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ON SOME PROBLEMS OF EXPANSIONS INVESTIGATED BY P. ERDŐS ET AL.

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Abstract. The objective of this article is twofold: First, we review the key results achieved by P. Erdős and his co-authors in the field of non-integer based expansions. On the other hand, in many places we present possible generalizations, and colourful, interesting examples; with the purpose of drawing attention of hopeful new researchers to this very interesting area.

1. Introduction, goals, notations

Non-integer based expansions of real numbers – hearing this title many people may think that this research field has been common for centuries. However, this is not the case; the systematic research on this area began only in the 1950s (with the mention that there were some "predecessor" publications).

The "official" start can be identified with the seminal papers of A. Rényi and W. Parry ([17], [16]); they introduced the basic concepts used here mostly even nowadays, as well as have generated the significant development of the whole area. By now, the examination of Rényi–Parry expansions has turned into a very extensive field of investigation, having extremely complicated connections.

The logical structure of this article (and the associated research) is based on the following questions:

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- What kind of expansions can be defined?
- How many different expansions can exist?
- What parts (groups of digits) may occur in general and specific expansions?
- Especially, what are the 1-expansions like?
- When do unique expansions exist, and how can the univoque set be characterized?

For the author this is an extended research area, based on a previous PhD thesis (supervised by I. Kátai). The new theorems and propositions presented in this paper are straightforward generalizations of the former results of P. Erdős, V. Komornik, I. Kátai and other researchers. Therefore, we always present the original result carefully, and in most cases the sketch of the original proof is given, too.

During the processing we put great emphasis on various colourful examples, the author's clear objective is the promotion of this area for the public.[1]

1.1. Integer bases

Let us consider first the features of the integer-based (number) systems! (For this summary we used papers [14] and [20].)

Let $b \geq 2$ be an integer (base of a number system), then the digits usable in b-adic expansions are:

$$\mathcal{A}_b = \{0, 1, \ldots, b-1\}. \tag{1.1}$$

For arbitrary α (nonnegative) real number there exists – at least one – b-based (or b-adic) expansion which can be written in the form:

$$\alpha = [\alpha] + \sum_{i=1}^{\infty} \frac{a_i}{b^i} = [\alpha].a_1 a_2 \cdots a_n \cdots , \tag{1.2}$$

here $a_i \in \mathcal{A}_b$, i.e. for the digits we have $0 \leq a_i < b$ for $i \geq 1$.

We note that form (1.2) can be considered as a sequence, and also as a number, the value of which is determined by the expression. In this paper we will use both interpretations.

The important properties of such expansions (number of possible expansions, uniqueness, periodicity) – are easy to explore and have been well-known

[1]Due to the great length, the originally planned paper has been divided into two parts, therefore, it will remain for part two – among others – the systematic study of unique expansions, using the Parry condition.

for quite a long time. The b-adic expansion is *unique for almost all* α (in Lebesgue-sense). We have *two different expansions* if and only if α's greedy (regular; see below) expansion is finite (i.e.: for countable many, but not for all rational numbers); here the infinite form ends with a pure $b-1$-tail.

By separating $[\alpha]$, the investigation is restricted to the set of fractions, i.e. to $[0,1]$. With this, however, the main characteristics of the analysed sequences and numbers (e.g.: number of possible expansions, set of the numbers with a unique expansion) will not be changed; in fact, the total number-line can be "filled in" based on the examination on the set of fractions.

There is one difference which is worth emphasising: on the set of fractions – especially – number 1 has only a unique expansion:

$$1 = \sum_{i=1}^{\infty} \frac{b-1}{b^i} = 0.(b-1)(b-1)(b-1)\ldots \tag{1.3}$$

1.2. Noninteger bases

Considering now the noninteger bases, the basic concepts can be introduced very similarly as above, as follows.

Let $\beta > 1$ (or $q > 1$) be a noninteger number (base). The digits usable in β-(based)expansions are:

$$(\mathcal{D}_\beta =)\mathcal{A}_\beta = \{0, 1, \ldots, [\beta]\}. \tag{1.4}$$

In this paper we will use the notations, which were used by the authors introducing the given concept originally, and which are widely accepted in the literature: $1/\beta = \Theta$, $[\beta] = k$.

As it is well-known, an arbitrary α (nonnegative) real number has (at least one) β-(based)expansion (with $a_i \in \mathcal{A}_\beta$) in the form

$$\alpha = \text{"separated integral part"} + \sum_{i=1}^{\infty} \frac{a_i}{\beta^i} = [\alpha].a_1a_2\cdots a_n\cdots. \tag{1.5}$$

We can introduce the set of fractions (with $a_i \in \mathcal{A}_\beta$) similarly as above:

$$\mathcal{F} = \left\{ x \middle| x = \sum_{i=1}^{\infty} \frac{a_i}{\beta^i} = \sum_{i=1}^{\infty} a_i \Theta^i \right\}. \tag{1.6}$$

Here $\mathcal{F}_{\min} = 0$ and $\mathcal{F}_{\max} = L$, where

$$L = k\Theta + k\Theta^2 + \cdots = \frac{k\Theta}{1-\Theta},$$

with $L \geq 1$ (which can be verified by easy calculations).

For $\alpha \in [0, L]$ we can give a fraction-expansion too, in the form

$$\alpha = \sum_{i=1}^{\infty} \frac{a_i}{\beta^i} = 0.a_1 a_2 \cdots a_n \cdots . \tag{1.7}$$

Commonly we work on the set $\mathcal{F}$, since we have no difference in substantial properties of the numbers and sequences (uniqueness, number of possible expansions).

As we mentioned above, expansions (1.5) and (1.7) can be interpreted as a sequence or as a number, too, the value of which is determined by the expansion.

