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ADDITIVE UNIQUENESS SETS FOR A PAIR OF MULTIPLICATIVE FUNCTIONS

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Dedicated to Professor Ha Huy Khoai on the occasion of his 70-th birthday

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Abstract. We give all solutions of those multiplicative functions f, g which satisfy

$$f\left(n^2+m^2+a+b\right)=g(n^2+a)+g(m^2+b) \quad \text{for all} \quad n,m\in\mathbb{N},$$

where a, b are non-negative integers with $a+b>0$. It is proved that if

$$g(a+36)+4g(a+25)-g(a+9)-g(a+4)-3g(a+1)\neq 0,$$

then

$$f(n)=n \quad \text{and} \quad g(m^2+a)=m^2+a, \quad g(m^2+b)=m^2+b$$

for all $n,m\in\mathbb{N},\ (n,2(a+b))=1$.

1. Introduction

Let $\mathcal{P}$, $\mathbb{N}$, $\mathbb{C}$ be the set of primes, positive integers and complex numbers, respectively. An arithmetic function $f:\mathbb{N}\to\mathbb{C}$ is said to be multiplicative if $(n,m)=1$ implies that $f(nm)=f(n)f(m)$. Let $\mathcal{M}$ denote the class of all multiplicative functions f with $f(1)=1$. For each non-negative integer a let

$$E_a=\{n^2+a \mid n\in\mathbb{N}\}.$$

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C. Spiro said that $E \subseteq \mathbb{N}$ is an additive uniqueness set for $\mathcal{M}$ if there is exactly one element $f \in \mathcal{M}$ which satisfies

$$f(n+m) = f(n) + f(m) \quad \text{for all} \quad n, m \in E.$$

In 1992, C. Spiro [7] showed that $E = \mathcal{P}$ is an additive uniqueness set for $\mathcal{M}$. In 1997, J.-M. De Koninck, I. Kátai and B. M. Phong [1] proved that if $f \in \mathcal{M}$ and

$$f(n^2 + p) = f(n^2) + f(p) \quad \text{for all} \quad n \in \mathbb{N},\ p \in \mathcal{P}$$

holds, then $f(n) = n$ for all $n \in \mathbb{N}$. Recently, in [6] we improve this result for two multiplicative functions, namely it is proved that if $f, g \in \mathcal{M}$ satisfy

$$f(p + m^2) = g(p) + g(m^2) \quad \text{and} \quad g(p^2) = g(p)^2$$

for all $p \in \mathcal{P}$ and $m \in \mathbb{N}$, then either

$$f(p + m^2) = 0, \quad g(p) = -1 \ \text{ and } \ g(m^2) = 1$$

for all primes p and $m \in \mathbb{N}$ or

$$f(n) = n \ \text{ and } \ g(p) = p, \quad g(m^2) = m^2$$

for all $p \in \mathcal{P},\ n, m \in \mathbb{N}$.

In the following we say that $A, B \subseteq \mathbb{N}$ is a pair of additive uniqueness sets (AU-sets) for $\mathcal{M}$ if $f \in \mathcal{M}$ satisfying

$$f(a+b) = f(a) + f(b) \quad \text{for all} \quad a \in A \quad \text{and} \quad b \in B,$$

implies $f(n) = n$ for all $n \in \mathbb{N}$. We are interested in characterizing all non-negative integers a and b such that $A = E_a$ and $B = E_b$ are AU-sets. It is proved in [4] that if a function $f \in \mathcal{M}$ with $f(4)f(9) \neq 0$ and $k \in \mathbb{N}$ satisfy the condition

$$f(n^2 + m^2 + k) = f(n^2) + f(m^2 + k) \quad \text{for all} \quad n, m \in \mathbb{N},$$

then $f(n) = n$ for all positive integers $n, (n, 2k) = 1$. K.-H. Indlekofer and B.M. Phong [2] proved that if $k \in \mathbb{N}$ and $f \in \mathcal{M}$ satisfy $f(2)f(5) \neq 0$ and

$$f\left(n^2 + m^2 + k + 1\right) = f(n^2 + 1) + f(m^2 + k) \quad \text{for all} \quad n, m \in \mathbb{N},$$

then $f(n) = n$ for all $n \in \mathbb{N}$, $(n, 2) = 1$.

Our main purpose in this paper is to give the answer for the general case.

Theorem. *Assume that non-negative integers a, b with $a+b>0$ and $f,\ g\in\mathcal{M}$ satisfy the condition*

$$f\left(n^2+m^2+a+b\right)=g(n^2+a)+g(m^2+b)\quad \textit{for all}\quad n,m\in\mathbb{N}. \tag{1}$$

Let

$$S_n=g(n^2+a)\quad \textit{and}\quad A=\frac{1}{120}(S_6+4S_5-S_3-S_2-3S_1).$$

Then the following assertions are true:

I. $A\in\{0,1\}$.

II. *If* $A=1$, *then*

$$g(m^2+a)=m^2+a,\ g(m^2+b)=m^2+b\ \textit{ for all } m\in\mathbb{N} \tag{2}$$

and

$$f(n)=n\ \textit{ for all }\ n\in\mathbb{N},\ (n,2(a+b))=1. \tag{3}$$

III. *If* $A=0$, *then there is a* $K\in\{1,2,3\}$ *such that* $S_{n+K}=S_n$ *for all* $n\in\mathbb{N}$.

III.1. *If* $K=1$, *then*

$$(f,g)\in\{(f_0,g_0),(f_1,g_1),(f_2,g_2)\},$$

where (f_i,g_i) *are given in Table 1:*

i	$g_i(n^2+a)$	$g_i(n^2+b)$	$f_i(n^2+m^2+a+b)$	*for*
0	$g_0(n^2+a)=0$	$g_0(n^2+b)=0$	$f_0(n^2+m^2+a+b)=0$	$\forall n,m\in\mathbb{N}$
1	$g_1(n^2+a)=0$	$g_1(n^2+b)=1$	$f_1(n^2+m^2+a+b)=1$	$\forall n,m\in\mathbb{N}$
2	$g_2(n^2+a)=1$	$g_2(n^2+b)=0$	$f_2(n^2+m^2+a+b)=1$	$\forall n,m\in\mathbb{N}$

Table 1

III.2. *If* $K=2$, *then*

$$(f,g)\in\{(f_3,g_3),(f_4,g_4),(f_5,g_5),(f_6,g_6)\},$$

where (f_i,g_i) *are defined as*

$$g_i(n^2+a)=\alpha_i\chi_2(n)+\beta_i,\ g_i(n^2+b)=\alpha_i\chi_2(n)+\gamma_i,$$

$$f_i(n^2+m^2+a+b)=\alpha_i\chi_2(n)+\alpha_i\chi_2(m)+\delta_i$$

and $\chi_2(n)$ *is the principal Dirichlet character (modulo 2). The values of* $\alpha_i,\beta_i,\gamma_i,\delta_i$ *are given in Table 2:*

i	(f_i, g_i)	α_i	β_i	γ_i	δ_i		*in the case*
3	(f_3, g_3)	c	$1-c$	0	$1-c$	$c \in \mathbb{C}$	$(a,b) \equiv (0,0) \pmod 2$
4	(f_4, g_4)	c	0	$-c$	$-c$	$c \in \mathbb{C}, c \neq 0$	$(a,b) \equiv (0,1) \pmod 2$
5	(f_5, g_5)	c	$-c$	0	$-c$	$c \in \mathbb{C}, c \neq 0$	$(a,b) \equiv (1,0) \pmod 2$
6	(f_6, g_6)	c	1	$-c$	$1-c$	$c \in \mathbb{C}, c \neq 0$	$(a,b) \equiv (1,1) \pmod 2$

Table 2

Here we write $(a,b) \equiv (x,y) \pmod m$ *if* $a \equiv x$ *and* $b \equiv y \pmod m$.

