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CHARACTERIZATION OF ARITHMETICAL FUNCTIONS WITH FUNCTIONAL EQUATION

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Abstract. The functional equation of type

$$f(\alpha + n^3 + m^3) = g(\alpha) + h(n^3) + h(m^3)$$

is investigated, where $n, m \in \mathbb{N}, \alpha \in \mathcal{A}$ and $\mathcal{A} \subseteq \mathbb{N}$ satisfies some conditions. It follows from our results that if $\mathcal{A} = \mathcal{P}$ (the set of all prime numbers), then there exist numbers A, D, Q such that $h(n^3) = An^3 + D$ and $g(p) = Ap + Q$ for every $p \in \mathcal{P}$, $n \in \mathbb{N}$. Similarly, if $\mathcal{A} = \{n^2 | n \in \mathbb{N}\}$, then $h(n^3) = An^3 + D$ and $g(m^2) = Am^2 + R$ for every $n, m \in \mathbb{N}$, where A, D, R are suitable numbers.

1. Introduction

Let $\mathcal{P}, \mathbb{N}$ and $\mathbb{C}$ be the set of primes, positive integers and complex numbers, respectively. For the sets $\mathcal{A},\ \mathcal{B} \subseteq \mathbb{N}$ we define $\mathcal{A}+\mathcal{B}, \mathcal{A}+2\mathcal{B}, \mathcal{A}-\mathcal{B}$ as follows:

$$\mathcal{A}+\mathcal{B} := \{a+b \mid a \in \mathcal{A}, b \in \mathcal{B}\}, \quad \mathcal{A}+2\mathcal{B} := \{a+b+b' \mid a \in \mathcal{A}, b, b' \in \mathcal{B}\}$$

and

$$\mathcal{A}-\mathcal{B} := \{a-b \mid a \in \mathcal{A}, b \in \mathcal{B}, a > b\}.$$

Let

$$\mathfrak{M} := \{p_1 + p_2 + p_3 \mid p_1, p_2, p_3 \in \mathcal{P}\}.$$

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Recently, by using the result of H. A. Helfgott [1] concerning the ternary Goldbach conjecture, I. Kátai and B. M. Phong [2] proved that if the functions $f:\mathfrak{M}\to\mathbb{C}$, $g:\mathcal{P}\to\mathbb{C}$ satisfy the condition

$$f\,(p_1+p_2+p_3)=g(p_1)+g(p_2)+g(p_3)$$

for every $p_1,p_2,p_3\in\mathcal{P}$, then there exist suitable constants $A,B\in\mathbb{C}$ such that

$$f(n)=An+3B\quad\text{and}\quad g(p)=Ap+B\quad\text{for all}\quad n\in\mathfrak{M},p\in\mathcal{P}.$$

It is proved in [3] that if the sets

$$\mathcal{A}=\{a_1<a_2<\cdots\}\subseteq\mathbb{N},\quad \mathcal{B}:=\{m^2\mid m\in\mathbb{N}\}$$

and the arithmetical functions $f:\mathcal{A}+\mathcal{B}\to\mathbb{C}$, $g:\mathcal{A}\to\mathbb{C}$ and $h:\mathcal{B}\to\mathbb{C}$ satisfy the equation

$$f(a+n^2)=g(a)+h(n^2)\quad\text{for all}\quad a\in\mathcal{A},n\in\mathbb{N},$$

then the assumption $8\mathbb{N}\subseteq\mathcal{A}-\mathcal{A}$ implies that there is a complex number A such that

$$g(a)=Aa+\widetilde{g}(a),\ \ h(n^2)=An^2+\widetilde{h}(n)\quad\text{and}\quad f(a+n^2)=A(a+n^2)+\widetilde{g}(a)+\widetilde{h}(n)$$

hold for all $a\in\mathcal{A},n\in\mathbb{N}$, furthermore

$$\widetilde{g}(a)=\widetilde{g}(b)\quad\text{if}\quad a\equiv b\pmod{120},\quad (a,b\in\mathcal{A})$$

and

$$\widetilde{h}(n)=\widetilde{h}(m)\quad\text{if}\quad n\equiv m\pmod{60},\quad (n,m\in\mathbb{N}).$$

By assuming the unknown hypothesis that every positive number of the form $8n$ is the difference of two primes, it follows from [3] that

$$F(p+n^2)=G(p)+H(n^2)\quad\text{for all}\quad p\in\mathcal{P},n\in\mathbb{N}.$$

then there are complex numbers A,A_2,D such that

$$G(p)=Ap+G(\overline{p})-A\overline{p},\quad G(2)=A+G(1)+A_2,$$
$$H(n^2)=An^2+A_2\chi_2(n)+D$$

and

$$F(p+n^2)=A(p+n^2)+G(\overline{p})-A\overline{p}+A_2\chi_2(n)+D$$

for all $p\in\mathcal{P}\setminus\{2\},n\in\mathbb{N}$, where $\chi_2(n)$ is the Dirichlet character (mod 2), that is $\chi_2(0)=0,\chi_2(1)=1$.

