

MULTIPLICATIVE FUNCTIONS WITH SMALL INCREMENT I.

Karl-Heinz Indlekofer (Paderborn, Germany)

Imre Kátaı and Bui Minh Phong

(Budapest, Hungary)

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Abstract. We prove that if f is a completely multiplicative function and

$$\sum_{n \leq x} \frac{|f(n+1) - f(n)|}{n} = O(\log x),$$

then either

$$\sum_{n \leq x} \frac{|f(n)|}{n} = O(\log x) \quad \text{or} \quad f(n) = n^{\sigma+it} \quad 0 < \sigma \leq 1, t \in \mathbb{R}.$$

1. Introduction

Let, as usual, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ be the set of positive integers, real and complex numbers, respectively. Let $\mathcal{M}$, $\mathcal{M}^*$ be the set of complex-valued multiplicative (completely multiplicative) functions. We say that $f \in \mathcal{M}_1$ (resp. $\mathcal{M}_1^*$), if $f \in \mathcal{M}$ (resp. $\mathcal{M}^*$) and $|f(n)| = 1$ for every $n \in \mathbb{N}$. Let $\mathcal{A}$, $\mathcal{A}^*$ be the set of real-valued additive (completely additive) functions.

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Let E be the set of those arithmetical function $e(n)$ ($n \in \mathbb{N}$), for which

$$(1.1) \quad \sup_{x \geq 1} \frac{1}{\log x} \sum_{n \leq x} \frac{|e(n)|}{n} < \infty.$$

Our purpose in this short paper is to prove the next assertion.

Theorem 1. *Let $f \in \mathcal{M}^*$, $\delta(n) = f(n+1) - f(n)$ ($n \in \mathbb{N}$). Assume that $\delta \in E$ and that $f \notin E$. Then*

$$f(n) = n^{\sigma+it}, \quad 0 < \sigma \leq 1, t \in \mathbb{R}.$$

The proof is based upon an important theorem due to O. Klurman which is cited now as

Lemma 1. (O. Klurman [4]) *If $f \in \mathcal{M}_1$ and*

$$(1.2) \quad \lim_{x \rightarrow \infty} \frac{1}{\log x} \sum_{n \leq x} \frac{|f(n+1) - f(n)|}{n} = 0,$$

then

$$f(n) = n^{it} \quad (n \in \mathbb{N}), \quad \tau \in \mathbb{R}.$$

A weaker assertion has been proved in [3], namely that, if $f \in \mathcal{M}$ and

$$\lim_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} |f(n+1) - f(n)| = 0,$$

then either

$$\lim_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} |f(n)| = 0 \quad \text{or} \quad f(n) = n^{\sigma+it} \quad 0 < \sigma \leq 1, t \in \mathbb{R}.$$

Remark 1. O. Klurman proved Lemma 1 for $f \in \mathcal{M}_1^*$, but using the method of Maucclair and Murata [5] one can prove that if (1.2) holds for $f \in \mathcal{M}_1$, then $f \in \mathcal{M}_1^*$.

Remark 2. In [1] and [6] the following assertion has been proved. Assume that $f, g \in \mathcal{M}$, for which

$$\sum_{n \leq x} |g(n+K) - f(n)| = O(x),$$

where K is an arbitrary positive integer. Then, either

$$\sum_{n \leq x} |g(n)| = O(x) \quad \text{and} \quad \sum_{n \leq x} |f(n)| = O(x),$$

or

$$f(n) = n^s U(n), g(n) = n^s V(n) \quad 0 < \operatorname{Re} s \leq 1,$$

furthermore

$$V(n + K) = U(n) \quad \text{for every } n \in \mathbb{N}.$$

Lemma 2. *If $f \in \mathcal{M}^*$ and (1.2) holds, then either*

$$\lim_{x \rightarrow \infty} \frac{1}{\log x} \sum_{n \leq x} \frac{|f(n)|}{n} = 0,$$

or

$$f(n) = n^{\sigma + it} \quad (n \in \mathbb{N}), \quad 0 < \sigma \leq 1, t \in \mathbb{R}.$$

Proof. Let

$$\Delta_K f(n) = \max_{|k| \leq K} |f(n+k) - f(n)|.$$

From (1.2) we have that

$$(1.3) \quad \lim_{x \rightarrow \infty} \frac{1}{\log x} \sum_{n \leq x} \frac{|\Delta_K f(n)|}{n} = 0$$

for every fixed K .