III.3. *If* $K = 3$, *then*

$$(f, g) \in \{(f_7, g_7), (f_8, g_8), \cdots, (f_{11}, g_{11})\},$$

where (f_i, g_i) *are defined as*

$$g_i(n^2 + a) = \alpha_i \chi_3(n) + \beta_i, \;\; g_i(n^2 + b) = \alpha_i \chi_3(n) + \gamma_i,$$

$$f_i(n^2 + m^2 + a + b) = \alpha_i \chi_3(n) + \alpha_i \chi_3(m) + \delta_i$$

and $\chi_3(n)$ *is the principal Dirichlet character (modulo 3). The values of* $\alpha_i, \beta_i, \gamma_i, \delta_i$ *are given in Table 3:*

i	(f_i, g_i)	α_i	β_i	γ_i	δ_i	*in the case*
7	(f_7, g_7)	-2	1	1	2	$(a,b) \equiv (1,1) \pmod 3$
8	(f_8, g_8)	-2	1	2	3	$(a,b) \equiv (1,2), (2,1) \pmod 3$
9	(f_9, g_9)	1	-1	0	-1	$(a,b) \equiv (2,3) \pmod 3$
10	(f_{10}, g_{10})	1	0	-1	-1	$(a,b) \equiv (3,2) \pmod 3$
11	(f_{11}, g_{11})	-2	3	0	3	$(a,b) \equiv (3,3) \pmod 3$

Table 3

2. Lemmas

We shall use the following results:

Lemma 1. *Let* a *and* b *be non-negative integers and* F, G *be arithmetical functions, for which the condition*

$$F(n^2 + m^2 + a + b) = G(n^2 + a) + G(m^2 + b) \tag{4}$$

is satisfied for all $n, m \in \mathbb{N}$. *For each* $j \in \mathbb{N}$ *let* $S_j := G(j^2 + a)$. *Then*

$$S_{n+12} = S_{n+9} + S_{n+8} + S_{n+7} - S_{n+5} - S_{n+4} - S_{n+3} + S_n \tag{5}$$

holds for all $n \in \mathbb{N}$ *and*

$$\begin{cases} S_7 &= 2S_5 - S_1, \\ S_8 &= 2S_5 + S_4 - 2S_1, \\ S_9 &= S_6 + 2S_5 - S_2 - S_1, \\ S_{10} &= S_6 + 3S_5 - S_3 - 2S_1, \\ S_{11} &= S_6 + 4S_5 - S_3 - S_2 - 2S_1, \\ S_{12} &= S_6 + 4S_5 + S_4 - S_2 - 4S_1. \end{cases} \tag{6}$$

Proof. This is Lemma 1 in [5]. ■

Lemma 2. *Let* a *and* b *be non-negative integers and* F, G *be arithmetical functions satisfying the condition* (4). *Let*

$$\begin{aligned} A &:= \frac{1}{120}(S_6 + 4S_5 - S_3 - S_2 - 3S_1), \\ \Gamma_2 &:= \frac{-1}{8}(S_6 - 4S_5 + 4S_4 - S_3 + 3S_2 - 3S_1), \\ \Gamma_3 &:= \frac{-1}{3}(S_6 - 2S_5 + 2S_3 - S_2), \\ \Gamma_4 &:= \frac{1}{4}(S_6 - 2S_4 - S_3 + S_2 + S_1), \\ \Gamma_5 &:= \frac{1}{5}(S_6 - S_5 - S_3 - S_2 + 2S_1), \end{aligned}$$

$$\begin{aligned} \Gamma &:= \frac{1}{4}(S_6 - 4S_5 + 2S_4 + 3S_3 + S_2 + S_1), \\ B_k &:= \Gamma_2\chi_2(k) + \Gamma_3\chi_3(k) + \Gamma_4\chi_4(k-1) + \Gamma_5\chi_5(k) + \Gamma, \end{aligned}$$

where $\chi_2(k) \pmod 2$, $\chi_3(k) \pmod 3$ *are the principal Dirichlet characters and* $\chi_4(k) \pmod 4$, $\chi_5(k) \pmod 5$ *are the real, non-principal Dirichlet characters, i.e.*

$$\chi_2(0) = 0, \chi_2(1) = 1,\ \chi_3(0) = 0,\ \chi_3(1) = \chi_3(2) = 1,\ \chi_4(0) = \chi_4(2) = 0,$$
$$\chi_4(1) = 1,\ \chi_4(3) = -1,\ \chi_5(2) = \chi_5(3) = -1,\ \chi_5(1) = \chi_5(4) = 1.$$

Then we have

$$S_k = Ak^2 + B_k \quad \text{for all} \quad k \in \mathbb{N}. \tag{7}$$

Proof. From the definition of B_k, we shall compute the values of B_k for $k = 1, 2, \cdots, 12$. We have

$$\begin{aligned}
B_1 &= -\frac{1}{120}S_6 - \frac{1}{30}S_5 + \frac{1}{120}S_3 + \frac{1}{120}S_2 + \frac{41}{40}S_1,\\
B_2 &= -\frac{1}{30}S_6 - \frac{2}{15}S_5 + \frac{1}{30}S_3 + \frac{31}{30}S_2 + \frac{1}{10}S_1,\\
B_3 &= -\frac{3}{30}S_6 - \frac{3}{10}S_5 + \frac{43}{40}S_3 + \frac{3}{40}S_2 + \frac{9}{40}S_1,\\
B_4 &= -\frac{2}{15}S_6 - \frac{8}{15}S_5 + S_4 + \frac{2}{15}S_3 + \frac{2}{15}S_2 + \frac{2}{5}S_1,\\
B_5 &= -\frac{5}{24}S_6 + \frac{1}{6}S_5 + \frac{5}{24}S_3 + \frac{5}{24}S_2 + \frac{5}{8}S_1,\\
B_6 &= \frac{7}{10}S_6 - \frac{6}{5}S_5 + \frac{3}{10}S_3 + \frac{3}{10}S_2 + \frac{9}{10}S_1,\\
B_7 &= -\frac{49}{120}S_6 + \frac{11}{30}S_5 + \frac{49}{120}S_3 + \frac{49}{120}S_2 + \frac{9}{40}S_1,\\
B_8 &= -\frac{8}{15}S_6 - \frac{2}{15}S_5 + S_4 + \frac{8}{15}S_3 + \frac{8}{15}S_2 - \frac{2}{5}S_1,\\
B_9 &= \frac{13}{40}S_6 - \frac{7}{10}S_5 + \frac{27}{40}S_3 - \frac{13}{40}S_2 + \frac{41}{40}S_1,\\
B_{10} &= \frac{1}{6}S_6 - \frac{1}{3}S_5 - \frac{1}{6}S_3 + \frac{5}{6}S_2 + \frac{1}{2}S_1,\\
B_{11} &= -\frac{1}{120}S_6 - \frac{1}{30}S_5 + \frac{1}{120}S_3 + \frac{1}{120}S_2 + \frac{41}{40}S_1,\\
B_{12} &= -\frac{1}{5}S_6 - \frac{4}{5}S_5 + S_4 + \frac{6}{5}S_3 + \frac{1}{5}S_2 - \frac{2}{5}S_1.
\end{aligned}$$