The equation $f(p+n^4+m^4)=g(p)+h(n^4)+h(m^4)$ is investigated in [5]. Some similar result was proved for this equation.

In this paper we shall prove the following

Theorem 1. *Assume that the sets*

$$\mathcal{A} = \{a_1 < a_2 < \cdots\} \subseteq \mathbb{N}, \quad \mathcal{B} := \{m^3 \mid m \in \mathbb{N}\}$$

and the arithmetical functions $f : \mathcal{A} + 2\mathcal{B} \to \mathbb{C}$, $g : \mathcal{A} \to \mathbb{C}$ *and* $h : \mathcal{B} \to \mathbb{C}$ *satisfy the equation*

$$f(\alpha + n^3 + m^3) = g(\alpha) + h(n^3) + h(m^3) \quad \textit{for all} \quad \alpha \in \mathcal{A}, n, m \in \mathbb{N}. \tag{1.1}$$

If

$$126 \in \mathcal{A} - \mathcal{A} \quad \textit{and} \quad 3402 \in \mathcal{A} - \mathcal{A}, \tag{1.2}$$

then there are complex numbers A, B, C, D *and functions* $F : \mathcal{A} + 2\mathcal{B} \to \mathbb{C}$, $G : \mathcal{P} \to \mathbb{C}$ *such that*

$$h(n^3) = An^3 + B\chi_7(n) + C\chi_3(n) + D \quad \textit{for all} \quad n \in \mathbb{N},$$

$$g(\alpha) = A\alpha + G(\alpha), \quad G(\alpha) = O(1)$$

and

$$f(\beta) = A\beta + F(\beta), F(\alpha + n^3 + m^3) = G(\alpha) + H(n) + H(m)$$

holds for $\alpha \in \mathcal{A}$, $n, m \in \mathbb{N}$ *and* $\beta \in \mathcal{A} + 2\mathcal{B}$, *where* $\chi_3(n) \pmod 3$, $\chi_7(n) \pmod 7$ *are non-principal Dirichlet characters, i.e.*

$$\chi_3(0) = 0, \chi_3(1) = 1, \chi_3(2) = -1,$$

$$\chi_7(0) = 0, \chi_7(1) = \chi_7(2) = \chi_7(4) = 1, \chi_7(3) = \chi_7(5) = \chi_7(6) = -1.$$

Corollary 1. *Let* $\mathcal{A} := \{M, M + 126, M + 3402\} \subseteq \mathbb{N}$. *If*

$$f(\alpha + n^3 + m^3) = g(\alpha) + h(n^3) + h(m^3) \quad \textit{for every} \quad \alpha \in \mathcal{A}, n, m \in \mathbb{N},$$

then all assertions of Theorem 1 *are satisfied.*

Corollary 2. *Assume that the arithmetical functions* f, g, h *satisfy the condition*

$$f(p + n^3 + m^3) = g(p) + h(n^3) + h(m^3) \quad \textit{for all} \quad p \in \mathcal{P}, n, m \in \mathbb{N}.$$

Then there are complex numbers A, D, Q *such that*

$$h(n^3) = An^3 + D, \quad g(p) = Ap + Q, \quad f(\beta) = A\beta + Q + 2D$$

hold for $p \in \mathcal{P}$, $n \in \mathbb{N}$ *and* $\beta \in \mathcal{P} + 2\mathcal{B}$.

By using a similar argument as in the proof of Theorem 1, we could prove the following

Theorem 2. *Assume that the sets*

$$\mathcal{A} = \{a_1 < a_2 < \cdots\} \subseteq \mathbb{N}, \quad \mathcal{B} := \{m^3 \mid m \in \mathbb{N}\}$$

and the arithmetical functions $f : \mathcal{A} + 2\mathcal{B} \to \mathbb{C}$, $g : \mathcal{A} \to \mathbb{C}$ *and* $h : \mathcal{B} \to \mathbb{C}$ *satisfy the equation* (1.1)*. If*

$$1008 \in \mathcal{A} - \mathcal{A} \quad \text{and} \quad 8064 \in \mathcal{A} - \mathcal{A}, \tag{1.3}$$

then all assertions of Theorem 1 *are satisfied.*

Corollary 3. *Let* $\mathcal{A} := \{M, M + 1008, M + 8064\} \subseteq \mathbb{N}$*. If*

$$f(\alpha + n^3 + m^3) = g(\alpha) + h(n^3) + h(m^3) \quad \text{for every} \quad \alpha \in \mathcal{A}, n, m \in \mathbb{N},$$

then all assertions of Theorem 1 *are satisfied.*

Corollary 4. *Assume that the arithmetical functions* f, g, h *satisfy the condition*

$$f(k^2 + n^3 + m^3) = g(k^2) + h(n^3) + h(m^3) \quad \text{for every} \quad k, n, m \in \mathbb{N}.$$

Then there are complex numbers A, D, R *such that*

$$h(n^3) = An^3 + D, \quad g(k^2) = Ak^2 + R, \quad f(\beta) = A(\beta) + R + 2D$$

hold for $k, n \in \mathbb{N}$ *and* $\beta \in \mathcal{P} + 2\mathcal{B}$*.*

Remark 1. It was proved in [4] that if $f : \mathbb{N} \to \mathbb{C}$ is multiplicative, and

$$f(p + m^3) = f(p) + f(m^3), \quad f(\pi^2) = f(\pi)^2$$

for all $p, \pi \in \mathcal{P}$ and $m \in \mathbb{N}$, then $f(n) = n$ for $n \in \mathbb{N}$.