We introduce the complementary sequence: for $(\varepsilon_i) = \varepsilon_1 \varepsilon_2 \dots$ let $(\overline{\varepsilon_i}) = (k - \varepsilon_1)(k - \varepsilon_2) \cdots = (k) - (\varepsilon_i)$, where with

$$\sum_{i=1}^{\infty} \varepsilon_i \Theta^i = x \text{ we have } \sum_{i=1}^{\infty} \overline{\varepsilon_i} \Theta^i = L - x = \overline{x}.$$

Based on the foregoing, the Reader might feel the apparatus we built up will be very likely unnecessary, since the characterization of possible expansions is relatively easy even in the noninteger-based cases (similarly to the integer-based cases). In contrast, however, even a short analysis shows that here we have a surprisingly varied and complex structure, and we face very interesting number theoretical, topological and ergodic challenges ([15], [20]).

2. Classical results

One of the first scientific papers, which was devoted to the analysis of the noninteger-based number systems, was published in 1936 ([13]). The author (A. J. Kempner) raises and examines here a number of interesting problems in connection with the possible expansions. The greediest (canonical, or simply greedy) and the laziest (lazy) expansions have already been introduced here, noting that between the two extreme strategies plenty of other options are conceivable. E.g.:

$$2 = 10.01000001\ldots_{(3/2)} \ \text{(can)} = 0.1111\ldots_{(3/2)} \ \text{(lazy)}. \tag{2.1}$$

It is also possible that the greedy and lazy expansions are the same, then obviously only this unique expansion can be defined.

Although some of the results of this article may seem a bit anachronistic today, the paper itself is still an extremely enjoyable and colourful reading material, written in a thought-provoking and entertaining style.

The greedy (or canonical, regular) expansion in the form as it is commonly known and used today, was introduced by A. Rényi, as a special case of f-expansions, in 1957 ([17]). In his seminal paper he proved, that this expansion (with fixed noninteger base, for arbitrary real number) is always unique, and he gave the producing algorithm, too. Here the largest possible digit is taken in every step (according to the greedy strategy).

With base number $\beta > 1$ and digit set (1.4) the following holds.

Theorem ([15], [17]). *Given $\beta > 1$ base number and $x \in [0, L]$, let's define a sequence of integers $(b_i) = (b_i(\beta, x))$ by the greedy algorithm: if $b_1, b_2, \dots, b_{n-1}$ have already been defined,*[2] *then let b_n be the largest integer less than β with*

$$\frac{b_1}{\beta} + \frac{b_2}{\beta^2} + \dots + \frac{b_n}{\beta^n} \le x. \tag{2.2}$$

Then (b_i) is an expansion of x.

Definition. The expansion (b_i) of the theorem is called the greedy (or regular) expansion of x in base β.

We note that the greedy form is the lexicographically largest expansion.

A. Rényi's classical example is the G-based system with $G = (\sqrt{5} + 1)/2 \approx$ ≈ 1.618 (G: golden ratio); here $G^2 = G + 1$ and using $g = 1/G = G - 1$ we have $g + g^2 = 1$, so, the greedy/regular expansion of 1 on $\mathcal{F}$ is $1 = 0.11$.

The author recommends the interested Reader to determine the greedy expansions of 1 and other numbers with this algorithm, in different systems. It is a very instructive task! Several such examples – among others in the G-based system, too – can be found in K. G. Hare's paper [10].

If in expansion (2.2) strict inequality is used instead of the allowing inequality, then we get the *quasi-greedy* (or *quasiregular*) expansion. Following from the construction the quasi-greedy form is always infinite (and so it is the lexicographically largest infinite expansion).

Similarly (to the greedy one), we can interpret and introduce the *lazy* expansion in a way, that in every step the smallest possible digit is chosen so, that the remainder has to be still legally expanded (see examples e.g. in [1] and [10]).

As it is already clear, in integer-based systems a real number can have exactly only one or two different expansions. In the noninteger-based cases the situation is more complicated. However, we can give here some classical results, which characterize these systems generally.

[2] We have no assumption for $n = 1$.

Proposition (G. A. Edgar, [5]). *Let $\beta > 1$ be a real base, and D be a finite (digit) set of real numbers, including 0 (not necessarily in form* (1.4)*). Then either some real numbers have no expansions in form* (1.5) *or some real numbers have more than one expansion in form* (1.5)*.*

Lemma (L. C. Eggan, C. L. Vanden Eynden, [6]). *For any noninteger base $\beta > 1$ with digit set* (1.4)*, there exist intervals in which each number has more than one expansion.*

To prove this lemma – based on [6] – for arbitrary base number β let us consider only the fraction-expansions. Then $0.1 < 0.0k^\infty$, since

$$0.0k^\infty_{(\beta)} = \frac{k/\beta^2}{1-1/\beta} > \frac{\beta-1}{\beta^2-\beta} = \frac{1}{\beta} = 0.1_{(\beta)}. \tag{2.3}$$

From this it follows that the numbers of the interval $[0.1, 0.0k^\infty]$ have at least two different expansions.

3. Expansion-trees in fixed bases

In this chapter we consider the problem of determining all possible expansions of given numbers, in fixed bases.

We refer here to an interesting early paper in this topic (Eggan–Eynden, 1966, [6]). Already in this work we can see several specific examples of such expansions.

The authors present here (among others) number $\xi = 0.(100)^\infty$ in the G-based system.[3] Here

$$\xi = 0.100100\ldots_{(G)} = \frac{1}{G} + \frac{1}{G^4} + \frac{1}{G^7} + \cdots = \sum_{i=0}^{\infty} \frac{1}{G^{3i+1}} = \frac{g}{1-g^3} = \frac{G^2}{G^3-1}.$$

Using identity $g + g^2 = 1$, in the expansion of ξ every part '100' can be rewritten into part '011', without changing the result. Since part '100' occurs countably often, therefore, in this way we get finally $2^{\aleph_0}$ (continuum) of different expansions, which can be represented by a complete binary tree. (Other expansions cannot be constructed, because if we anywhere "break" the rules governing the triple blocks, already a number greater or less than ξ is obtained. E.g. $0.0101^\infty = g^2 + \frac{g^4}{1-g} \approx 0.7639 < \xi \approx 0.8090$.)

This construction can be easily adapted to other systems, and so many similar examples can be constructed. Let us consider e.g. the system,[4] where

[3] This example is also presented in paper [1].
[4] For the bases of some special systems we will introduce private notations.