Assume first that there exists such a prime power $Q = q^\ell$ for which $|f(Q)| = \rho < 1$.

Let

$$n = Qn_1 + a_0, \quad \text{if } 0 \leq a_0 < Q, q \nmid n_1$$

and

$$n = (a_0 + Q) + Q(n_1 - 1), \quad \text{if } 0 \leq a_0 < Q, q \mid n_1.$$

Then

$$|f(n)| \leq |\Delta_K f(n)| + \rho |f(n_1)|$$

in the first case, and

$$|f(n)| \leq |\Delta_K f(n)| + \rho |f(n_1)| + |\Delta_K f(n_1)|$$

in the second case. Thus

$$s(x) := \sum_{n \leq x} \frac{|f(n)|}{n} \leq \rho \sum_{\substack{n_1 \leq \frac{x}{Q} \\ q \nmid n_1}} \frac{|f(n_1)|}{n_1} + \rho \sum_{\substack{n_1 \leq \frac{x}{Q} \\ q | n_1}} \frac{|f(n_1)|}{n_1} + \epsilon(x) \log x,$$

where $\epsilon(x) \searrow 0$ as $x \rightarrow \infty$.

Thus

$$(1.4) \quad s(x) \leq \rho s\left(\frac{x}{Q}\right) + \epsilon(x) \log x$$

for every $x \geq Q$. This implies that $|f(n)| \geq 1$ for every $n \in \mathbb{N}$.

If $Q^r \leq x < Q^{r+1}$, then

$$s(x) < \rho^{r-1} s(Q) + \sum_{\ell=0}^{r-1} \rho^\ell \epsilon\left(\frac{x}{Q^\ell}\right) \log \frac{x}{Q^\ell},$$

consequently

$$\frac{s(x)}{\log x} \rightarrow 0 \quad (x \rightarrow \infty).$$

Let us assume now that $F(n) = |f(n)| \geq 1$ ($n \in \mathbb{N}$). Let $u(n) = \log F(n)$. Then $u(n) \geq 0$, $u \in \mathcal{A}^*$,

$$|u(n+1) - u(n)| \leq |F(n+1) - F(n)| \leq |f(n+1) - f(n)|,$$

consequently

$$\lim_{x \rightarrow \infty} \frac{1}{\log x} \sum_{n \leq x} \frac{|u(n+1) - u(n)|}{n} = 0.$$

Hence, applying a theorem of Kátai [2] we obtain that $u(n) = \sigma \log n$. Then

$$F(n) = n^\sigma, F(n+1) - F(n) = \sigma n^{\sigma-1} + O(n^{\sigma-2})$$

and

$$\sum_{n \leq x} \frac{|F(n+1) - F(n)|}{n} = o(\log x)$$

implies that $\sigma < 1$. Let $f(n) = n^\sigma g(n)$, $g \in \mathcal{M}_1^*$. If $g \in \mathcal{M}_1^*$, then

$$\liminf_{x \rightarrow \infty} \frac{1}{\log x} \sum_{n \leq x} \frac{|g(n+1) - g(n)|}{n} = 0,$$

and Lemma 1 implies Lemma 2. ■

2. Proof of Theorem 1

Let

$$h(x) = \sum_{n \leq x} \frac{|f(n)|}{n}.$$

Assume first that there exists a prime q such that $|f(q)| = \rho < 1$. From (1.1) we obtain that

$$\sum_{n \leq x} \frac{|\Delta_K f(n)|}{n} = O(\log x),$$

if K is fixed, and so

$$h(x) \leq \sum_{n \leq x} \frac{q|f(q)||f(n_1)|}{qn_1} + O(\log x).$$

Then

$$h(x) \leq \rho h\left(\frac{x}{q}\right) + B \log x \quad (x \geq 2q),$$

whence we obtain that $h(x) = O(\log x)$, $f \in E$.