Remark 2. Theorem 1 and Theorem 2 remain valid if f, g, h maps into an arbitrary Abelian group.

Remark 3. We hope that Theorem 1 and Theorem 2 remain valid if f, g, h satisfy (1.1) without (1.2) and (1.3).

2. Lemmas

In this section, we assume that the arithmetical functions f, g, h satisfy (1.1) and (1.2). Let

$$S_n := h(n^3).$$

Lemma 1. *For every $n \in \{5, 6, \cdots, 24\}$, we have*

$$S_n = c_1(n)S_1 + c_2(n)S_2 + c_3(n)S_3 + c_4(n)S_4, \tag{2.1}$$

where $c_1(n), c_2(n), c_3(n), c_4(n)$ are given in the following table:

Table 1

n	$(c_1(n), c_2(n), c_3(n), c_4(n))$	n	$(c_1(n), c_2(n), c_3(n), c_4(n))$
5	$(-2,\ \frac{1}{2},\ 1,\ \frac{3}{2})$	15	$(-53,\ \frac{1}{2},\ 0,\ \frac{107}{2})$
6	$(-3,\ 0,\ 1,\ 3)$	16	$(-64,\ 0,\ 0,\ 65)$
7	$(-\frac{9}{2},\ -\frac{1}{4},\ \frac{1}{2},\ \frac{21}{4})$	17	$(-78,\ \frac{1}{2},\ 1,\ \frac{155}{2})$
8	$(-8,\ 1,\ 0,\ 8)$	18	$(-92,\ \frac{1}{2},\ 0,\ \frac{185}{2})$
9	$(-11,\ \frac{1}{2},\ 0,\ \frac{23}{2})$	19	$(-108,\ -\frac{1}{2},\ 1,\ \frac{217}{2})$
10	$(-15,\ -\frac{1}{2},\ 1,\ \frac{31}{2})$	20	$(-127,\ \frac{1}{2},\ 1,\ \frac{253}{2})$
11	$(-21,\ 1,\ 0,\ 21)$	21	$(-\frac{293}{2},\ \frac{1}{4},\ \frac{1}{2},\ \frac{587}{4})$
12	$(27,\ 0,\ 1,\ 27)$	22	$(-168,\ 0,\ 0,\ 169)$
13	$(-34,\ -\frac{1}{2},\ 1,\ \frac{69}{2})$	23	$(-193,\ 1,\ 0,\ 193)$
14	$(-\frac{87}{2},\ \frac{3}{4},\ \frac{1}{2},\ \frac{173}{4})$	24	$(-219,\ 0,\ 1,\ 219)$

Proof. First we note from (1.2) that there are $u_1, u_1', u_2, u_2' \in \mathcal{A}$ such that

$$126 = u_1 - u_1' \quad \text{and} \quad 3402 = u_2 - u_2'.$$

Let

$$E_1 := g(u_1) - g(u_1') \quad \text{and} \quad E_2 := g(u_2) - g(u_2').$$

For numbers $a, b, c, d \in \mathbb{N}$, we define I_1, I_2 as follows:

$$I_1 := \{(a, b, c, d)|\ \ a^3 + b^3 - c^3 - d^3 = 126 = u_1 - u_1'\}$$

and

$$I_2 := \{(a, b, c, d)|\ \ a^3 + b^3 - c^3 - d^3 = 3402 = u_2 - u_2'\}.$$

It is obvious from (1.1) that

$$S_a + S_b - S_c - S_d = E_i \quad \text{if} \quad (a, b, c, d) \in I_i \ \ (i = 1, 2). \tag{2.2}$$

By applying Maple program, we computed that the following elements (a, b, c, d) are

in I_1: (4, 4, 1, 1), (1,6,3, 4), (5, 9, 6, 8), (9, 9, 1, 11), (9, 12, 10, 11), (10, 11, 2, 13), (11, 13, 3, 15), (11, 19, 4, 20), (12, 15, 4,17), (13, 17, 5, 19), (14, 19, 6, 21), (15, 21, 7, 23), (16, 23, 8, 25), (20, 22, 21, 21), (18, 22, 9, 25), (6,27,13, 26), (17, 25, 9, 27), (12, 29, 23, 24), (18, 27, 10, 29)

and the following elements (a, b, c, d) are

in I_2: (12, 12, 3, 3), (1, 17, 8, 10), (2, 25, 4, 23), (3, 18, 9, 12), (5, 22, 8, 19), (7, 20, 13, 14), (15, 27, 18, 24).