Consequently, we obtain from (6) and $A = \frac{1}{120}(S_6 + 4S_5 - S_3 - S_2 - 3S_1)$ that

$$\begin{aligned}
&A \cdot k^2 + B_k = S_k \quad \text{for all} \quad 1 \le k \le 6,\\
&A \cdot 7^2 + B_7 = 2S_5 - S_1 = S_7,\\
&A \cdot 8^2 + B_8 = 2S_5 + S_4 - 2S_1 = S_8,\\
&A \cdot 9^2 + B_9 = S_6 + 2S_5 - S_2 - S_1 = S_9,\\
&A \cdot 10^2 + B_{10} = S_6 + 3S_5 - S_3 - 2S_1 = S_{10},\\
&A \cdot 11^2 + B_{11} = S_6 + 4S_5 - S_3 - S_2 - 2S_1 = S_{11},\\
&A \cdot 12^2 + B_{12} = S_6 + 4S_5 + S_4 - S_2 - 4S_1 = S_{12}.
\end{aligned}$$

Therefore, we proved that (7) holds for $1 \le k \le 12$.

Assume that $Ak^2 + B_k = S_k$ holds for $n \le k \le n + 11$, where $n \ge 1$. Then we deduce from (5) that

$$\begin{aligned}
S_{n+12} &= S_{n+9}+S_{n+8}+S_{n+7}-S_{n+5}-S_{n+4}-S_{n+3}+S_n = \\
&= A\Big[(n+9)^2+(n+8)^2+(n+7)^2-(n+5)^2-(n+4)^2-(n+3)^2+n^2\Big]+ \\
&+\Big[B_{n+9}+B_{n+8}+B_{n+7}-B_{n+5}-B_{n+4}-B_{n+3}+B_n\Big] = \\
&= A(n+12)^2+B_{n+12},
\end{aligned}$$

which proves that (7) holds for $n+12$ and so it is true for all n. In the last relation we have used

$$\begin{aligned}
&B_{n+9}+B_{n+8}+B_{n+7}-B_{n+5}-B_{n+4}-B_{n+3}+B_n = \\
&=\Gamma_2\Big[\sum_{k=n+6}^{n+9}\chi_2(k)-\sum_{k=n+3}^{n+6}\chi_2(k)+\chi_2(n)\Big]+ \\
&+\Gamma_3\Big[\sum_{k=n+7}^{n+9}\chi_3(k)-\sum_{k=n+3}^{n+5}\chi_3(k)+\chi_3(n)\Big]+ \\
&+\Gamma_4\Big[\sum_{k=n+6}^{n+9}\chi_4(k-1)-\sum_{k=n+3}^{n+6}\chi_4(k-1)+\chi_4(n-1)\Big]+ \\
&+\Gamma_5\Big[\sum_{k=n+6}^{n+10}\chi_5(k)-\sum_{k=n+2}^{n+6}\chi_5(k)-\chi_5(n+10)+\chi_5(n+2)+\chi_5(n)\Big]+\Gamma = \\
&=\Gamma_2\chi_2(n)+\Gamma_3\chi_3(n)+\Gamma_4\chi_4(n-1)+\Gamma_5\chi_5(n+2)+\Gamma = \\
&=\Gamma_2\chi_2(n+12)+\Gamma_3\chi_3(n+12)+\Gamma_4\chi_4(n+11)+\Gamma_5\chi_5(n+12)+\Gamma = B_{n+12}.
\end{aligned}$$

Lemma 2 is proved. ■

3. Proof of the parts (I) and (II) of Theorem

Proof of (I). Assume that non-negative integers a, b with $a+b>0$ and $f,\ g\in\mathcal{M}$ satisfy the condition (1). For each $\ell\in\mathbb{N}$, let

$$I_\ell := \{n\in\mathbb{N}\ \mid\ (2n+1,4\ell+1)=1\}.$$

It is easy to show that

$$\Big[n^2+a\Big]\Big[(n+1)^2+a\Big]=\Big[n(n+1)+a\Big]^2+a$$

and

$$(n^2+a,(n+1)^2+a)=1 \quad \text{for all} \quad n\in I_a.$$

Now we apply Lemma 2 with $F = f$ and $G = g$. Then for $n \in I_a$, we have

$$g(n^2+a)g((n+1)^2+a) = g\Big[(n^2+a)((n+1)^2+a)\Big] = g\Big[\Big(n(n+1)+a\Big)^2+a\Big],$$

which proves

$$S_n S_{n+1} = S_{n(n+1)+a} \quad \text{for all} \quad n \in I_a, \tag{8}$$

therefore we get from (7) that

$$(An^2+B_n)(A(n+1)^2+B_{n+1}) = A[n(n+1)+a]^2 + B_{n(n+1)+a} \tag{9}$$

holds for all $n \in I_a$. By the definition of B_k we have

$$B_{k+60} = B_k \quad \text{for all} \quad k \in \mathbb{N},$$

consequently

$$|B_k| \le L := \max\big(|B_1|, \cdots, |B_{60}|\big).$$

Thus, (9) implies

$$\Big(A + \frac{B_n}{n^2}\Big)\Big(A + \frac{B_{n+1}}{(n+1)^2}\Big) = A\Big[1 + \frac{a}{n(n+1)}\Big]^2 + \frac{B_{n(n+1)+a}}{n^2(n+1)^2},$$

which with $n \to \infty$ gives

$$A^2 = A, \text{i.e.} \;\; A \in \{0,1\}.$$

Proof of (II). $A = 1$**.** We obtain from (9) that

$$(B_n + B_{n+1} - 2a)n^2 + 2(B_n - a)n + B_n + B_nB_{n+1} - B_{n(n+1)+a} - a^2 = 0,$$

holds for all $n \in I_a$. For each $n \in I_1$ and $m \in \mathbb{N}$ let

$$N(n,m) := 60(4a+1)m + n.$$

Since $N(n,m) \in I_a$ and $B_{N(n,m)} = B_n$, we infer from the above relation that

$$(B_n+B_{n+1}-2a)N(n,m)^2+2(B_n-a)N(n,m)+B_n+B_nB_{n+1}-B_{n(n+1)+a}-a^2 = 0$$

is satisfied for all $n \in I_a,\ m \in \mathbb{N}$, which implies that

$$B_n = a \quad \text{for all} \quad n \in I_a.$$

Let

$$J := \{j \in \mathbb{N} \mid (2j+1, 60) = 1\} = \{3, 5, 6, 8, 9, 11, 14, \cdots\}.$$

For each $j \in J$ let

$$n_j := 60x_j + j \quad (x_j \in \mathbb{N})$$

such that

$$2n_j + 1 = 120x_j + (2j+1) \in \mathcal{P} \quad \text{and} \quad 2n_j + 1 > 4a + 1.$$