$1 = 0.22$ (then $T = \beta = \sqrt{3}+1$ and $t = \Theta = \frac{\sqrt{3}-1}{2}$). Here for number $\xi' = 0.(100)^\infty$ parts '100' can be rewritten into '022' in every position (see Figure 1.); in the nodes the (parts of) digits in the expansions are displayed. (Similarly as above, we get so all of the possible expansions.)

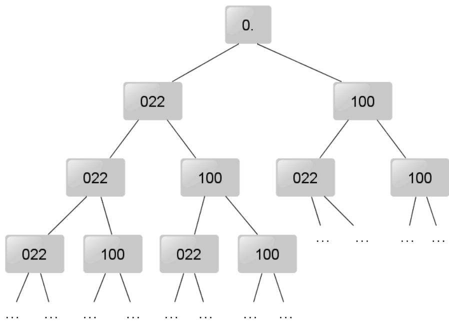

Figure 1. Expansions of $\xi' = 0.(100)^\infty$ in system with base $\sqrt{3}+1$

Our observations can be summarized as follows. Let $k \geq 2$ be an integer. Let us denote the positive solution of the equation $k\Theta + k\Theta^2 = 1$ by Θ'_k, and the reciprocal value of this number by β'_k (this is the base of the system, and the positive solution of the equation $-k\beta + k\beta^2 = 1$). Thus

$$\Theta'_k = \frac{-k+\sqrt{k^2+4k}}{2k} \quad \text{and} \quad \beta'_k = \frac{k+\sqrt{k^2+4k}}{2}.$$

Proposition 3.1. *Let us consider the expansions of number $\xi = 0.(100)^\infty$ in the β'_k-based systems. Here parts '100' can be rewritten into parts '0kk' in every position. In this way we get finally $2^{\aleph_0}$ of different expansions, which can be represented by a full binary tree. Further ξ-expansions cannot be constructed.*

In the above-mentioned work ([6]) the authors – briefly – mentioned as well, that in the G-based system number g has an infinite number of periodic expansions. Roughly 15 years later, this problem arose again in a somewhat different approach, and then P. Erdős et al. presented all expansions of 1 in this system ([7]).[5] Their results are as follows.

Proposition ([7], [15]). *In the G-based system number* 1 *has countably many different expansions* (*on* $\mathcal{F}$).[6] *One of them is periodic* (*quasigreedy/quasi-regular*)*:*

$$1 = \frac{1}{G} + \frac{1}{G^3} + \frac{1}{G^5} + \frac{1}{G^7} + \cdots,$$

[5]This was the first such type of "complete" investigation.
[6]See Theorem [7]/1. below, too.

and besides this there exist for all $N = 0, 1, 2, \dots$ the two expansions:

$$1 = \left(\sum_{i=1}^{N} \frac{1}{G^{2i-1}} \right) + \frac{1}{G^{2N+1}} + \frac{1}{G^{2N+2}} \quad \textit{and} \quad 1 = \left(\sum_{i=1}^{N} \frac{1}{G^{2i-1}} \right) + \left(\sum_{i=2N+2}^{\infty} \frac{1}{G^{i}} \right).$$

In the original proof ([7]) the authors worked so, that they took the usable digits one after an other in the expansion: If $\varepsilon_1 = 0$, then the only way to continue is $1 = g^2 + g^3 + g^4 + \dots$. If $\varepsilon_1 = \varepsilon_2 = 1$, then also obligatorily only the expansion $1 = g + g^2$ is possible. The analysis can be continued similarly.

It is very intuitive, if according to the statement, we list the individual expansions:

Expansion of 1		Category, remark
$g + g^3 + g^5 + g^7 + \dots$	$0.(10)^\infty$	quasiregular
$g + g^2$	0.11	$N = 0$, case a); greedy
$g^2 + g^3 + g^4 + \dots$	0.01^∞	$N = 0$, case b); lazy
$g + g^3 + g^4$	0.1011	$N = 1$, case a)
$g + g^4 + g^5 + g^6 + \dots$	0.1001^∞	$N = 1$, case b)
$g + g^3 + g^5 + g^6$	0.101011	$N = 2$, case a)
$g + g^3 + g^6 + g^7 + g^8 + \dots$	0.101001^∞	$N = 2$, case b)
$\dots$	$\dots$	$\dots$

It is easy to observe that starting from the greedy form – always replacing the last 1-digit appropriately, using the greedy relation – arise all of the cases type-a) (these are the "greedy-clones"); and similarly, with the lazy relation, all of the cases type-b) (the "lazy-clones").

This observation may be used to construct all expansions of 1 in other – easy-to-describe – systems, too.

Let $k \geq 2$ be an integer. Let us denote the positive solution of $k\Theta + \Theta^2 = 1$ by Θ_k; and the reciprocal value of this number by β_k (this is the base of the system, and it is the positive solution of the equation $-k\beta + \beta^2 = 1$).[7] Thus

$$\Theta_k = \frac{-k + \sqrt{k^2 + 4}}{2} \quad \text{and} \quad \beta_k = \frac{k + \sqrt{k^2 + 4}}{2} = k + \Theta_k.$$

Hence, in these systems the greedy expansion of 1 (on $\mathcal{F}$) is $1 = 0.k1$, and the lazy form is $1 = 0.(k-1)k^\infty$ (the verification of the latter is left to the Reader). E. g. in the case $k = 2$ we have $S = \beta_2 = \sqrt{2} + 1$, $s = \Theta_2 = \sqrt{2} - 1$, the greedy expansion of 1 is $1 = 0.21$, and the lazy form is $1 = 0.12^\infty$.

Proposition 3.2. *In systems with base β_k number* 1 *has the following expansions:*

[7] These β_k-s separate the small and big cases, when determining the univoque numbers [11].

(i) $1 = 0.k1$ (*regular/greedy*)
(ii) $1 = 0.(k-1)k^\infty$ (*lazy*)
(iii) $1 = 0.(k0)^\infty$ (*quasiregular*)
(iv) $1 = 0.(k0)^i k1$ (*"greedy-clones"*)
(v) $1 = 0.(k0)^i(k-1)k^\infty$ (*"lazy-clones"*).