Since $(4, 4, 1, 1) \in I_1$ and $(12, 12, 3, 3) \in I_2$, then from (2.2) we have $E_1 = 2(S_4 - S_1), E_2 = 2(S_{12} - S_3)$. It is clear from (2.2) that for all $(a, b, c, d) \in I_i \ \ (i = 1, 2)$, we have

$$S_a + S_b - S_c - S_d = E_i \quad (i = 1, 2).$$

Thus we obtain the system of 24 equations with 28 unknowns, namely $S_1, S_2, \cdots, \cdots, S_{27}$ and S_{29} are unknowns. We solve this linear system and we get solutions as follows:

$$S_5 = -2S_1 + \tfrac{1}{2}S_2 + S_3 + \tfrac{3}{2}S_4, \qquad S_6 = -3S_1 + S_3 + 3S_4,$$

$$S_7 = -\tfrac{9}{2}S_1 - \tfrac{1}{4}S_2 + \tfrac{1}{2}S_3 + \tfrac{21}{4}S_4, \qquad S_8 = -8S_1 + S_2 + 8S_4,$$

$$S_9 = -11S_1 + \tfrac{1}{2}S_2 + \tfrac{23}{2}S_4, \qquad S_{10} = -15S_1 - \tfrac{1}{2}S_2 + S_3 + \tfrac{31}{2}S_4$$

$$S_{11} = -21S_1 + S_2 + 21S_4, \qquad S_{12} = -27S_1 + S_3 + 27S_4,$$

$$S_{13} = -34S_1 - \tfrac{1}{2}S_2 + S_3 + \tfrac{69}{2}S_4, \qquad S_{14} = -\tfrac{87}{2}S_1 + \tfrac{3}{4}S_2 + \tfrac{1}{2}S_3 + \tfrac{173}{4}S_4,$$

$$S_{15} = -53S_1 + \tfrac{1}{2}S_2 + \tfrac{107}{2}S_4, \qquad S_{16} = -64S_1 + 65S_4,$$

$$S_{17} = -78S_1 + \tfrac{1}{2}S_2 + S_3 + \tfrac{155}{2}S_4, \qquad S_{18} = -92S_1 + \tfrac{1}{2}S_2 + \tfrac{185}{2}S_4,$$

$$S_{19} = -108S_1 - \tfrac{1}{2}S_2 + S_3 + \tfrac{217}{2}S_4 \qquad S_{20} = -127S_1 + \tfrac{1}{2}S_2 + S_3 + \tfrac{253}{2}S_4,$$

$$S_{21} = -\tfrac{293}{2}S_1 + \tfrac{1}{4}S_2 + \tfrac{1}{2}S_3 + \tfrac{587}{4}S_4, \qquad S_{22} = -168S_1 + 169S_4,$$

$$S_{23} = -193S_1 + S_2 + 193S_4, \qquad S_{24} = -219S_1 + S_3 + 219S_4$$

$$S_{25} = -247S_1 + 248S_4, \qquad S_{26} = -279S_1 + \tfrac{1}{2}S_2 + S_3 + \tfrac{557}{2}S_4,$$

$$S_{27} = -312S_1 + S_3 + 312S_4, \qquad S_{29} = -387S_1 + S_2 + 387S_4.$$

Thus, we proved that (2.1) holds with the $c_1(n), c_2(n), c_3(n), c_4(n)$ that are given in Table 1. We note from these values that

$$E_1 = 2(S_4 - S_1) \quad \text{and} \quad E_2 = 2(S_{12} - S_3) = 54(S_4 - S_1).$$

Lemma 1 is proved. ■

Lemma 2. *We have*

$$(2.3)\qquad S_{n+24} = S_{n+17} + S_{n+16} + S_{n+15} - S_{n+9} - S_{n+8} - S_{n+7} + S_n + E,$$

for all $n \in \mathbb{N}$, *where* $E := 48(S_4 - S_1)$.

Proof. It is easy to check that

$$(2.4)\qquad (2n+7t)^3 + (n+8t)^3 - (2n+9t)^3 - n^3 = 126t^3$$

holds for all $n, t \in \mathbb{N}$. Thus, by applying (2.4) with $t = 1$ and $t = 3$, we have

$$(2n+7)^3 + (n+8)^3 - (2n+9)^3 - n^3 = 126 = u_1 - u_1'$$

and

$$(2n+21)^3 + (n+24)^3 - (2n+27)^3 - n^3 = 3402 = u_2 - u_2'.$$

Thus, we infer from (2.2) that

$$(2.5)\qquad \begin{cases} S_{2n+9} - S_{2n+7} - S_{n+8} + S_n & = -E_1 = -2(S_4 - S_1) \\ S_{2n+27} - S_{2n+21} - S_{n+24} + S_n & = -E_2 = -54(S_4 - S_1) \end{cases}$$

for all $n \in \mathbb{N}$. Since

$$S_{2n+27} - S_{2n+21} = \Big(S_{2n+27} - S_{2n+25}\Big) + \Big(S_{2n+25} - S_{2n+23}\Big) + \Big(S_{2n+23} - S_{2n+21}\Big),$$

