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MULTIPLICATIVE FUNCTIONS WITH SMALL INCREMENT II.

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Abstract. We prove that if f is a multiplicative function satisfying the relations

$$\overline{\lim_{n\to\infty}} \frac{1}{\log x} \sum_{n\le x} \frac{|f(n)|}{n} = \infty, \quad \overline{\lim_{n\to\infty}} \frac{1}{\log x} \sum_{n\le x} \frac{|f(n+K)-f(n)|}{n} < \infty,$$

then there are real numbers σ, t with $(0<\sigma\le 1)$ and a Dirichlet character $\chi \pmod K$ such that $f(n) = n^{\sigma+it}\chi(n)$.

1. Introduction

Let, as usual, $\mathcal{P}$, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ be the set of primes, positive integers, real and complex numbers, respectively. Let $\mathcal{M}$, $\mathcal{M}^*$ be the set of complex-valued multiplicative (completely multiplicative) functions. We say that $f \in \mathcal{M}_1$ (resp. $\mathcal{M}_1^*$), if $f \in \mathcal{M}$ (resp. $\mathcal{M}^*$) and $|f(n)| = 1$ for every $n \in \mathbb{N}$. For an arithmetical function f, $f: \mathbb{N} \to \mathbb{C}$ let

$$\Delta_K f(n) := f(n+K) - f(n), \quad K \in \mathbb{N},$$

where $K \in \mathbb{N}$.

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In [1] we proved the following assertion.

Theorem A. *Let $f \in \mathcal{M}$, assume that*

$$\sum_{n\le x} |\Delta_K f(n)| = O(x), \tag{1.1}$$

then, either

$$\sum_{n\le x} |f(n)| = O(x), \tag{1.2}$$

or

$$f(n) = n^{\sigma+it}\chi(n) \quad \text{for every} \quad n \in \mathbb{N}, (n, K) = 1,$$

where $0 < \sigma \le 1, t \in \mathbb{R}$ and χ is a Dirichlet character $\pmod K$.

In [1] and [7] the following result is proved.

Theorem B. *Assume that the functions $f, g \in \mathcal{M}$ and a number $K \in \mathbb{N}$ satisfy*

$$\sum_{n\le x} |g(n+K) - f(n)| = O(x).$$

Then, either

$$\sum_{n\le x} |g(n)| = O(x), \quad \sum_{n\le x} |f(n)| = O(x)$$

or there are functions $F, G \in \mathcal{M}$ and a complex number s such that

$$f(n) = n^s F(n), \quad g(n) = n^s G(n), 0 < \Re s \le 1$$

and $G(n+K) - F(n) = 0$ for every $n \in \mathbb{N}$.

Let $\mathcal{M}_K$ be the set of those multiplicative functions f, for which $f(n) = 0$ if $(n, K) > 1$.

Our purpose in this paper is to investigate those $f \in \mathcal{M}_K$ for which

$$\overline{\lim_{n\to\infty}} \frac{1}{\log x} \sum_{n\le x} \frac{|f(n)|}{n} = \infty, \tag{1.3}$$

and

$$\overline{\lim_{n\to\infty}} \frac{1}{\log x} \sum_{n\le x} \frac{|\Delta_K f(n)|}{n} < \infty. \tag{1.4}$$

Theorem 1. *Let $f \in \mathcal{M}_K$ be such a function for which* (1.3), (1.4) *hold. Then*

$$f(n) = n^{\sigma+it}\chi(n), \quad 0 \le \sigma \le 1, t \in \mathbb{R}$$

and χ is a Dirichlet character $\pmod{K}$.

Remark 1. This theorem for $K = 1$ was proved in [3].