There are no other 1*-expansions in these systems.*

Proof. Suppose we have found a new 1-expansion, which does not belong to our list. Then – taking into account the facts mentioned above – this must be in the form $0.(k0)^i d\dots$ (separating every $k0$ part, for $i \geq 1$). In case $d = k-1$ the production would be a "lazy-clone" (the sequence would have been legally continued only in this way); case $d \leq k-2$ cannot be fulfilled either, because in this case the produced number would be less than 1; thus $d = k$.

The expansion cannot end here, since $0.(k0)^i k \neq 1$; let e be the digit following d. For $e = 1$ we would have a "greedy-clone"; cases $e = 2, \dots, k$ are not possible either (we would get so a number greater than 1); so $e = 0$. But now we have a contradiction with the fact that all parts $k0$ were separated. ∎

In Figure 2. we show these expansions in the $S = \sqrt{2}+1$-based system.

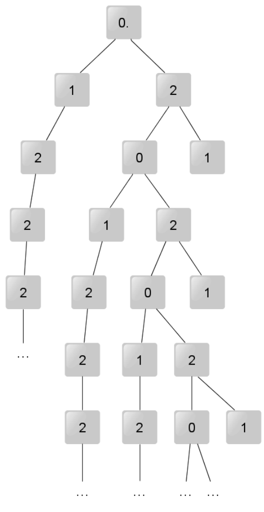

Figure 2. Expansions of 1 in the $\sqrt{2}+1$-based system

Now we are in the position that we are able to determine the expansions of several specific numbers. It is an obvious idea to extend the results obtained so far in the following manner.

Let $\eta \leq L$. If η's known greedy expansion – on the set of fractions – is $\eta = 0.\eta_1\eta_2\ldots_{(\beta)}$, then $0.0\eta_1\eta_2\ldots_{(\beta)}$ will be clearly the expansion of $\eta\Theta$. This is similarly true for any other expansions $\eta = 0.\eta_1'\eta_2'\ldots_{(\beta)}$ (if there exist more than one). It remains a question, however, that whether by inserting a digit 0 we really receive all expansions of $\eta\Theta$.

As it is well-known, if a number have two different expansions, these can differ in the first different position *by exactly one*. (Based on [6] for this the following short proof can be given. In such cases we have clearly $k \geq 2$ and $0.2 \leq 0.0k^\infty$. This is the same as $2\Theta \leq \frac{k\Theta^2}{1-\Theta}$ or $2(1-\Theta) \leq k\Theta$. But this yields $1 = 2(1-1/2) < 2(1-\Theta) \leq k\Theta = [\beta]\Theta < 1$, which is a contradiction.)

So, if we can find for $\eta\Theta$ such an expansion in which the first fraction-digit differs from 0, then this digit must be only 1. Such an expansion can really exist in the cases when $\eta_1 = k$, since $0.1 < 0.0k^\infty$ holds. Moreover, based on (2.3), it is also clear that $0.0(k-1)^\infty < 0.1$, so if $\eta_1 = k-1$, then we surely cannot find another such expansion.

However, if we expand η originally *not* on the set of fractions, we get the expansions of type $\eta = 1.\eta_1''\eta_2''\eta_3''\ldots_{(\beta)}$ already at the beginning of the procedure; e.g. following the greedy algorithm. So, all these forms will occur among the expansions of $\eta\Theta$, by shifting the digits to the right.

To sum up, we have proved the following statement:

Proposition 3.3. *We fix a base $\beta > 1$. Let us assume that in this system all possible expansions of a number $\eta \in (0, L)$ are given* (*not only on the set of fractions*)*. Then we can construct all possible expansions of numbers $\Theta^i\eta$ $(i \geq 1)$ too, by shifting the digits appropriately to the right, and supplementing the* (*beginning of the*) *sequences expanding η by a necessary number of zero digits.*

A completely similar result can be formulated in relation to numbers of form $k\Theta + \cdots + k\Theta^i + \Theta^i\eta$. Besides these, we can make similar constructions working with numbers $c_1\Theta + \cdots + c_i\Theta^i + \Theta^i\eta$, where $c_j \in \{1, 2, \ldots, k-1\}$.[8]

4. Expansions of 1

Up to this point we have presented how to give the expansion tree – by a formula or in a graphic way – in a case of a fixed base, for specific numbers or for groups of numbers.

Our objective is now to solve a more general problem; namely, let us give the number of possible expansions for a specific number x, concerning all (or as many as possible) β-bases! (First we work with bases $\beta \in (1, 2)$.)

[8] The base idea can be seen e.g. in paper [11].

Like many other similar problems, this question arose first specially in relation to number 1. Going back into the 1980s, the mathematicians working in this field believed at that time generally that considering number 1, it was possible to give infinitely many different expansions (1.7), for all bases β in $(1,2)$, i.e. in expansion

$$1 = \frac{\varepsilon_1}{\beta} + \frac{\varepsilon_2}{\beta^2} + \frac{\varepsilon_3}{\beta^3} + \cdots \tag{4.1}$$

can always appear infinitely many different sequences of digits $\varepsilon_i \in \{0,1\}$.

That's why it was surprising, when in 1991 P. Erdős and others proved that for a continuum of bases in $1 < \beta < 2$ only one such expansion exists (and moreover, in many cases, the number of possible expansions is countable). Their result (of that time) is as follows:

Theorem ([7]/1)**.** a) *For every base number* $1 < \beta < G$ *there exist* $2^{\aleph_0}$ (*continuum*) *different expansions for* 1 *in form* (4.1)*;*

b) *We can find* (*at least*) *countably many bases* $G \leq \beta < 2$*, for which there exist countably many different expansions for number* 1 *in form* (4.1)*;*

c) *There exist* $2^{\aleph_0}$ (*continuum*) *base numbers* $G < \beta < 2$*, for which the expansion of* 1 *is unique in form* (4.1)*.*

The original idea of the proof of case a) is the following ([7] and ([15]).