Thus, $n_j \in I_a$, and so $B_{n_j} = a$ for all $j \in J$. Since the sequence $\{B_k\}_{k=1}^{\infty}$ is a periodic (modulo 60), therefore

$$B_j = B_{60x_j+j} = B_{n_j} = a \quad \text{for all} \quad j \in J.$$

Consequently

$$\begin{cases} (B_3 =) & -\frac{3}{30}S_6 - \frac{3}{10}S_5 + \frac{43}{40}S_3 + \frac{3}{40}S_2 + \frac{9}{40}S_1 = a, \\ (B_5 =) & -\frac{5}{24}S_6 + \frac{1}{6}S_5 + \frac{5}{24}S_3 + \frac{5}{24}S_2 + \frac{5}{8}S_1 = a, \\ (B_6 =) & \frac{7}{10}S_6 - \frac{6}{5}S_5 + \frac{3}{10}S_3 + \frac{3}{10}S_2 + \frac{9}{10}S_1 = a, \\ (B_8 =) & -\frac{8}{15}S_6 - \frac{2}{15}S_5 + S_4 + \frac{8}{15}S_3 + \frac{8}{15}S_2 - \frac{2}{5}S_1 = a, \\ (B_{11} =) & -\frac{1}{120}S_6 - \frac{1}{30}S_5 + \frac{1}{120}S_3 + \frac{1}{120}S_2 + \frac{41}{40}S_1 = a \\ (A =) & \frac{1}{120}S_6 + \frac{1}{30}S_5 - \frac{1}{120}S_3 - \frac{1}{120}S_2 - \frac{1}{40}S_1 = 1. \end{cases}$$

The solutions of this system are:

$$S_1 = 1+a, S_2 = 2^2+a, S_3 = 3^2 = a, S_4 = 4^2+a, S_5 = 5^2+a \quad \text{and} \quad S_6 = 6^2+a.$$

These relations with the next lemma prove (II) of our theorem. ■

Lemma 3. (Theorem 1, [5]) *Assume that non-negative integers a, b with $a+b>0$ and $f,\ g \in \mathcal{M}$ satisfy the condition* (1). *If either*

$$g(i^2+a) = i^2+a \quad \textit{or} \quad g(j^2+b) = j^2+b \quad \textit{for} \quad i,j = 1,2,\ldots,6$$

then

$$g(m^2+a) = m^2+a,\ g(m^2+b) = m^2+b \ \ \textit{for all } m \in \mathbb{N}$$

and

$$f(n) = n \quad \textit{for all} \quad n \in \mathbb{N},\ (n, 2(a+b)) = 1.$$

The proof of (II) is completed. ■

4. Proof of the part (III): $A = 0$.

From (7) we have

$$S_n = g(n^2 + a) = B_n \quad \text{for all} \quad n \in \mathbb{N}.$$

Since

$$A = \frac{1}{120}(S_6 + 4S_5 - S_3 - S_2 - 3S_1) = 0,$$

we have

$$S_6 = -4S_5 + S_3 + S_2 + 3S_1,$$

consequently

$$\begin{aligned} S_n = {} & \frac{1}{2}(2S_5 - S_4 - S_2)\chi_2(n) + (2S_5 - S_3 - S_1)\chi_3(n) + \\ & + \frac{1}{2}(-2S_5 - S_4 + S_2 + 2S_1)\chi_4(n-1) + (-S_5 + S_1)\chi_5(n) + \\ & + \frac{1}{2}(-4S_5 + S_4 + 2S_3 + S_2 + 2S_1). \end{aligned} \tag{10}$$

It is obvious that $S_{n+60} = S_n$ for all $n \in \mathbb{N}$.

Lemma 4. *Let a and b be non-negative integers and $f, g \in \mathcal{M}$ satisfying the condition* (1). *Assume that $K \in \mathbb{N}$ such that*

$$S_{n+K} = S_n \quad \textit{for all} \quad n \in \mathbb{N}.$$

Let

$$J_\ell(K) := \{j \in \mathbb{N} \mid (2j+1, K, 4\ell + 1) = 1\},$$

$$\mathcal{L}(K) := \{(u, v) \mid u, v \in \mathbb{N},\ (2u+1, K, 4(v^2 + a + b) + 1) = 1\}$$

and

$$D := g(b+1) - g(a+1).$$

Then

$$S_j S_{j+1} = S_{j(j+1)+a} \quad \textit{for all} \quad j \in J_a(K), \tag{11}$$

$$(S_j + D)(S_{j+1} + D) = S_{j(j+1)+b} + D \quad \textit{for all} \quad j \in J_b(K), \tag{12}$$

and

$$\Big(S_u + S_v + D\Big)\Big(S_{u+1} + S_v + D\Big) = S_{u(u+1)+v^2+a+b} + S_v + D \tag{13}$$

for all $(u, v) \in \mathcal{L}(K)$.

Proof of Lemma 4. First we prove (11). For each $j \in J_a(K)$ we have $(2j+1, K, 4a+1) = 1$, consequently there is a $x_j \in \mathbb{N}$ such that

$$(2Kx_j + (2j+1), 4a+1) = 1.$$

Let $n_j := Kx_j + j$. Then $(2n_j + 1, 4a+1) = 1$ and so $n_j \in I_a$. From (8) we have

$$S_{n_j} S_{n_j+1} = S_{n_j(n_j+1)+a},$$

which with $n_j \equiv j \pmod K$ proves (11).

Now we prove (12) and (13). First we deduce from (1) that

$$f(n^2+m^2+a+b) = g(n^2+a) + g(m^2+b) = g(n^2+b) + g(m^2+a),$$

consequently

$$g(n^2+b) - g(n^2+a) = g(m^2+b) - g(m^2+a) = g(b+1) - g(a+1) := D$$

for all $n, m \in \mathbb{N}$. Then

$$(14) \qquad \begin{cases} g(n^2+b) = S_n + D \quad \text{for all} \quad n \in \mathbb{N}, \\ f(n^2+m^2+a+b) = S_n + S_m + D \quad \text{for all} \quad n, m \in \mathbb{N}. \end{cases}$$

For each $j \in J_b(K)$ we have

$$(2j+1, K, 4b+1) = 1 \quad \text{and} \quad (2Kx_j + (2j+1), 4b+1) = 1$$

for some $x_j \in \mathbb{N}$. As we seen above, for $n_j := Kx_j + j$, we have $(2n_j+1, 4b+1) = 1$ and

$$\left(n_j^2 + b, (n_j+1)^2 + b\right) = (2n_j+1, 4b+1) = 1.$$

Since $g \in \mathcal{M}$, we obtain

$$g(n_j^2+b)g((n_j+1)^2+b) = g\Big[\big(n_j^2+b\big)\big((n_j+1)^2+b\big)\Big] = g\Big[\Big(n_j(n_j+1)+b\Big)^2+b\Big],$$

which with (14) and the fact $n_j \equiv j \pmod K$ proves (12).

Now we prove (13). For each pair $(u, v) \in \mathcal{L}(K)$, there is a $x_u \in \mathbb{N}$ such that $(2Kx_u + 2u + 1, 4(v^2+a+b)+1) = 1$. Let $n_u = Kx_u + u$. Then

$$\begin{aligned} &(n_u^2 + v^2 + a + b, (n_u+1)^2 + v^2 + a + b) = \\ &= (n_u^2 + v^2 + a + b, 2n_u + 1) = (2n_u + 1, 4(v^2+a+b)+1) = \\ &= (2Kx_u + 2u + 1, 4(v^2+a+b)+1) = 1 \end{aligned}$$

and so $f \in \mathcal{M}$ implies

$$f\Big(n_u^2+v^2+a+b\Big)f\Big((n_u+1)^2+v^2+a+b\Big)=$$
$$=f\Big((n_u^2+v^2+a+b)((n_u+1)^2+v^2+a+b)\Big)=$$
$$=f\Big[\Big(n_u(n_u+1)+v^2+a+b\Big)^2+v^2+a+b\Big].$$

This with (14) shows that

$$\Big(S_{n_u}+S_v+D\Big)\Big(S_{n_u+1}+S_v+D\Big)=S_{n_u(n_u+1)+v^2+a+b}+S_v+D,$$

and so (13) is proved because the condition $n_u \equiv u \pmod K$ implies

$$S_{n_u}=S_u \quad \text{and} \quad S_{n_u(n_u+1)+v^2+a+b}=S_{u(u+1)+v^2+a+b}.$$

Lemma 4 is proved. ∎

Lemma 5. *Let a and b be non-negative integers and $f, g \in \mathcal{M}$ satisfying the condition (1). Let $S_n = g(n^2+a)$. If $S_{n+1}=S_n$ for all $n \in \mathbb{N}$, then*

(15) $$(f,g) \in \{(f_0,g_0),(f_1,g_1),(f_2,g_2)\},$$

where $(f_0,g_0),(f_1,g_1),(f_2,g_2)$ are given in Table 1.