it follows directly from (2.5) that

$$S_{n+24} = S_{n+17} + S_{n+16} + S_{n+15} - S_{n+9} - S_{n+8} - S_{n+7} + S_n + E,$$

for all $n \in \mathbb{N}$, where

$$E = E_2 - 3E_1 = [54(S_4 - S_1)] - 3[2(S_4 - S_1)] = 48(S_4 - S_1).$$

Lemma 2 is proved. ■

Lemma 3. *We have*

$$(2.6)\qquad \mathcal{S}_n = An^3 + B\chi_7(n) + C\chi_3(n) + D \quad \textit{for every} \quad n \in \mathbb{N},$$

where

$$\begin{cases} A = \dfrac{S_4 - S_1}{63}, & B = \dfrac{2S_1 + 7S_2 - 14S_3 + 5S_4}{28}, \\ C = \dfrac{8S_1 - 9S_2 + S_4}{18}, & D = \dfrac{2S_1 + S_2 + 2S_3 - S_4}{4} \end{cases}$$

and $\chi_3(n) \pmod 3$, $\chi_7(n) \pmod 7$ *are non-principal Dirichlet characters, i.e.*

$$\chi_3(0) = 0, \chi_3(1) = 1, \chi_3(2) = -1,$$

$$\chi_7(0) = 0, \chi_7(1) = \chi_7(2) = \chi_7(4) = 1, \chi_7(3) = \chi_7(5) = \chi_7(6) = -1.$$

Proof. With the help of computer, using the definition of A, B, C, D and Lemma 1, one can check that (2.6) is true for positive integers $1 \le n \le 24$.

Assume that (2.6) holds for $n = k, \cdots, k+23$, where $k \ge 1$. We prove that (2.6) holds for $n = k + 24$.

From (2.3), and using the assumption of induction, we obtain that

$$S_{k+24} = S_{k+17} + S_{k+16} + S_{k+15} - S_{k+9} - S_{k+8} - S_{k+7} + S_k + E =$$

$$= A\Big[(k+17)^3 + (k+16)^3 + (k+15)^3 - (k+9)^3 - (k+8)^3 - (k+7)^3 + k^3\Big] +$$

$$= B\Big[\chi_7(k+17) + \chi_7(k+16) + \chi_7(k+15) - \chi_7(k+9) - \chi_7(k+8) -$$

$$-\chi_7(k+7) + \chi_7(k)\Big] + C\Big[\chi_3(k+17) + \chi_3(k+16) + \chi_3(k+15) -$$

$$-\chi_3(k+9) - \chi_3(k+8) - \chi_3(k+7) + \chi_3(k)\Big] + D + E =$$

$$= A\Big[(k+24)^3 - 3024\Big] + B\chi_7(k+17) + C\chi_3(k) + D + E =$$

$$= A(k+24)^3 + B\chi_7(k+24) + C\chi_3(k+24) + (3024A - E) + D =$$

$$= A(k+24)^3 + B\chi_7(k+24) + C\chi_3(k+24) + D,$$

which proves that (2.6) holds for $n = k + 24$, and so it is true for every $n \in \mathbb{N}$. The proof of (2.6) is finished.

Lemma 3 is proved. ■

Lemma 4. *Let $M \in \mathbb{N}, M \equiv 0 \pmod 6$. Then the equation*

$$x^3 + y^3 - z^3 - t^3 = M \tag{2.7}$$

is solvable in $\mathbb{N}$.

Proof. Let $M = 6.2^\alpha m, \alpha \ge 0, (m, 2) = 1$. One can check easily that

$$(x, y, z, t) = \begin{cases} \left(2^\alpha + \frac{m+1}{2}, 2^\alpha - \frac{m+1}{2}, 2^\alpha + \frac{m-1}{2}, 2^\alpha - \frac{m-1}{2}\right) & \text{if } 2^{\alpha+1} > m \\ \left(2^\alpha + \frac{m+1}{2}, \frac{m-1}{2} - 2^\alpha, 2^\alpha + \frac{m-1}{2}, \frac{m+1}{2} - 2^\alpha\right) & \text{if } 2^{\alpha+1} < m \end{cases}$$

is a solution of (2.7) except if $m = 2^{\alpha+1} \pm 1$.

Assume that $M = 6.2^{\alpha}m = 3.2^{\alpha+1}(2^{\alpha+1} \pm 1)$. If $\alpha \le 3$, then the solutions of (2.7) are:

$$(x,y,z,t,M) = \begin{cases} (11,20,4,21,6), \quad (2,4,3,3,18) & \text{if } \alpha = 0, \\ (1,14,8,13,36), \quad (9,11,10,10,60) & \text{if } \alpha = 1, \\ (3,20,10,19,168), \quad (1,7,4,4,216) & \text{if } \alpha = 2, \end{cases}$$

as one can see easily.