Remark 2. The proof is based upon a theorem of O. Klurman [5] which is referred here as

Lemma 1. *Let $f \in \mathcal{M}^*$,* $|f(n)| = \begin{cases} 1 & \text{if} \quad (n,K) = 1 \\ 0 & \text{if} \quad (n,K) > 1 \end{cases}$

and

$$\frac{1}{\log 2x} \sum_{n \le x} \frac{|\Delta_K f(n)|}{n} \to 0 \quad (x \to \infty). \tag{1.5}$$

Then

$$f(n) = n^{i\tau}\chi(n) \quad \textit{for every} \quad n \in \mathbb{N}, \tau \in \mathbb{R},$$

where χ is a Dirichlet character $\pmod{K}$. *If f is such a function, then* (1.5) *holds true.*

Remark 3. By using the method of J.-L Mauclaire and L. Murata [6], we have

Lemma 2. *If $f \in \mathcal{M}$,*

$$|f(n)| = \begin{cases} 1 & \text{if} \quad (n,K) = 1 \\ 0 & \text{if} \quad (n,K) > 1 \end{cases}$$

and (1.5) *holds, then $f \in \mathcal{M}^*$, consequently*

$$f(n) = n^{i\tau}\chi(n) \quad \textit{for every} \quad n \in \mathbb{N}, \tau \in \mathbb{R}$$

where χ is a Dirichlet character $\pmod{K}$.

2. Proof of Lemma 2

Let q be a prime, $q \nmid K$. It is enough to prove that $f(q^k) = f(q)^k$ for every q and $k = 1, 2, \ldots$.

Let

$$S_\ell(q) = q^{\ell-1} + \cdots + q + 1 \quad (\ell = 1, 2, \ldots).$$

Thus

$$S_\ell(q) = qS_{\ell-1}(q) + 1 \quad (\ell = 2, 3, \ldots).$$

Since $|f(m)| = 1$ if $(m, qK) = 1$, therefore

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{\substack{m\le x\\ (m,Kq)=1}} \frac{|f(m)|}{m} = c_q > 0. \tag{2.1}$$

Let $A \in \mathbb{N}$. Since

$$|\Delta_{AK} f(n)| \le \sum_{j=0}^{A-1} |\Delta_K f(m + jK)|,$$

therefore

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{\substack{m\le x\\ (m,Kq)=1}} \frac{|f(q^k m + KS_k(q)) - f(q^k)f(m)|}{mq} = 0. \tag{2.2}$$

Furthermore,

$$f(q^\ell m + KS_\ell(q)) - f\Big(q^\ell m + K(S_\ell(q) - 1)\Big) =$$

$$= f(q^\ell m + KS_\ell(q)) - f(q)f\Big(q^{\ell-1} m + KS_{\ell-1}(q)\Big).$$

Applying this for $\ell = 1, \ldots, k$, we obtain that

$$\frac{1}{\log x} \sum_{\substack{m\le x\\ (m,Kq)=1}} \frac{|f(q^k m + KS_k(q)) - f(q)^k f(m)|}{mq} \to 0,$$

which by (2.2) implies that

$$|f(q^k) - f(q)^k| \sum_{\substack{m\le x\\ (m,Kq)=1}} \frac{|f(m)|}{m} = O(\log x),$$

and this by (2.1) implies that $f(q^k) = f(q)^k$ for every $k \in \mathbb{N}$.

3. Proof of Theorem 1

If f is a solution , then $F(n) := |f(n)|$ is a solution also.

Let $p \in \mathcal{P}, (p, K) = 1, Q = p^\alpha$. For some $n \in \mathbb{N}$, let $n \equiv \ell_n K \pmod{Q}$, $\ell_n \in \{0, \dots, Q-1\}$ and let n_1 be defined by the relation $n = Qn_1 + \ell_n K$. Then

$$|F(n) - F(Qn_1)| = |\Delta_{K\ell_n} F(Qn_1)|. \tag{3.1}$$

If $p|n_1$, then $(p, n_1 - K) = 1$, and so

$$\begin{aligned} F(Qn_1)) &= F(Q(n_1 - K)) + \Delta_{QK} F(Q(n_1 - K)) = \\ &= F(Q)F(n_1 - K) + \Delta_{QK} F(Q(n_1 - K)) = \\ &= F(Q)F(n_1) - F(Q)\Delta_K F(n_1 - K) + \Delta_{QK} F(Q(n_1 - K)), \end{aligned}$$

so, hence and from (3.1), we get

$$\begin{aligned} |F(n) - F(Q)F(n_1)| \le F(Q)|\Delta_K F(n_1 - K)| + \\ + |\Delta_{QK} F(Q(n_1 - K))| + |\Delta_{K\ell_n} F(Qn_1)|. \end{aligned} \tag{3.2}$$

Let us assume that Q is fixed. $n_1 = m$ at the mapping $n \to n_1$ occurs for Q distinct n, furthermore $n_1 \le n/Q$ and $n_1 \ge 1$ if $n \ge QK$.