Proof. By our assumption, we have $S_n = s$ for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ such that

$$(2n+1,4a+1)=(2n+1,4b+1)=(2n+1,4(a+b+1)+1)=1.$$

Then we have

$$(n^2+a,(n+1)^2+a)=1,(n^2+b,(n+1)^2+b)=1$$

and

$$(n^2+a+b+1,(n+1)^2+a+b+1)=1,$$

consequently

$$S_nS_{n+1}=S_{n(n+1)+a},$$
$$(S_n+D)(S_{n+1}+D)=S_{n(n+1)+a}+D$$

and

$$(S_n+S_1+D)(S_{n+1}+S_1+D)=S_{n(n+1)+a+b+1}+S_1+D.$$

Since

$$g(n^2+a)=s,\ g(m^2+D)=s+D$$

and

$$f(n^2+m^2+a+b)=2s+D \quad \text{for all} \quad n,m\in\mathbb{N},$$

we get from the above relations that

$$s^2=s,\ \ (s+D)^2=s+D \quad \text{and} \quad (2s+D)^2=2s+D.$$

It is clear to see that all solutions of this system are:

$$(s,D)\in\{(0,0),(0,1),(1,-1)\}.$$

Thus, (15) is true and Lemma 5 is proved. ■

In the following we say that the sequence $\{S_n\}_{n=1}^{\infty}$ is trivial, if there is a number s such that $S_n=s$ for all $n\in\mathbb{N}$.

Lemma 6. *Let a and b be non-negative integers and $f,g\in\mathcal{M}$ satisfying the condition (1). Let $S_n=g(n^2+a)$. If $S_{n+4}=S_n$ and $\{S_n\}_{n=1}^{\infty}$ is not trivial, then $S_{n+2}=S_n$ is satisfied for all $n\in\mathbb{N}$ and*

$$(f,g)\in\{(f_3,g_3),(f_4,g_4),(f_5,g_5),(f_6,g_6)\},$$

where $(f_3,g_3),(f_4,g_4),(f_5,g_5),(f_6,g_6)$ are given in Table 2.

Proof. From our assumption and Lemma 4, we have $K=4$ and (11)–(13) hold for all $j,u,v\in\mathbb{N}$. Thus, we obtain from (11) and (12) that

$$S_2(S_3-S_1)=S_2S_3-S_1S_2=S_{a+2}-S_{a+2}=0,$$

$$S_4(S_3-S_1)=S_3S_4-S_4S_5=S_a-S_a=0$$

and

$$\begin{aligned}(S_2+D)(S_3-S_1)&=[(S_2+D)(S_3+D)-D]-[(S_1+D)(S_2+D)-D]=\\&=S_{b+2}-S_{b+2}=0.\end{aligned}$$

We shall prove that

$$S_3=S_1 \quad \text{and} \quad S_4=S_2. \tag{16}$$

Assume that $S_3\neq S_1$. Then the above relations imply that $S_2=S_4=0$ and $D=0$. By applying (13) with $(u,v)=(3,1),(3,3),(4,1)$ and $(u,v)=(4,3)$, we have

$$S_{a+b+2}=(S_3+S_1+D)(S_4+S_1+D)-(S_1+D):=x_1,$$

$$S_{a+b+2} = (S_3 + S_3 + D)(S_4 + S_3 + D) - (S_3 + D) := x_2,$$

$$S_{a+b+2} = (S_4 + S_1 + D)(2S_1 + D) - (S_1 + D) := x_3$$

and

$$S_{a+b+2} = (S_4 + S_3 + D)(S_1 + S_3 + D) - (S_3 + D) := x_4.$$

Consequently

$$(S_3 - S_1)(S_1 + 2S_3 + S_4 + 2D - 1) = x_1 - x_2 = 0,$$

$$(S_3 - S_1)(S_1 + S_4 + D) = x_1 - x_3 = 0,$$

and

$$(S_3 - S_1)(S_1 + S_3 + D - 1) = x_4 - x_1 = 0.$$

Since $S_3 \neq S_1$, $S_2 = S_4 = D = 0$, we have $2S_3 - 1 = 0, S_1 = 0, S_3 - 1 = 0$, which are impossible. Thus, the first assertion of (16) is proved.

Assume that $S_3 = S_1$. Now we apply (13) with $(u, v) = (1, 2), (1, 4), (3, 2)$ and $(u, v) = (3, 4)$ to get

$$S_{a+b+2} = (S_1 + S_2 + D)(2S_2 + D) - (S_2 + D),$$

$$S_{a+b+2} = (S_1 + S_4 + D)(S_2 + S_4 + D) - (S_4 + D),$$

$$S_{a+b} = (S_3 + S_2 + D)(S_4 + S_2 + D) - (S_2 + D)$$

and

$$S_{a+b} = (S_3 + S_4 + D)(2S_4 + D) - (S_4 + D).$$

The first two relations imply that

$$(S_2 - S_4)(S_4 + 2S_2 + S_1 + 2D - 1) = 0$$

and the last two equations give

$$(S_2 - S_4)(2S_4 + S_2 + S_1 + 2D - 1) = 0.$$

Since

$$(S_2 - S_4)(2S_4 + S_2 + S_1 + 2D - 1) - (S_2 - S_4)(S_4 + 2S_2 + S_1 + 2D - 1) = (S_2 - S_4)^2,$$

we have $S_4 = S_2$. Thus, (16) is proved.

In the following we may assume that (16) is true. Then the sequence $\{S_n\}_{n=1}^{\infty}$ satisfies the condition $S_{n+2} = S_n$ for all $n \in \mathbb{N}$ and we infer from (10) that

$$S_n = (-S_2 + S_1)\chi_2(n) + S_2 \in \{S_1, S_2\},$$

where $S_1 \neq S_2$, because the sequence $\{S_n\}_{n=1}^{\infty}$ is not trivial. We obtain from (11)-(13) that

$$S_a = S_1S_2, S_b = (S_1 + D)(S_2 + D) - D$$

and

$$S_{a+b} = (S_1+S_2+D)(2S_2+D)-S_2-D, S_{a+b+1} = (S_1+S_2+D)(2S_1+D)-S_1-D.$$

The solutions of S_n are given in the parities of a and b.

(Ia) If $(a, b) \equiv (0, 0) \pmod 2$, then $(f, g) = (f_3, g_3)$.