Since for base G we have $1 = g + g^2$, thus $g^n = g^{n+1} + g^{n+2}$. Using this $g^n = 2g^{n+2} + g^{n+3} = g^{n+2} + 2g^{n+3} + g^{n+4}, \ldots$ so finally $g^n = g^{n+2} + g^{n+3} + g^{n+4} + \ldots$ holds (for all indices n). If $1 < \beta < G$, then $1 < \Theta + \Theta^2$, and so we have $\Theta^n < \Theta^{n+2} + \Theta^{n+3} + \ldots$, which implies that we can find an index m with

$$\Theta^n < \Theta^{n+2} + \Theta^{n+3} + \cdots + \Theta^{n+m}.$$

Let us constitute an index sequence $\{n_j\}$, for which $n_{j+1} - n_j > m$. Then sequence $\{\Theta^n \mid n \neq n_j\} = \{\lambda_i\}$ satisfies the following two conditions:

$$\lambda_1 > \lambda_2 > \ldots, \text{ and } \lambda_n < \lambda_{n+1} + \lambda_{n+2} + \ldots .$$

Thus, sub-sums of $\sum_{n=1}^{\infty} \lambda_n$ run through the interval $[0, \sum_1^{\infty} \lambda_n]$. If we choose n_1 large enough, then $\sum_{n=1}^{\infty} \lambda_n > 1 > \sum_{j=1}^{\infty} \Theta^{n_j}$ will hold.

From this it follows that for arbitrary sub-sum $\sum_{j=1}^{\infty} \varepsilon_j \Theta^{n_j}$ ($\varepsilon_j = 0$ or 1) we can find a sequence $\{\delta_n\}$ $0-1$, for which

$$\sum_{j=1}^{\infty} \varepsilon_j \Theta^{n_j} + \sum_{n=1}^{\infty} \delta_n \lambda_n = 1,$$

and so the desired $2^{\aleph_0}$ different expansions are constructed.

Vividly described, the point here is, that if $1 < \beta < G$, then the expansion tree becomes more arborescent than that is in the G-based system, namely a binary sub-tree can be identified in it. This ensures the existence of the continuum different expansions.

Analysing special systems, similar behaviour can be seen in cases $k < \beta < < \beta_k$, too.

Sub-case $\beta = G$ from part b) is already presented above. For cases $\beta > G$ let us consider the following clever argument ([7]). Fix a base β, for which $1 = \Theta + \Theta^2 + \cdots + \Theta^m$ holds, with an index $m \geq 3$ (for every m we get different bases). Similarly as in part a), then

$$\Theta^n > \Theta^{n+2} + \Theta^{n+3} + \Theta^{n+4} + \dots , \tag{4.2}$$

i.e. two consecutive zero digits are not allowed. Therefore, however, for 1-expansions we have only the following possibilities:

$$1 = \Theta + \Theta^2 + \cdots + \Theta^m,$$
$$1 = \Theta + \cdots + \Theta^{m-1} + \Theta^{m+1} + \Theta^{m+2} + \cdots + \Theta^{2m},$$
$$1 = \Theta + \cdots + \Theta^{m-1} + \Theta^{m+1} + \cdots + \Theta^{2m-1} + \Theta^{2m+1} + \Theta^{2m+2} + \cdots + \Theta^{3m},$$
$$\vdots$$
$$1 = \sum_{n \geq 1; m \nmid n} \Theta^n.$$

To prove that really no other expansions can exist, let us consider the case when we choose a digit 0 into position $m-1$. But then using (4.2)

$$\Theta + \Theta^2 + \cdots + \Theta^{m-2} + \Theta^m + \Theta^{m+1} + \Theta^{m+2} + \cdots < 1 = \Theta + \Theta^2 + \cdots + \Theta^m.$$

If we have written digits 1 in the first $m-1$ positions already, and digit 0 in the m-th position, then in the following we have to choose $\varepsilon_{m+1} = \varepsilon_{m+2} = = \cdots = \varepsilon_{2m-1} = 1$, since with $\varepsilon_{2m-1} = 0$ using (4.2) it follows

$$\Theta + \cdots + \Theta^{m-1} + \Theta^{m+1} + \cdots + \Theta^{2m-2} + \Theta^{2m} + \Theta^{2m+1} + \Theta^{2m+2} + \cdots <$$
$$< \Theta + \cdots + \Theta^{m-1} + \Theta^{m+1} + \Theta^{m+2} + \cdots + \Theta^{2m} = 1.$$

The analysis can be continued similarly. By this, the existence of the countably many expansions is proved.

Generalization for $k = 2$: Case S has already been treated above.

Let now base β be fixed so, that $1 = 2\Theta + \Theta^2 + \cdots + \Theta^m$ holds, for an index $m \geq 3$. Then we have

$$\Theta^n > \Theta^{n+1} + 2\Theta^{n+2} + 2\Theta^{n+3} + 2\Theta^{n+4} + \dots . \tag{4.3}$$

For the 1-expansions now we have the following possibilities:

$$\begin{aligned}
&1 = 2\Theta + \Theta^2 + \cdots + \Theta^m,\\
&1 = 2\Theta + \Theta^2 + \cdots + \Theta^{m-1} + 2\Theta^{m+1} + \Theta^{m+2} + \cdots + \Theta^{2m},\\
&1 = 2\Theta + \Theta^2 + \cdots + \Theta^{m-1} + 2\Theta^{m+1} + \Theta^{m+2} + \cdots +\\
&\quad + \Theta^{2m-1} + 2\Theta^{2m+1} + \Theta^{2m+2} + \cdots + \Theta^{3m}, \quad \cdots .
\end{aligned}$$

With the same argument as above, we can conclude that no other expansions exist. Namely, let us decrement the last "stable" digit 1, then by (4.3) we have

$$2\Theta + \Theta^2 + \cdots + \Theta^{m-2} + 2\Theta^m + 2\Theta^{m+1} + 2\Theta^{m+2} + \cdots <$$
$$< 2\Theta + \Theta^2 + \cdots + \Theta^{m-2} + \Theta^{m-1} + \Theta^m = 1$$

If in the first $m-1$ positions we have already written digits $2, 1, \dots, 1$, and in the m-th position digit 0, then in the following we have to choose $\varepsilon_{m+1} = 2$, $\varepsilon_{m+2} = \cdots = \varepsilon_{2m-1} = 1$, since with $\varepsilon_{2m-1} = 0$ by (4.3) we have

$$2\Theta + \Theta^2 + \cdots + \Theta^{m-1} + 2\Theta^{m+1} + \Theta^{m+2} + \cdots + \Theta^{2m-2} + 2\Theta^{2m} + 2\Theta^{2m+1} + \cdots <$$
$$< 2\Theta + \Theta^2 + \cdots + \Theta^{m-1} + 2\Theta^{m+1} + \Theta^{m+2} + \cdots + \Theta^{2m-2} + \Theta^{2m-1} + \Theta^{2m} = 1.$$

Continuing similarly, the existence of the countably many expansions derives.

