{p, L\}$.

We are going to demonstrate that

$$\gcd(aq_1, m_2, \ldots, m_r) = \gcd(m_1, m_2, \ldots, m_{r-1}, aq_1) =$$
$$= \gcd(m_1, \ldots, m_{i-1}, aq_1, m_{i+1}, \ldots, m_r) = d$$

for every $2 \leq i \leq r - 1$.

As $d \mid a$ we have $d \mid \gcd(aq_1, m_2, \ldots, m_r)$, $d \mid \gcd(m_1, m_2, \ldots, m_{r-1}, aq_1)$ and $d \mid \gcd(m_1, \ldots, m_{i-1}, aq_1, m_{i+1}, \ldots, m_r)$ for every $2 \leq i \leq r - 1$.

We show that $\gcd(d', aq_1, m_2, \ldots, m_r) = \gcd(d', m_1, m_2, \ldots, m_{r-1}, aq_1) =$
$= \gcd(d', m_1, \ldots, m_{i-1}, aq_1, m_{i+1}, \ldots, m_r) = 1$ for every $2 \leq i \leq r - 1$.

Suppose that it is false and for example $\gcd(d', aq_1, m_2, \ldots, m_r) > 1$. Then $\gcd(d', m_2, \ldots, m_r) > 1$ and as $d' | m_{r+1}$ and $d' \mid n$, hence

$$\gcd(n, m_2, \ldots, m_r, m_{r+1}) > 1$$

contradicting our assumption $\gcd(n, m_2, \ldots, m_r, m_{r+1}) = 1$. Hence

$$\gcd(d', aq_1, m_2, \ldots, m_r) = 1$$

and similarly we can show that

$$\gcd(d', m_1, m_2, \ldots, m_{r-1}, aq_1) = \gcd(d', m_1, \ldots, m_{i-1}, aq_1, m_{i+1}, \ldots, m_r) = 1$$

for every $2 \leq i \leq r - 1$. As $q_1 > p$, $\gcd(q_1, m_i) = 1$ for every $1 \leq i \leq r$.

In Case 2, as $\gcd(m_1, m_2, \ldots, m_r) = d$, but $2d \mid (m_1 + m_2 + \ldots + m_r)$, there must be at least two among the numbers $m_1, m_2, \ldots, m_r$ which are not divisible by $2d$. Therefore in both Case 1 and Case 2 we have $\gcd(aq_1, m_2, \ldots, m_r) = \gcd(m_1, m_2, \ldots, m_{r-1}, aq_1) = \gcd(m_1, \ldots, m_{i-1}, aq_1, m_{i+1}, \ldots, m_r) = d$ for every $2 \le i \le r-1$.

Next we show that $m_1 + m_2 + \cdots + m_r \equiv aq_1 \pmod{2^{s+1}}$, where $d = 2^s$ is the largest power of 2 which is a factor of d.

In Case 1: $(m_1 + m_2 + \cdots + m_r)/d \equiv 1 \equiv d' \pmod 2$, hence $m_1 + m_2 + \cdots + m_r \equiv d \equiv dd' \pmod{2^{s+1}}$.

In Case 2: $(m_1 + m_2 + \cdots + m_r)/d \equiv 0 \equiv 2d' \pmod 2$, hence $m_1 + m_2 + \cdots + m_r \equiv d \equiv 2dd' \pmod{2^{s+1}}$.

As $aq_1 \ge q_1 \ge L$, the above implies $R_{m_1,m_2,\ldots,m_r}(aq_1) > k$. As for any r-tuple of primes $p_1, p_2, \ldots, p_r$ such that $aq_1 = m_1p_1 + m_2p_2 + \cdots + m_rp_r$ we have $n = m_1p_1 + m_2p_2 + \cdots + m_rp_r + m_{r+1}q_2$, $R_{m_1,m_2,\ldots,m_{r+1}}(n) > k$ follows. ∎

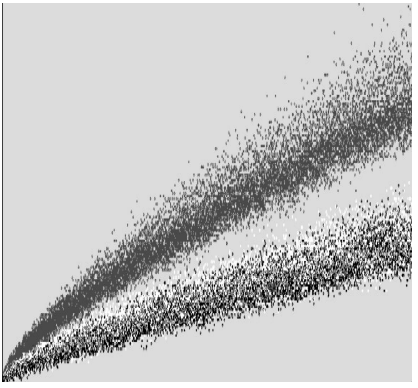
3. Computational results

To support our statements we developed some programs directly checking the conjecture up to 10^9. We carried out a triple-checking, namely we implemented three different methods, which we are planning to publish in details in a future paper. According to Proposition 2.7 it is sufficient to verify Conjecture 2.6 for relative prime values of m_1 and m_2. We checked Conjecture 2.6 for all possible cases when $1 \le m_1, m_2 \le 25$ and $\gcd(m_1, m_2) = 1$ until $(n = 10^9)$. In Table 1 we summarized the cases $1 \le m_1 \le 25, 1 \le m_2 \le 14$. The values of m_2 from 1 to 14 are found in the first row whereas, the first column contains the values of m_1 from 1 to 25. A letter x is shown when m_1, m_2 are not relative prime. Otherwise a cell of the ith row and jth column contains the smallest value of n from which the conjecture was found true for $m_1 = i, m_2 = j$ until $n = 10^9$ by our program.