Thus, we assume that $M = 3.2^{\alpha+1}(2^{\alpha+1} \pm 1)$ and $\alpha \ge 3$. In this case, if $n = 2^{\alpha-3}(2^{\alpha+1} \pm 1)$, then

$$(n+3)^3 + (n-3)^3 - (n+1)^3 - (n-1)^3 = 48n = 3.2^{\alpha+1}(2^{\alpha+1} \pm 1) = M.$$

Lemma 4 is proved. ∎

3. Proof of Theorem 1

Let

$$H(n) := B\chi_7(n) + C\chi_3(n) + D \quad \text{for every} \quad n \in \mathbb{N}.$$

Then, from Lemma 3, we have $h(n^3) = S_n = An^3 + H(n)$ and so $H(n)$ is bounded.

Now we prove that

$$g(\alpha) = A\alpha + G(\alpha) \quad \text{and} \quad G(\alpha) = O(1) \quad \text{for every} \quad \alpha \in \mathcal{A}.$$

For each $\alpha \in \mathcal{A}$ we denote by $\overline{\alpha}$ the smallest element of $\mathcal{A}$, for which $\alpha - \overline{\alpha} \equiv 0 \pmod 6$. It is shown in Lemma 4 that there are $a,b,c,d \in \mathbb{N}$ such that

$$\alpha - \overline{\alpha} = a^3 + b^3 - c^3 - d^3.$$

Then from (1.1) we have

$$g(\overline{\alpha}) + S_a + S_b = g(\alpha) + S_c + S_d,$$

which implies that

$$g(\alpha) = A\Big[a^3 + b^3 - c^3 - d^3\Big] + B\Big(\chi_7(a) + \chi_7(b) - \chi_7(c) - \chi_7(d)\Big) + \\ + C\Big(\chi_3(a) + \chi_3(b) - \chi_3(c) - \chi_3(d)\Big) + g(\overline{\alpha}) = A\alpha + G(\alpha),$$

where

$$G(\alpha)=g(\overline{\alpha})-A\overline{\alpha}+B\Big(\chi_7(a)+\chi_7(b)-\chi_7(c)-\chi_7(d)\Big)+$$
$$+C\Big(\chi_3(a)+\chi_3(b)-\chi_3(c)-\chi_3(d)\Big).$$

It is obvious that $G(\alpha)=O(1)$ for all $\alpha\in\mathcal{A}$.

Finally, if we define $F(\beta):=f(\beta)-A\beta$, then we have

$$F(\alpha+n^3+m^3)=f(\alpha+n^3+m^3)-A(\alpha+n^3+m^3)=$$
$$=g(\alpha)+h(n^3)+h(m^3)-A(\alpha+n^3+m^3)=G(\alpha)+H(n)+H(m)$$

for all $\alpha\in\mathcal{A}, n, m\in\mathbb{N}$.

Theorem 1 is proved. ■

4. Proof of Theorem 2

We shall use an argument which is similar but more complicated than that was used in the proof of Theorem 1.

First we note from (1.3) that there are $v_1, v_1', v_2, v_2'\in\mathcal{A}$ such that

$$1008=v_1-v_1' \quad \text{and} \quad 8064=v_2-v_2'.$$

Let

$$e_1:=g(v_1)-g(v_1') \quad \text{and} \quad e_2:=g(v_2)-g(v_2').$$

We denote by J_1 and J_2 the following sets:

$$J_1:=\{(a,b,c,d)|\ \ a^3+b^3-c^3-d^3=1008\}$$

and

$$J_2:=\{(a,b,c,d)|\ \ a^3+b^3-c^3-d^3=8064\}.$$

It is obvious from (1.1) that

$$S_a+S_b-S_c-S_d=e_i \quad \text{if} \quad (a,b,c,d)\in J_i \ \ (i=1,2). \tag{4.1}$$

By applying Maple program, we computed that the following 33 elements (a,b,c,d) are

in J_1: (8, 8, 2, 2), (1, 18, 9, 16), (1, 23, 8, 22), (2, 12, 6,8), (2, 19, 3, 18), (3, 13, 6, 10), (3, 29, 18, 26), (5, 30, 12, 29), (6, 11, 3, 8), (6, 23, 15, 20), (8, 26, 17, 23), (8, 39, 23, 36), (10, 18, 12, 16), (11, 34, 19, 32), (16, 17, 1, 20), (17, 22, 9, 24), (18, 18, 2, 22), (20, 22, 4, 26), (20, 28, 21, 27), (21, 24, 5, 28), (22, 26, 6, 30), (22, 38, 8, 40), (23, 28, 7, 32), (24, 30, 8, 34), (25, 32, 9, 36), (26, 34, 10, 38), (28, 38, 12, 42), (29, 40, 13, 44), (30, 42, 14, 46), (31, 44, 15, 48), (32, 46, 16, 50) (33, 41, 3, 47), (38, 38, 17, 47)

and the following 14 elements belong

to J_2: (16, 16, 4, 4), (1, 23, 9, 15), (1, 23, 2, 16), (2, 46, 16, 44), (3, 25, 9, 19), (9, 39, 11, 37), (11, 19, 1, 5), (11, 31, 15, 27), (13, 29, 21, 21), (15, 33, 19, 29), (15, 41, 17, 39), (21, 35, 27, 29), (27, 43, 1, 45), (29, 43, 22, 44).