More generally, by building cases $k \geq 3$ in, the following assertion results:

Theorem 4.1. *We can find* (*at least*) *countably many bases* $\beta_k \leq \beta < k+1$ *for which countably many different expansions exist in form* (4.1) *for number* 1. *For a fixed* m*, these expansions are as follows:*

$$\begin{aligned}
&1 = k\Theta + \Theta^2 + \cdots + \Theta^m,\\
&1 = k\Theta + \Theta^2 + \cdots + \Theta^{m-1} + k\Theta^{m+1} + \Theta^{m+2} + \cdots + \Theta^{2m},\\
&1 = k\Theta + \Theta^2 + \cdots + \Theta^{m-1} + k\Theta^{m+1} + \Theta^{m+2} + \cdots +\\
&\quad + \Theta^{2m-1} + k\Theta^{2m+1} + \Theta^{2m+2} + \cdots + \Theta^{3m}, \quad \cdots .
\end{aligned}$$

Proof. This can be carried out very similarly, as it was presented above, by using inequality

$$\Theta^n > (k-1)\Theta^{n+1} + k\Theta^{n+2} + k\Theta^{n+3} + k\Theta^{n+4} + \dots . \tag{4.4}$$ ■

To the proof of case c): We will revert to this topic in the planned next part of this article, when the unique expansions will be handled in more detail. Here we just mention that the interesting paper [12] deals with this subject, too.

P. Erdős and I. Joó – just shortly after describing results in [7] – also showed that the number of different 1-expansions (for arbitrary many bases) may even be a fixed positive integer ([8], 1992)!

Theorem ([8]). *For every integer $n \geq 1$ we can find $2^{\aleph_0}$ (continuum) base numbers β in $(1, 2)$, for which in these systems number 1 has exactly n different expansions in form (4.1).*

The authors gave the production of the base numbers β and that of the corresponding expansions, too. For a fixed positive integer n let

$$1 = \sum_{i=1}^{\infty} \varepsilon_i \Theta^i = 0.1^9(0^91)^{n-1}(0a001)^{\infty}, \tag{4.5}$$

here a can be chosen freely to 0 and 1 in each cycle. Then in (4.5), single digits 1 at the end of the middle block can be exchanged into the full digit-sequence (ε_i) in (4.5) (starting with a sub-block containing only pure 1-digits); and totally $n-1$ pieces of different substitutions can be carried out.

As an example, for $n = 2$ in cases $a = 0$ or $a = 1$ the substitution carries out as follows:

	block 1.	block 2.	"mixed" part	repeating part
	11111111$\cancel{1}$	0000000001	$0a0010a$	$(0010a)^{\infty}$
+		1111111110	0000000	$(010a0)^{\infty}$
	111111110	1111111111	$0a0010a$	$(011aa)^{\infty}$

An extra substitution cannot be performed, hence there are exactly two different expansions.

The original proof of the theorem was presented in three steps ([8]).

a) If in an expansion (4.5) there is a digit 1 (m-th position), and in the subsequent eight positions somewhere we have at least one digit 1, then ε_m cannot be changed into 0 (without changing the preceding digits), because

$$0.\varepsilon_1\varepsilon_2 \ldots \varepsilon_{m-1}01^{\infty} < 1. \tag{4.6}$$

To prove this, let us fix $\varepsilon_m = 0$, and from the $m+1$-th position let us add the shifted digit-sequence (ε_i) to the original one. Then, in each of the next eight positions there will be surely at least one digit 1, but at least one place we will have two 1-s. Here we replace again, by adding the shifted sequence. Going on similarly, finally all digits 0 will be eliminated, and in all positions $n \geq m+1$, we will have at least one digit 1, but in several places even more (at most n). This expansion-sequence gives exactly 1, so (4.6) really holds.

b) If in expansion (4.5) there is a digit 0 (m-th position), and in the subsequent eight positions somewhere we have at least one digit 0, then we cannot change ε_m into 1 (without changing the preceding digits), because

$$0.\varepsilon_1\varepsilon_2 \ldots \varepsilon_{m-1}10^{\infty} > 1. \tag{4.7}$$

Let be $\varepsilon_m = 1$, and from the $m+1$-th position let us add the shifted sequence (ε_i) to the original one, but with opposite sign. Then, somewhere in the next eight positions – at least in one place – digit -1 will not be neutralized by an original 1, so it remains -1; moreover, in position $m+9$ we will have a digit ≤ 0. Let us eliminate digits -1-s from positions $\leq m+8$, by adding the appropriately shifted sequence again, with opposite sign. So, in position $m+9$ we will get a digit ≤ -1. Going on similarly, finally in all positions $\geq m+1$ here will be digits ≤ 0, but in several places even < 0-s will appear. This expansion-sequence gives exactly 1, so (4.7) holds.

c) Let us find now in expansion (4.5) the first digit, which can be changed. According to point b), this can be only a 1, and from point a) it follows that it must be included in those $n-1$ digits, which are followed by 9 pieces of digits 0. Let us consider the m-th such digit, which is ε_{10m-1}. Let us introduce the concept of m-th expansion, for the one in which we leave ε_{10m-1}, and instead of it we write in the appropriately shifted version of (4.5) (starting from position $10m$); getting so an additional 1-expansion. Then the m-th expansion is unique by fixing the first $10m-1$ digits of it, and we have no other 1-expansions besides (4.5) and the m-th expansions.