Since

$$|\Delta_{Ks} F(m)| \le \sum_{j=0}^{s-1} |\Delta_K F(m + jK)|,$$

from (3.2) we obtain that

$$\frac{1}{\log x} \sum_{n \le x} \frac{1}{n} |F(n) - F(Q)F(n_1)| < C \tag{3.3}$$

if $x \ge 2$. C may depend on Q.

Let

$$E_F(x) = \sum_{n \le x} \frac{|F(n)|}{n}.$$

From (3.3) we obtain that

$$E_F(x) \le F(Q) E_F\Big(\frac{x}{Q}\Big) + c_1, \tag{3.4}$$

and

$$F(Q) E_F\Big(\frac{x - QK}{Q}\Big) - C \le E_F(x). \tag{3.5}$$

If $F(Q) < 1$, then from (3.4) we obtain that $\limsup E_F(x) < \infty$, which contradicts our assumption. Thus $F(n) \ge 1$ for every n coprime to K.

Let $F(Q) = Q^\eta,\ \eta \ge 0$. If $\eta = 0$, then $E_F(x) = O(\log x)$ easily follows from (3.4).

Assume that $\eta > 0$. Let $A > C + 1$, x_0 be so large that $E_F(x_0) \ge A$. Let

$$x_{\nu+1} = Qx_\nu + QK \quad (\nu = 1, 2, \ldots).$$

Thus from (3.5)

$$E_F(x_{\nu+1}) \ge Q^\eta E_F(x_\nu) - E_F(x_{\nu+1}) \ge Q^\eta E_F(x_\nu) - C \quad (\nu = 1, 2, \ldots),$$

and this implies that

$$E_F(x_{\nu+1}) \ge c_2 Q^{(\nu+1)\eta},$$

where c_2 is a positive constant, and so

$$E_F(x) > c_3 x^\eta$$

with some positive constant c_3 if x is large.

On the other hand, from (3.4) we obtain that

$$E_F(x) \le c_4 x^\eta$$

with some positive constant c_4 if x is large. Hence we obtain that

$$F(Q) = Q^\eta \quad \eta \in \mathbb{R}, \eta \ge 0,$$

consequently η is a constant,

$$F(n) = \begin{cases} n^\eta & \text{if} \quad (n, K) = 1 \\ 0 & \text{if} \quad (n, K) > 1. \end{cases}$$

If $\eta = 0$, from (3.4) we obtain that $E_f(x) = O(\log x)$.

Since $\Delta_K f(n) \asymp n^{\eta-1}$, therefore $\eta \le 1$.

Let us write $g(n) = f(n) \cdot n^{-\eta}$. Then $g(n)$ is multiplicative, $|g(n)| = 1$ if $(n, K) = 1$ and $g(n) = 0$, if $(n, K) > 1$.

We have

$$\begin{aligned} \Delta_K f(n) &= (n+K)^\eta g(n+K) - n^\eta g(n) = \\ &= n^\eta \Delta_K g(n) + \Big[(n+K)^\eta - n^\eta\Big] g(n+K), \end{aligned}$$

and so

$$\sum_{n \le x} \frac{n^\eta |\Delta_K g(n)|}{n} \le \sum_{n \le x} \frac{|\Delta_K f(n)|}{n} + \sum_{n \le x} \frac{C n^{\eta-1}}{n} = O(\log x).$$

Consequently we obtain that

$$\frac{1}{\log x} \sum_{\substack{n \le x \\ (n,K)=1}} \frac{|\Delta_K g(n)|}{n} \to 0 \quad (x \to \infty).$$

From Lemma 1, 2 we obtain that

$$g(n) = n^{i\tau} \chi(n), \quad \tau \in \mathbb{R},$$

where χ is a Dirichlet character $\pmod K$.

Theorem 1 is proved. ■

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