Since $(8,8,2,2) \in J_1$ and $(16,16,4,4) \in J_2$, then from (4.1) we have $e_1 = 2(S_8 - S_2), e_2 = 2(S_{16} - S_4)$. Thus, from above values of J_1, J_2 and from (4.1), we obtain the system of 45 equations with 49 unknowns, namely $S_1, S_2, \cdots, S_{48}$ and S_{50} are unknowns. We solve this linear system and with computer one can check that

$$\mathcal{S}_n = An^3 + B\chi_7(n) + C\chi_3(n) + D \tag{4.2}$$

holds for all $n \leq 48$, where A, B, C, D are given in Lemma 3. We also have $e_1 = 16(S_4 - S_1)$ and $e_2 = 128(S_4 - S_1)$.

Finally, by applying (2.4) with $t = 2^3$ and $t = 4^3$, we get by a similar argument as in the proof of Lemma 2 that

$$S_{n+32} = S_{n+25} + S_{n+23} - S_{n+9} - S_{n+7} + S_n + e \quad \text{for every} \quad n \in \mathbb{N}, \tag{4.3}$$

where $e = e_2 - 2e_1 = 96(S_4 - S_1)$. From (4.3) we obtain that (4.2) holds for every $n \in \mathbb{N}$.

The remaining assertions of Theorem 2 are obtained on the same way as in the proof of Theorem 1, we omit it.

Theorem 2 is proved. ■

5. Proofs of corollaries

Corollary 1 and Corollary 3 are direct consequences of Theorem 1 and Theorem 2, respectively.

Proof of Corollary 2. It is true that $126 = 131 - 5 \in \mathcal{P} - \mathcal{P}$ and $3402 = 3407 - 5 \in \mathcal{P} - \mathcal{P}$, therefore the condition (1.2) and all assertions of Theorem 1 hold.

For numbers $a, b, c, d \in \mathbb{N}$ and $p, q \in \mathcal{P}$, we define T as follows:

$$T := \{(p, a, b, q, c, d) |\ \ p + a^3 + b^3 = q + c^3 + d^3\}.$$

It is obvious from (1.1) that

$$g(p) + S_a + S_b = g(q) + S_c + S_d \quad \text{if} \quad (p, a, b, q, c, d) \in T. \tag{5.1}$$

We computed that the following elements (p, a, b, q, c, d) are in T: $(17, 1, 1, 3, 2, 2)$, $(5, 1, 3, 17, 2, 2)$, $(19, 1, 1, 5, 2, 2)$, $(7, 1, 3, 19, 2, 2)$, $(59, 1, 1, 7, 3, 3)$, $(59, 1, 2, 3, 1, 4)$.

Repeated use of (5.1) gives

$$\begin{aligned}
&g(17) = g(3) + 2S_2 - 2S_1,\\
&g(5) = g(17) + 2S_2 - (S_1 + S_3) = g(3) + 4S_2 - 3S_1 - S_3,\\
&g(19) = g(5) + 2S_2 - 2S_1 = g(3) + 6S_2 - 5S_1 - S_3,\\
&g(7) = g(19) + 2S_2 - (S_1 + S_3) = g(3) + 8S_2 - 6S_1 - 2S_3,\\
&g(59) = g(7) + 2S_3 - 2S_1 = g(3) + 8S_2 - 8S_1.
\end{aligned}$$

Thus we have

$$S_4 = g(59) + S_2 - g(3) = 9S_2 - 8S_1.$$

Since $(29, 1, 1, 3, 1, 3)$, $(37, 2, 3, 7, 1, 4)$, $(11, 1, 3, 37, 1, 1)$ and $(29, 3, 3, 11, 2, 4)$ are in T, we obtain from (5.1) that

$$\begin{aligned}
&g(29) = g(3) + S_3 - S_1 = g(3) - S_1 + S_3,\\
&g(37) = g(7) + S_1 + S_4 - S_2 - S_3 = g(3) + 16S_2 - 13S_1 - 3S_3,\\
&g(11) = g(37) + 2S_1 - (S_1 + S_3) = g(3) + 16S_2 - 12S_1 - 4S_3,\\
&19S_1 + 7S_3 - 26S_2 = g(29) + 2S_3 - (g(11) + S_2 + S_4) = 0.
\end{aligned}$$

This gives

$$S_3 = \frac{26}{7}S_2 - \frac{19}{7}S_1.$$

Consequently

$$A = \frac{S_4 - S_1}{63} = \frac{9S_2 - 8S_1 - S_1}{63} = \frac{S_2 - S_1}{7},$$

$$B = \frac{2S_1 + 7S_2 - 14(\frac{26}{7}S_2 - \frac{19}{7}S_1) + 5(9S_2 - 8S_1)}{28} = 0,$$

$$C = \frac{8S_1 - 9S_2 + S_4}{18} = \frac{8S_1 - 9S_2 + (9S_2 - 8S_1)}{18} = 0$$

and

$$D = \frac{2S_1 + S_2 + 2S_3 - S_4}{4} = -\frac{S_2 - 8S_1}{7}.$$

Now we prove that

$$g(p) = Ap + Q \quad \text{for every} \quad p \in \mathcal{P}, \tag{5.2}$$

where

$$Q := g(2) - 2A.$$

It is obvious that (5.2) holds for $p = 2$. Now we prove that (5.2) holds for $p = 3$.