The original idea of the theorem *can be adapted* to the systems, where $k=2$. Let us define β with the equation

$$1 = \sum_{i=1}^{\infty} \varepsilon_i \Theta^i = 0.2^9(0^9 1)^{n-1}(0a001)^{\infty} \tag{4.8}$$

(the role of a is the same, as above). Then e.g. for $n=2$ the following substitution can be carried out:

	block 1.	block 2.	"mixed" part	repeating part
	22222222$\cancel{2}$	0000000001	$0a0010a$	$(0010a)^{\infty}$
+		2222222220	0000000	$(010a0)^{\infty}$
	222222221	2222222221	$0a0010a$	$(011aa)^{\infty}$

It is not hard to prove that other expansions do not exist.

Now we formulate this result even generally. For simplicity, let us remain only by cases $n=2$.

Theorem 4.2. *For all integers $k \geq 2$ we can find $2^{\aleph_0}$ base numbers β in $(k, k+1)$ so, that in the systems number* 1 *has exactly two different expansions. The expansions are given by* (4.9). *Here*

$$1 = \sum_{i=1}^{\infty} \varepsilon_i \Theta^i = 0.k^9 0^9 1(0a001)^{\infty}, \tag{4.9}$$

where the role of a is the same, as it was above.

Proof. Analogously to the original theorem, following points a), b) and c) above.

a) If in an expansion (4.9) there is a digit d (m-th position; here $1 \leq d \leq k$, but we need only cases $d = 1$ and $d = k$), and in the subsequent eight positions somewhere we have at least one digit d, then ε_m cannot be changed into $d - 1$ (without changing the preceding digits), because

$$0.\varepsilon_1\varepsilon_2\ldots\varepsilon_{m-1}(d-1)k^{\infty} < 1. \tag{4.10}$$

Let $\varepsilon_m = d - 1$, and from the $m+1$-th position let us add the shifted digit-sequence (ε_i) to the original one. Then, in each of the next eight positions there will be at least a k-value digit, but at least in one place we will have a digit $k + d$. Here we replace again, and add the shifted sequence. Going on similarly, finally in all positions with indices $n \geq m + 1$, we will have at least a k-value digit, but in several places even greater value appear (at most $n \cdot k - 1$). This sequence generates exactly 1, so (4.10) really holds.

b) If in an expansion (4.9) there is a digit 0 (m-th position), and in the subsequent eight positions somewhere we have at least one digit 0, then we cannot change ε_m into 1 (without changing the preceding digits), because

$$0.\varepsilon_1\varepsilon_2\ldots\varepsilon_{m-1}10^{\infty} > 1. \tag{4.11}$$

Let be $\varepsilon_m = 1$, and from the $m+1$-th position let us add the shifted sequence (ε_i) to the original one, but with opposite sign. Then in the following eight positions $-k$-value and $-k+1$-value digits will appear, respectively. Continuing the elimination results, that in every position $\geq m + 1$ there is a digit ≤ 0, but in several positions digits even < 0. This expansion-sequence gives exactly 1, so (4.11) really holds.

c) Let us find now in expansion (4.5) the first digit, which can be changed. According to points a) and b), this can be only a k, and from a) it follows that this must be only that, which is followed by nine digits 0 (so, this digit is in position 9). Based on the facts mentioned above, this digit can be decreased by 1, and after it (starting from position 10) we can write in the appropriately shifted version of (4.9); thereby getting an additional 1-expansion.

This clearly can be carried out; if, however, the first 9 digits are fixed, then the expansion is already uniquely determined, so we cannot write down a new expansion (additionally to the two already existing). ∎

We note that base numbers determined by expansion (4.9) are all close to $k + 1$ (roughly by 1-2 thousandth below it).

5. Number of expansions – general

After the investigation of 1-expansions now our next objective is – for intervals of base numbers – to characterize comprehensively (or possibly: even to give) the number of expansions of elements in $[0, L]$.

Let us look into the Eggan-Eynden paper again ([6], 1966)! As we will see, the following important result can be considered as a "first arrival", leading up to the forthcoming far-reaching investigations.[9]

Theorem ([6]/2)**.** a) *For bases $\beta > G$ there exist infinitely many numbers, which have only one β-based expansion.*

b) *For bases $G \geq \beta (> 1)$ every positive number has infinitely many different β-based expansions.*

The original proof of the authors is a constructive one, in fact they produce infinitely many proper numbers/forms ([6]).

a1) Let first be $G < \beta < 2$. Then number $\xi_1 = 0.(01)^\infty$ has only one expansion. Let us assume the contrary, that another expansion exists, and this can be written in form $\xi_1 = 0.a_1a_2a_3\ldots$. Then from $\xi_1 < 0.1$ we have $a_1 = a_2 = 0$. Using this $\xi_1 = 0.(01)^\infty = 0.00a_3 \cdots \leq 0.001^\infty$. Let us subtract $0.00(01)^\infty$ from both sides, this results $0.01 \leq 0.0(01)^\infty$. However, by multiplying β, from this derives $\xi_1 \geq 0.1$, which is a contradiction.

a2) If $\beta > 2$, then number $\xi_2 = 0.1^\infty$ has only one expansion, too. Let us suppose the contrary, that another expansion exists, in form $\xi_2 = 0.a_1a_2a_3\ldots$. We may assume that $a_1 \neq 1$. If $a_1 = 0$, then $0.0k^\infty > 0.1^\infty$, so $0.0(k-1)^\infty > 0.1$ must hold; whilst if $a_1 = 2$, then $0.1^\infty > 0.2$, so $0.01^\infty > 0.1$ is necessary. But

$$0.01^\infty \leq 0.0(k-1)^\infty = \frac{(k-1)/\beta^2}{1-1/\beta} = \frac{k-1}{\beta^2-\beta} < \frac{\beta-1}{\beta^2-\beta} = \frac{1}{\beta} = 0.1_{(\beta)},$$

i.e. we have surely $a_1 = 1$.