In this case, $(S_a, S_b, S_{a+b}, S_{a+b+1}) = (S_2, S_2, S_2, S_1)$ and so

$$\begin{cases} S_a = S_1S_2 & = S_2 \\ S_b = (S_1 + D)(S_2 + D) - D & = S_2 \\ S_{a+b} = (S_1 + S_2 + D)(2S_2 + D) - S_2 - D & = S_2 \\ S_{a+b+1} = (S_1 + S_2 + D)(2S_1 + D) - S_1 - D & = S_1. \end{cases}$$

This is equivalent to

$$\begin{cases} S_2(S_1 - 1) & = 0 \\ (S_2 + D)(S_1 + D - 1) & = 0 \\ (2S_2 + D)(S_1 + S_2 + D - 1) & = 0 \\ (2S_1 + D)(S_1 + S_2 + D - 1) & = 0, \end{cases}$$

and the last two equations with the condition $S_1 \neq S_2$ imply that $S_1 + S_2 + D - 1 = 0$. If $S_2 = 0$, then $D(D - 1) = 0$ and $S_1 + D - 1 = 0$, which imply that $DS_1 = 0$. But $S_1 \neq S_2 = 0$, we have $D = 0$ and $S_1 = -D + 1 = 1$. Thus we proved that $S_2 = 0$ implies $(S_1, S_2, D) = (1, 0, 0)$. If $S_2 \neq 0$, then $S_1 = 1$ and $S_2 + D = 0$. Thus, $g(n^2 + a) = S_n = c\chi_2(n) + (1 - c) = g_3(n^2 + a)$, $g(n^2 + b) = S_n + D = c\chi_2(n) = g_3(n^2 + b)$. The case (Ia) is proved.

(Ib) If $(a, b) \equiv (0, 1) \pmod 2$, then $(f, g) = (f_4, g_4)$.

Assume that $(a, b) \equiv (0, 1) \pmod 2$.

Then $(S_a, S_b, S_{a+b}, S_{a+b+1}) = (S_2, S_1, S_1, S_2)$ and we have the system of equations

$$\begin{cases} S_2(S_1 - 1) & = 0 \\ (S_1 + D)(S_2 + D - 1) & = 0 \\ (2S_2 + D - 1)(S_1 + S_2 + D) & = 0 \\ (2S_1 + D - 1)(S_1 + S_2 + D) & = 0. \end{cases}$$

Similarly as above, the last two equations imply $S_1 + S_2 + D = 0$. If $S_2 \neq 0$, then $S_1 = 1$, $(1+D)(S_2+D-1) = 0$, $1+S_2+D = 0$, and from this we obtain

$$-2(D+1) = (1+D)(S_2+D+1) - 2(D+1) = (1+D)(S_2+D-1) = 0.$$

This implies $D = -1$ and $S_2 = 1+S_2-1 = S_1+S_2+D = 0$, which is impossible. Thus we proved that $S_2 = 0$, consequently $S_1 + D = 0$ and $(S_1, S_2, D) = (c, 0, -c)$, where $c \neq 0$. Thus, $(S_n, D) = (c\chi_2(n), -c)$ and the assertion (Ib) is proved.

(Ic) If $(a, b) \equiv (1, 0) \pmod 2$, then $(f, g) = (f_5, g_5)$.

In this case we have $(S_a, S_b, S_{a+b}, S_{a+b+1}) = (S_1, S_2, S_1, S_2)$, and similarly we get

$$\begin{cases} S_1(S_2-1) & = 0 \\ (S_1+D-1)(S_2+D) & = 0 \\ S_1+S_2+D & = 0. \end{cases}$$

It is obvious that if $S_1 = 0$, then $S_2 + D = 0$. Thus $(f, g) = (f_5, g_5)$ and (Ic) is true. If $S_1 \neq 0$, then $S_2 = 1$, and $(S_1 + D - 1)(1 + D) = 0, S_1 + 1 + D = 0$ imply $D = -1$, $S_1 = 0$. This is impossible.

(Id) If $(a, b) \equiv (1, 1) \pmod 2$, then $(f, g) = (f_6, g_6)$.

Assume that $(a, b) \equiv (1, 1) \pmod 2$.

Then $(S_a, S_b, S_{a+b}, S_{a+b+1}) = (S_1, S_1, S_2, S_1)$ and the system of equations is the following:

$$\begin{cases} S_1(S_2-1) & = 0 \\ (S_1+D)(S_2+D-1) & = 0 \\ S_1+S_2+D-1 & = 0. \end{cases}$$

Similarly as in the case (Ia), if $S_1 = 0$, then $D = 0$ and $S_2 = 1$. If $S_1 \neq 0$, then $S_2 = 1$ and $S_1 + D = 0$. Consequently $S_n = (S_1 - 1)\chi_2(n) + 1$, $g(n^2 + b) = = (S_1 - 1)\chi_2(n) + 1 - S_1$ and so (Id) holds for $c = S_1 - 1 \neq 0$.

The proof of Lemma 6 is completed. ∎

Lemma 7. *Let a and b be non-negative integers and $f, g \in \mathcal{M}$ satisfying the condition* (1). *Let $S_n = g(n^2 + a)$. If $S_{n+3} = S_n$ and $\{S_n\}_{n=1}^{\infty}$ is not trivial, then*

$$(f, g) \in \{(f_7, g_7), \cdots, (f_{11}, g_{11})\},$$

where $(f_7, g_7), \cdots, (f_{11}, g_{11})$ are given in Table 3.

Proof. Assume that $S_{n+3} = S_n$. Then $K = 3$ and it is obvious that $2, 3 \in J_a(3)$, $2, 3 \in J_b(3)$. We prove that

$$S_2 = S_1. \tag{17}$$

Assume that $S_2 \neq S_1$. Then we infer from (11) that

$$S_3(S_2 - S_1) = S_2S_3 - S_3S_4 = S_a - S_a = 0$$

and

$$(S_3+D)(S_2-S_1) = [(S_2+D)(S_3+D)-D]-[(S_3+D)(S_4+D)-D] = S_b-S_b = 0,$$

which imply $S_3 = 0$, $D = 0$.

On the other hand, we have $(2,1), (3,1), (2,2), (3,2) \in \mathcal{L}(3)$ and so we get from (13) that

$$S_{a+b+1} = (S_2 + S_1 + D)(S_3 + S_1 + D) - S_1 - D,$$

$$S_{a+b+1} = (S_3 + S_1 + D)(S_1 + S_1 + D) - S_1 - D,$$

$$S_{a+b+1} = (2S_2 + D)(S_3 + S_2 + D) - S_2 - D$$

and

$$S_{a+b+1} = (S_3 + S_2 + D)(S_1 + S_2 + D) - S_2 - D$$

The first and second relations imply that

$$S_1(S_2 - S_1) = (S_3 + S_1 + D)(S_2 - S_1) = S_{a+b+1} - S_{a+b+1} = 0$$

and so

$$S_1 = 0, S_{a+b+1} = (S_3 + S_1 + D)(S_1 + S_1 + D) - S_1 - D = 0.$$

This with the third relation gives $S_2 = \frac{1}{2}$, because

$$0 = S_{a+b+1} = (2S_2 + D)(S_3 + S_2 + D) - S_2 - D = S_2(2S_2 - 1).$$

Then the last relation implies

$$S_{a+b+1} = (S_3 + S_2 + D)(S_1 + S_2 + D) - S_2 - D = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4},$$

which is contradicted by the fact that $S_{a+b+1} = 0$. Therefore, (17) is proved.

Since $\{S_n\}_{n=1}^{\infty}$ is not trivial sequence, we assume that $S_3 \neq S_1$. Then we have

$$S_n = (S_1 - S_3)\chi_3(n) + S_3 \quad \text{for all} \quad n \in \mathbb{N},$$

where $\chi_3(n)$ is the principal Dirichlet character $\pmod 3$.