Since $(47, 1, 2, 2, 3, 3) \in T$, $(61, 1, 1, 47, 2, 2) \in T$, $(61, 1, 2, 5, 1, 4) \in T$ and $(13, 3, 3, 2, 1, 4) \in T$, therefore

$$\begin{aligned}
g(47) &= g(2) + 2S_3 - S_2 - S_1 = g(2) - \frac{45}{7}S_1 + \frac{45}{7}S_2 = 47A + Q,\\
g(61) &= g(47) + 2S_2 - 2S_1 = g(2) - \frac{59}{7}S_1 + \frac{59}{7}S_2 = 61A + Q,\\
g(2) &- g(3) + A = g(61) + S_2 - g(5) - S_4 = 0\\
g(13) &= g(2) + S_2 + S_4 - 2S_3 = g(2) - \frac{11}{7}S_1 + \frac{11}{7}S_2 = 13A + Q.
\end{aligned}$$

These relations with the above computations show that

$$g(3) = g(2) + A = 3A + Q.$$

Therefore the above computations show that (5.2) holds for $p \le 19$.

We shall complete the proof of (5.2). For each $p \in \mathcal{P}, p > 19$ let $\overline{p} \in \{5, 7\}$ be that integer for which $p - \overline{p} \equiv 0 \pmod 6$. In Lemma 4 we proved that there are $a, b, c, d \in \mathbb{N}$ such that

$$p - \overline{p} = c^3 + d^3 - a^3 - b^3.$$

Then from (5.1) we have

$$g(p) + S_a + S_b = g(\overline{p}) + S_c + S_d,$$

which implies

$$g(p) = A\Big[c^3 + d^3 - a^3 - b^3\Big] + A\overline{p} + Q = A(p - \overline{p}) + A\overline{p} + Q = Ap + Q.$$

The proof of Corollary 2 is completes. ∎

Proof of Corollary 4. Let now

$$\mathcal{A} := \{n^2 \in \mathbb{N}\}.$$

Since $1008 = 32^2 - 4^2 \in \mathcal{A} - \mathcal{A}$ and $8064 = 90^2 - 6^2 \in \mathcal{A} - \mathcal{A}$, therefore the condition (1.3) and so all assertions of Theorem 2 hold, i. e.

$$h(n^3) = An^3 + B\chi_7(n) + C\chi_3(n) + D \quad \text{for every} \quad n \in \mathbb{N},$$

where A, B, C, D and $\chi_7(n), \chi_3(n)$ are defined in Lemma 3.

For numbers $a, b, c, d \in \mathbb{N}$ and $u, v \in \mathbb{N}$, we define $\mathcal{H}$ as follows:

$$\mathcal{H} := \{(u, a, b, v, c, d) |\ \ u^2 + a^3 + b^3 = v^2 + c^3 + d^3\}.$$

It is obvious from (1.1) that

$$g(u^2) + S_a + S_b = g(v^2) + S_c + S_d \quad \text{if} \quad (u, a, b, v, c, d) \in R. \tag{5.3}$$

The following elements (u, a, b, v, c, d) are in $\mathcal{H}$: $(1, 1, 7, 2, 5, 6)$, $(2, 1, 3, 4, 2, 2)$, $(1, 3, 10, 2, 8, 8)$, $(1, 3, 11, 4, 7, 10)$. Thus, an application of (5.3) gives

$$g(2^2) = g(1^2) + S_1 + S_7 - S_5 - S_6, \quad g(4^2) = g(2^2) + S_1 + S_3 - 2S_2$$

and

$$g(2^2) = g(1^2) + S_3 + S_{10} - 2S_8, \quad g(4^2) = g(1^2) + S_3 + S_{11} - S_7 - S_{10}.$$

These imply with (2.6) that

$$S_1 + S_7 - S_5 - S_6 - (S_3 + S_{10} - 2S_8) = 0 \tag{5.4}$$

and

$$g(1^2) + S_3 + S_{11} - S_7 - S_{10} - [g(2^2) + S_1 + S_3 - 2S_2] = 0,$$

consequently

$$2S_1 - 2S_2 - S_5 - S_6 + 2S_7 + S_{10} - S_{11} = 0. \tag{5.5}$$

From (5.4) and (5.5) we infer that

$$S_3 = -\frac{19}{7}S_1 + \frac{26}{7}S_2 \quad \text{and} \quad S_4 = -8S_1 + 9S_2,$$

which give

$$A = \frac{S_2 - S_1}{7}, \quad B = C = 0 \quad \text{and} \quad D = -\frac{S_2 - 8S_1}{7}.$$

Finally, without any important change in the proof Corollary 2, we could prove that $g(n^2) = An^2 + (g(1) - A)$ holds for all $n \in \mathbb{N}$, which with $R = g(1) - A$ proves Corollary 4.

Corollary 4 is proved. ■

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