We can assume that $|f(q)| \geq 1$ holds for every prime q .

Let $F(n) := |f(n)|$. Then $F(n+1) - F(n) \in E$. Let q_1, q_2 be primes, $F(q_1) = q_1^{\lambda_1}$, $F(q_2) = q_2^{\lambda_2}$. We shall prove that $\lambda_1 = \lambda_2$.

Assume indirectly that $\lambda_1 < \lambda_2$.

It is clear that

$$h(x) = q_1^{\lambda_1} h\left(\frac{x}{q_1}\right) + B_1 \log x, \quad B_1 \text{ is bounded,}$$

$$h(x) = q_2^{\lambda_2} h\left(\frac{x}{q_2}\right) + B_2 \log x, \quad B_2 \text{ is bounded.}$$

Hence

$$h(x) = q_1^{k_1 \lambda_1} h\left(\frac{x}{q_1^{k_1}}\right) + O(q_1^{k_1 \lambda_1} \log x)$$

and

$$h(x) = q_2^{k_2 \lambda_2} h\left(\frac{x}{q_2^{k_2}}\right) + O(q_2^{k_2 \lambda_2} \log x).$$

If k_1, k_2 are appropriately chosen, then

$$q_2^{k_2} < q_1^{k_1} < q_2^{k_2+1}, q_1^{k_1 \lambda_1} < \frac{1}{4} q_2^{k_2 \lambda_2},$$

so

$$h\left(\frac{x}{q_1^{k_1}}\right) = h(x)q_1^{-k_1\lambda_1} + O(\log x)$$

and

$$h\left(\frac{x}{q_2^{k_2}}\right) = h(x)q_2^{-k_2\lambda_2} + O(\log x),$$

consequently

$$0 \leq h\left(\frac{x}{q_2^{k_2}}\right) - h\left(\frac{x}{q_1^{k_1}}\right) = \left(\frac{1}{q_2^{k_2\lambda_2}} - \frac{1}{q_1^{k_1\lambda_1}}\right)h(x) + O(\log x),$$

whence

$$-\frac{3}{4}h(x) + O(q_1^{k_1\lambda_1} \log x) \geq 0.$$

Thus $h(x) = O(\log x)$, which contradicts to our assumption.

Hence we have $F(n) = n^\lambda$. If $\lambda = 0$, then $f \in E$. Since $F(n+1) - F(n) \in E$, we obtain that $0 < \lambda \leq 1$. Let

$$f(n) = n^\lambda U(n), U \in \mathcal{M}_1^*.$$

Then

$$|U(n+1) - U(n)| = \left| \frac{f(n+1)}{(n+1)^\lambda} - \frac{f(n)}{n^\lambda} \right| \leq \frac{|f(n+1) - f(n)|}{n^\lambda} + \left| 1 - \left(1 + \frac{1}{n}\right)^\lambda \right|.$$

Thus

$$\sum_{n \leq x} \frac{|U(n+1) - U(n)|}{n} \leq \sum_{n \leq x} \frac{|f(n+1) - f(n)|}{n^{\lambda+1}} + c \sum_{n \leq x} \frac{1}{n^2}$$

the right hand side is bounded as $x \rightarrow \infty$. The conditions of Lemma 1 hold for U , thus $U(n) = n^{it}$, $t \in \mathbb{R}$.

This completes the proof of Theorem 1. ■

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K.-H. Indlekofer

Faculty of Informatics
 Electrical Faculty of Informatics
 University of Paderborn
 D-33098 Paderborn
 Warburger Str. 100
 Germany
 k-heinz@uni-paderborn.de

I. Kátai and B. M. Phong

Department of Computer Algebra
 Faculty of Informatics
 Eötvös Loránd University
 H-1117 Budapest, Pázmány Péter sétány 1/C
 Hungary
 katai@inf.elte.hu
 bui@inf.elte.hu