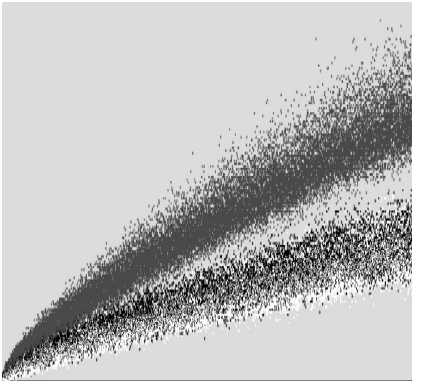
The plot of the function $R(n)$ assigning to each even integer $n \ge 4$ the number of its Goldbach partitions is known as Goldbach's comet. Analogously, for any fixed positive integers m_1, m_2 we shall call the plot of the function $R_{m_1,m_2}(n)$ assigning to each n satisfying the conditions of Conjecture 2.6 the number of its generalized Goldbach partitions with coefficients m_1, m_2, a *generalized Golbach's comet*. Figure 1 shows some comets for different pairs m_1, m_2. Investigating the anatomy of such comets can be an interesting challenge for the near future.

Table 1: This table contains the minimal values from which the conjecture is true up to 10^9

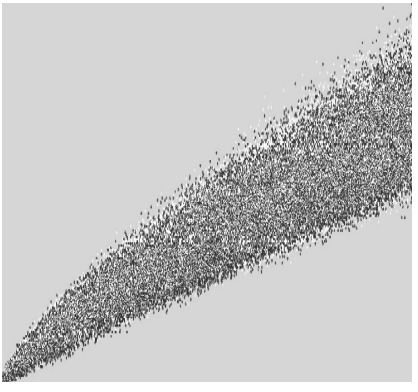
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	4	7	14	79	26	17	38	51	32	31	56	29	118	165
2	7	x	19	x	167	x	93	x	119	x	675	x	721	x
3	14	19	x	59	64	x	100	253	x	137	142	x	326	463
4	79	x	59	x	363	x	1693	x	631	x	2385	x	4075	x
5	26	167	64	363	x	193	464	1337	278	x	1522	959	4584	3383
6	17	x	x	x	193	x	425	x	x	x	1363	x	1727	x
7	38	93	100	1693	464	425	x	2713	758	2459	2298	2375	12328	x
8	51	x	253	x	1337	x	2713	x	1637	x	6511	x	18463	x
9	32	119	x	631	278	x	758	1637	x	817	2072	x	3010	2791
10	31	x	137	x	x	x	2459	x	817	x	7493	x	11053	x
11	56	675	142	2385	1522	1363	2298	6511	2072	7493	x	3625	13020	11295
12	29	x	x	x	959	x	2375	x	x	x	3625	x	11455	x
13	118	721	326	4075	4584	1727	12328	18463	3010	11053	13020	11455	x	17831
14	165	x	463	x	3383	x	x	x	2791	x	11295	x	17831	x
15	52	179	x	1297	x	x	1196	1403	x	x	1648	x	3806	2923
16	475	x	857	x	4891	x	10467	x	7661	x	25725	x	32509	x
17	528	2373	556	7761	2544	2449	8108	22275	3970	13817	18406	15103	28878	43425
18	41	x	x	x	1189	x	6845	x	x	x	6895	x	11243	x
19	454	1759	620	12169	3084	3137	17848	19429	7502	14623	35256	8741	30786	56239
20	111	x	1249	x	x	x	8389	x	3809	x	17913	x	25917	x
21	92	277	x	1541	662	x	x	3523	x	3817	4024	x	6544	x
22	403	x	821	x	7527	x	10733	x	11123	x	x	x	49633	x
23	834	2211	2206	24501	9222	4903	13494	47251	7456	22997	44208	16741	44448	42713
24	101	x	x	x	4231	x	6589	x	x	x	9709	x	14225	x
25	298	2401	842	7183	x	2491	8622	14083	6662	x	31638	10481	25664	23449



(a) $m_1 = 1, m_2 = 50$

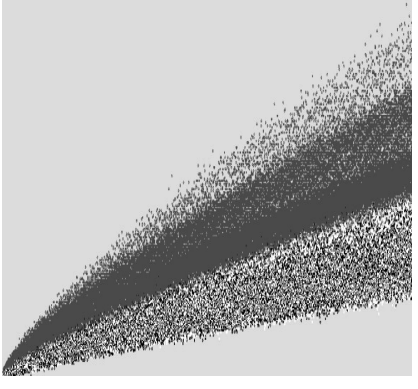


(b) $m_1 = 1, m_2 = 100$



(c) $m_1 = 30, m_2 = 31$

(d) $m_1 = 5, m_2 = 7$



(e) $m_1 = 11, m_2 = 13$

(f) $m_1 = 14, m_2 = 25$

Figure 1: The Comets for some m_1, m_2 values

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