The original proof in parts a1) and a2) ends with the conclusion that forms $\beta^{-i}\xi_1$ and $\beta^{-i}\xi_2$ are even unique ($\forall i \geq 1$).

b) Then $\beta \leq G$, so $k = 1$. To the examined number ξ let us consider $\overline{\xi}$, where $\overline{\xi} = L - \xi = 0.11\cdots - \xi$. Their expansions are complements of each other, so these two numbers have the same number of expansions.

The authors distinguish in this section three cases as follows.

b1) All expansions of number ξ_1 are eventually constant (1 or 0).

[9] Their investigation was carried out on the whole number line, not in $[0, L]$, but – as we have already mentioned – this does not influence the point.

Let us assume that $\xi_1 = 0.a_1a_2\dots a_u 1$. Then based on lemma presented in Section 2, we can write $\xi_1 = 0.a_1a_2\dots a_u 0b_1b_2\dots$, too; here the b_i-s are not all zero. So, for ξ_1 we have found an expansion, where the index of the constant part is greater than before. Repeating this procedure, the constant part can be pushed out arbitrarily far away.

When the expansion of ξ_1 ends with 1^∞, then we work similarly with $\overline{\xi_1}$.

b2) Number ξ_2 has an expansion, in which parts 100 or 011 appear infinitely many times.

Let us assume, that in the expansion part 100 appears infinitely many times (in the opposite case we work with $\overline{\xi_2}$, in the same manner); and $\xi_2 = 0.a_1a_2\dots a_u 100a_{u+4}\dots$. From $\beta \le G$ it follows $0.011 \ge 0.100$, so we can write even $\xi_2 = 0.a_1a_2\dots a_u 0b_1b_2\dots$ (for some b_i-s). This change can be carried out in infinitely many places, and by this we always get new expansions.

b3) Number ξ_3 has an expansion, which ends with $(01)^\infty$.

From $\beta \le G$ it follows $0.1 \le 0.(01)^\infty$. Thus, $0.(01)^\infty = 0.1b_2b_3\dots$, for some b_i-s. In the expansion of ξ_3 this change can be carried out in infinitely many places, and by this we always get new expansions.

To the points we add the following comments.

a1) In the topic of numbers with unique expansions – cases $\beta \in (G, 2)$ – the first exhaustive, revealing investigation was carried out by Z. Daróczy and I. Kátai ([3] and [4]; 1993 and 1995). They called the numbers with unique expansions (and the sequences representing them) *univoque*. In paper [3] they proved that if $\beta \in (G, q(2))$ (here base $q(2)$ is determined by $1 = 0.1101$, $q(2) \approx$ ≈ 1.7549), then for each base the following univoque elements can be found: number $\xi_1 = 0.(01)^\infty$ above and its variants $0.0\dots 0(01)^\infty$ and $0.1\dots 1(01)^\infty$, respectively. (This is a so-called stable segment.)

a2) The extension of the investigation of univoque numbers for cases $\beta > 2$ was started a few years later (I. Kátai and G. Kallós). Results formulated in paper [11] also prove that if $\beta \in (2, S)$, then only number $\xi_2 = 0.1^\infty$ presented above and its variants $0.0\dots 01^\infty$ and $0.2\dots 21^\infty$ are univoque. If $\beta \in (S, q'(2))$ (here base $q'(2)$ is determined by $1 = 0.2102$), then already $\xi_2 = 0.(02)^\infty$ can be an appropriate ”raw material” and so on.

The result of point b) was improved significantly in 1990 (P. Erdős, I. Joó and V. Komornik, [9]). Based on their former result on 1-expansions (Theorem [7]/1; above), the authors developed here the investigation to the whole interval $[0, L]$ in cases $1 < \beta < G$. With this we can say that the analysis is now – from such a point of view – complete for the given bases:

Theorem ([9]/3). *If $1 < \beta < G$ and $0 < x < L$, then number x has $2^{\aleph_0}$ (continuum) different expansions.*

Here – besides the original proof ([9]) – it is worth seeing the elegant, modified new version, presented by V. Komornik in 2011 ([15]).

The result of this theorem cannot be transformed automatically to cases either $\beta > G$ or $\beta > 2$, since here we can always find numbers with only one expansion, too. However, besides this it is true, that the existence of the continuum many expansions fails only in "relatively few" cases. For bases $\beta < 2$ the proof was given by N. Sidorov, in 2003:

Theorem ([18]/1). *For arbitrary fixed bases $\beta \in (1, 2)$ – so essentially: for bases $\beta \in [G, 2)$ – almost every number $x \in [0, L]$ has continuum many different β-based expansions.*

The same result was proven a few years later generally – with an ergodic approach – for arbitrary $\beta > 1$; including even the cases $\beta > 2$ (K. Dajani, M. de Vries, 2007, [2]).

N. Sidorov gave even a complete classification for base numbers $\beta \in (1, 2)$ following that how many such elements can be found in $[0, L]$, which have *no* uncountably many different expansions.

Theorem ([19]/3.6). *The set of such x-s in $[0, L]$ which have less than continuum different expansions...*

- *... consist of only the endpoints 0 and L, when $\beta < G$;*
- *... is countably infinite – the subset of $\mathbb{Q}(\beta)$ –, if $G \le \beta < \beta_*$;*
- *... is uncountable, with Hausdorff dimension 0, when $\beta = \beta_*$;*
- *... is uncountable, with Hausdorff dimension > 0, but < 1, if $\beta_* < \beta < 2$.*

Here number β_* is the so-called Komornik–Loreti-constant, with $\beta_* \approx 1.7872$ ([15]).

In Figure 3. we present the intervals corresponding to the points of the theorem, with appropriate colouring (only 0 and L – dotted blue; countably infinite – dashed red; continuum with Hausdorff dimension 0 – lime big dot; continuum with Hausdorff dimension $0 < D_H < 1$ – brown). On the x-axis we can see bases β, the inner separating points are G and β_*, respectively.

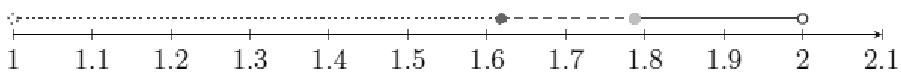

Figure 3. Numbers in $[0, L]$ with less than continuum distinct expansions

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