One can check by using (11)-(12) that

$$S_a = S_1 S_3, \quad S_b = (S_1 + D)(S_3 + D) - D, \tag{18}$$

furthermore by applying (14) with $(u, v) = (2, 3),\ (2, 1)$, we have

$$S_{a+b} = (S_1 + S_3 + D)(2S_3 + D) - S_3 - D \tag{19}$$

and

$$S_{a+b+1} = (2S_1 + D)(S_3 + S_1 + D) - S_1 - D. \tag{20}$$

∘ First we consider the case when $a + b \equiv 0 \pmod 3$. It is obvious that $(1, 2) \in \mathcal{L}(3)$, and so

$$S_{a+b} = (2S_1 + D)^2 - S_1 - D. \tag{21}$$

This with (19) leads to $(S_1 - S_3)(2S_3 + 4S_1 + 3D - 1) = 0$, and so

$$2S_3 + 4S_1 + 3D - 1 = 0. \tag{22}$$

On other hand, $a + b \equiv 0 \pmod 3$ gives

$$S_{a+b} - S_3 = (2S_3 + D)(S_1 + S_3 + D - 1) = 0$$

and

$$S_{a+b+1} - S_1 = (2S_1 + D)(S_1 + S_3 + D - 1) = 0.$$

These imply that

$$S_1 + S_3 + D - 1 = 0. \tag{23}$$

One can check from (22) and (23) that

$$S_3 = S_1 + 2 \quad \text{and} \quad D = -2S_1 - 1.$$

Since $a + b \equiv 0 \pmod 3$, there are three possibilities:

(i) $(a, b) \equiv (1, 2) \pmod 3$,

(ii) $(a, b) \equiv (2, 1) \pmod 3$,

(iii) $(a, b) \equiv (3, 3) \pmod 3$.

In the case (i), we have $1 \in J_a(3)$, $S_a = S_1$ and $S_b = S_1$, consequently

$$S_1(S_3 - 1) = S_1(S_1 + 1) = 0 \quad \text{and} \quad S_1^2 - S_1 - 2 = S_1 S_2 - S_3 = S_{a+2} - S_3 = 0.$$

These imply that $S_1 = -1$, and so $S_3 = S_1 + 2 = 1$, $D = -2S_1 - 1 = 1$. Thus $(S_1, S_3, d) = (-1, 1, 1)\}$ and $(f, g) \in \{(f_8, g_8)\}$.

In the case (ii), we also have $1 \in J_b(3)$, $S_a = S_1$, $S_b = S_1$, and so it follows that $(f, g) \in \{(f_8, g_8)\}$.

In the case (iii), we have $1 \in J_a$, consequently $S_a = S_1 S_3 = S_3$ and $S_1^2 = S_{a+2} = S_1$. It is obvious from $S_3 \neq S_1$ and $S_1 S_3 = S_3$ that $S_1 \neq 0$. Therefore, $S_1^2 = S_1$ implies that $S_1 = 1$, and so $S_3 = S_1 + 2 = 3$, $D = -2S_1 - 1 = -3$. This shows that $g(n^2 + a) = S_n = -2\chi_3(n) + 3 = g_{11}(n^2 + a)$, $g(n^2 + b) = -2\chi_3(n) = g_{11}(n^2 + b)$.

∘ Now we consider the case when $a + b \equiv 1 \pmod 3$. We have $(1, v) \in \mathcal{L}(3)$ for all $v \in \mathbb{N}$, therefore (21) and (22) are true, furthermore $(1, 3) \in \mathcal{L}(3)$ with (13) gives

$$S_{a+b+2} = (S_1 + S_3 + D)^2 - S_3 - D. \tag{24}$$

Thus, we have

$$S_{a+b} - S_1 = (S_2 + S_1 + D)(2S_3 + D) - S_3 - S_1 - D = (2S_3 + D - 1)(S_1 + S_3 + D) = 0,$$

$$S_{a+b} - S_1 = (2S_1 + D)^2 - 2S_1 - D = (2S_1 + D)(2S_1 + D - 1) = 0,$$

$$S_{a+b+1} - S_1 = (2S_1 + D)(S_1 + S_3 + D - 1) = 0$$

and

$$S_{a+b+2} - S_3 = (S_1 + S_3 + D)^2 - 2S_3 - D = 0.$$

It is clear to see from the second and third relation that $S_1 = -\frac{D}{2}$, and so we have

$$(2S_3 + D - 1)(2S_3 + D) = 0 \quad \text{and} \quad (2S_3 + D)(2S_3 + D - 4) = 0.$$

Since $S_3 \neq S_1 = -\frac{D}{2}$, we have $2S_3 + D \neq 0$. Consequently

$$2S_3 + D - 1 = 0 \quad \text{and} \quad 2S_3 + D - 4 = 0,$$

which are impossible.

∘ Finally we consider the case when $a + b \equiv 2 \pmod 3$. Then we have

$$S_{a+b} - S_1 = (S_1 + S_3 + D)(2S_3 + D) - S_3 - D - S_1 = (2S_3 + D - 1)(S_1 + S_3 + D) = 0$$

and

$$S_{a+b+1} - S_3 = (2S_1 + D)(S_3 + S_1 + D) - S_3 - S_1 - D = (2S_1 + D - 1)(S_1 + S_3 + D) = 0,$$

which show that $S_1 + S_3 + D = 0$.

Since $a + b \equiv 2 \pmod 3$, there are three possibilities:

(iv) $(a, b) \equiv (1, 1) \pmod 3$,

(v) $(a,b) \equiv (2,3) \pmod 3$,

(vi) $(a,b) \equiv (3,2) \pmod 3$.

In the case (iv), we have $S_a = S_1$, $S_b = S_1$. It is clear to see that if $S_1 = 0$, then $S_3 + D = 0$ and $D = 0$, consequently $S_3 = 0$, which is impossible. If $S_1 \neq 0$, then $S_3 = 1$, which implies that $S_1 + 1 + D = 0$ and $(S_1 + D)D = 0$. Thus, $-D = (S_1 + 1 + D)D - D = (S_1 + D)D = 0$ and $S_1 = -D - 1 = -1$. Hence we have $(f,g) = (f_7, g_7)$.

In the case (v), we have $S_a = S_1S_3 = S_1$, $S_b = S_3$ and if $S_1 \neq 0$, then $S_1S_3 = S_1$ implies $S_3 = 1$ and $S_1 + D + 1 = 0$. We infer from the fact

$$\begin{aligned}0 = S_b - S_3 =&(S_1 + D)(S_3 + D) - D - S_3 = (S_1 + D - 1)(S_3 + D) = \\ &(S_1 + D + 1)(S_3 + D) - 2(S_3 + D) = -2(S_3 + D)\end{aligned}$$

that $D = -S_3 = -1$, which is contradicted by the fact that $S_1 = -(S_3+D) = 0$.

Thus, $S_1 = 0$ and $S_3 = -D, D \neq 0$. Since $1 \in J_b(3)$, we infer from (12) that

$$(S_1 + D)^2 - D = S_{b+2} = S_1,\ (S_1 + D)(S_1 + D - 1) = 0,$$

which with $D \neq 0$ implies that $D = 1$. Then $(S_1, S_3, D) = (0, -1, 1)$ and $(f,g) = (f_9, g_9)$.

In the case (vi), we have $1 \in J_a$,$S_a = S_1S_3 = S_3$, $S_b = S_1$ and $S_1^2 = S_{a+2} = = S_1$. Then $S_1S_3 = S_3$ and $S_1 \neq S_3$ imply $S_1 \neq 0$. Then $S_1^2 = S_1$ implies $S_1 = 1$, and so $S_3 + D = -S_1 = -1$. Finally, we infer from

$$0 = S_b - S_1 = (S_1 + D)(S_3 + D) - D - S_1 = (S_3 + D - 1)(S_1 + D) = -2(S_1 + D)$$

that $D = -S_1 = -1$ and $S_3 = -S_1 - D = 0$. Thus, we have $(S_1, S_3, D) = = (1, 0, -1)$ and $(f,g) = (f_{10}, g_{10})$.

Lemma 7 is proved. ■

Proof of (III).

Assume that the non-negative integers a and b and $f, g \in \mathcal{M}$ satisfy the condition (1), furthermore $A = 0$ and (10) hold. Let $S_n = g(n^2 + a), D = g(b + 1) - g(a + 1)$.

First we note from (10) that $K = 60$ and $S_{11} = S_1, S_{12} = S_4 + S_3 - S_1$. Since $3, 11 \in J_a(60)$, we infer from (11) that

$$(S_4 - S_1)(S_3 - S_1) = S_3S_4 - S_{11}S_{12} = S_{a+12} - S_{a+12} = 0.$$

There are two possibilities: (I) $S_4 \neq S_1, S_3 = S_1$ and (II) $S_4 = S_1$.

Case I: $S_4 \neq S_1, S_3 = S_1$.

We shall prove that $S_5 = S_1$.

One can check from (10) that

$$S_8 = 2S_5 + S_4 - 2S_1, S_9 = -2S_5 + 3S_1, S_{23} = 2S_5 - S_1$$

and

$$S_{24} = -2S_5 + S_4 + 2S_1,$$

which with (11) imply

$$4(S_4 - S_1)(S_5 - S_1) = S_{23}S_{24} - S_8S_9 = S_{a+12} - S_{a+12} = 0,$$

because

$$\begin{aligned} S_{23}S_{24} - S_8S_9 &= (2S_5 - S_1)(-2S_5 + S_4 + 2S_1)- \\ &- (2S_5 + S_4 - 2S_1)(-2S_5 + 3S_1) = 4(S_4 - S_1)(S_5 - S_1). \end{aligned}$$

Thus, we proved that $S_5 = S_1$.

Since $S_5 = S_3 = S_1$, the sequence $\{S_n\}_{n=1}^{\infty}$ has the form $\{S_1, S_2, S_1, S_4, \cdots\}$, and so $K = 4$. Consequently, all solutions are given in Lemma 6.

Case II: $S_4 = S_1$.

We deduce from (10) that $K = 60$, furthermore

$$S_8 = 2S_5 - S_1, \quad S_9 = -2S_5 + S_3 + 2S_1, \quad S_{10} = -S_5 + S_2 + S_1$$

and

$$S_{14} = -2S_5 + S_2 + 2S_1, \quad S_{15} = -S_5 + S_3 - S_1.$$

Since $3, 8, 9, 14 \in J_a(60)$, we infer from (11) that

$$2(S_5 - S_1)(2S_5 - S_3 - S_1) = S_3S_4 - S_8S_9 = S_{a+12} - S_{a+12} = 0$$

and

$$(S_5 - S_1)(S_3 - S_2) = S_9S_{10} - S_{14}S_{15} = S_{a+30} - S_{a+30} = 0.$$

Case (II.a): $S_4 = S_1, \; S_5 \neq S_1$**.**

In this case the above relations imply

$$S_3 = S_2 \quad \text{and} \quad S_5 = \frac{S_3 + S_1}{2}$$

and so we get from (10) that

$$S_n = \Big(\frac{S_1 - S_2}{2}\Big)\chi_5(n) + \Big(\frac{S_1 + S_2}{2}\Big) \quad \text{for all} \quad n \in \mathbb{N}. \tag{25}$$

If $S_2 = S_1$ then $S_n = S_1$ for all $n \in \mathbb{N}$. Hence by Lemma 5 we get all solutions of (f, g).

Assume now that $S_2 \neq S_1$. Since $(u, v) \in \mathcal{L}(5)$ for $(u, v) \in \{(1, 2), (4, 1), (2, 5)\}$, an application of (13) for these pairs, we have

$$S_{a+b+1} = (S_1 + S_2 + D)(2S_2 + D) - (S_2 + D) := y_1,$$

$$S_{a+b+1} = (S_4 + S_1 + D)(S_5 + S_1 + D) - (S_1 + D) =$$
$$= (2S_1 + D)(S_1 + S_5 + D) - (S_1 + D) := y_2,$$

and

$$S_{a+b+1} = (S_2+S_5+D)(S_3+S_5+D)-(S_5+D) = (S_2+S_5+D)^2-(S_5+D) := y_3,$$

which imply

$$y_1 - y_2 = \frac{1}{2}(S_2 - S_1)(4S_2 + 6S_1 + 5D - 2) = 0$$

and

$$y_1 - y_3 = \frac{-1}{4}(S_2 - S_1)(S_2 - S_1 + 2) = 0.$$

These imply $S_1 = 1 - \frac{D}{2}$ and $S_2 := -1 - \frac{D}{2}$, consequently we get from (25) that

$$S_n = \chi_5(n) - \frac{D}{2} \quad \text{for all} \quad n \in \mathbb{N}.$$

It is obvious that $(5, 5) \in \mathcal{L}(5)$, we get from (13) that

$$S_{a+b} = (S_5 + S_5 + D)(S_6 + S_5 + D) - (S_5 + D) =$$

$$= (2S_5 + D)(S_1 + S_5 + D) - (S_5 + D) = -\frac{D}{2},$$

consequently $5|a + b$. Thus,we have $(5, 4(a + b + 2 \cdot 3 + 2^2) + 1) = (5, 41) = 1$ and $(2, 2) \in \mathcal{L}(5)$. An application of (13) with $(u, v) = (2, 2)$ implies

$$S_{a+b} = (S_2+S_2+D)(S_3+S_2+D)-(S_2+D) = (-2)(-2)-(-1+\frac{D}{2}) = 5-\frac{D}{2}.$$

This is impossible.

Case (II.b): $S_5 = S_4 = S_1$

The sequence $\{S_n\}_{n=1}^{\infty}$ has the form $\{S_1, S_2, S_3, S_1, \cdots\}$, and so $K = 12$. We have

$$(S_2 - S_1)(S_3 - S_1) = S_9 S_{10} - S_5 S_6 = S_{a+6} - S_{a+6} = 0,$$

and so there are two possibilities:

○ (i) $S_2 = S_1$ and ○ (ii) $S_3 = S_1$.

In the case (i), we have

$$S_n = (S_1 - S_3)\chi_3(n) + S_3 \quad \text{for all} \quad n \in \mathbb{N},$$

where $\chi_3(n)$ is the principal Dirichlet character $\pmod 3$. Thus, $S_{n+3} = S_n$ for all $n \in \mathbb{N}$, consequently Lemma 7 gives all solutions of (f, g).

Now assume that (ii) is true. Then the sequence $\{S_n\}_{n=1}^{\infty}$ has the form $\{S_1, S_2, S_1, S_1, \cdots\}$, and so $K = 4$. Lemma 6 gives all solutions of (f, g).

The proof of (III) is completed and the theorem is proved. ■

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