ON CONVOLUTED SUMS

Jean-Marie De Koninck[1] (Québec, Canada)

Imre Kátai (Budapest, Hungary)

Dedicated to the memory of C.A. Corrádi

Communicated by Ferenc Schipp

(Received February 5, 2016; accepted March 11, 2016)

Abstract. Given a complex valued multiplicative function f such that $|f(n)| = 1$ for each $n \in \mathbb{N}$, let $h_f(n) := \sum_{\nu=1}^{n-1} f(\nu)f(n-\nu)$. We investigate under which conditions we have $h_f(n) = o(n)$ for almost all positive integers n as $n \to \infty$.

1. Introduction and notation

Let $\mathcal{M}_1$ stand for the set of those multiplicative functions f which are such that $|f(n)| = 1$ for each $n \in \mathbb{N}$. Then, given $f \in \mathcal{M}_1$, consider the corresponding convoluted sum

$$h_f(n) := \sum_{\nu=1}^{n-1} f(\nu)f(n-\nu).$$

This function was studied by Corrádi and Kátai [2] in 1969 in the case where the function f takes the values ± 1 only, and more recently by De Koninck, Germán and Kátai [4] in the case where $|f(n)| \leq 1$ for each $n \in \mathbb{N}$. Here, we are interested in establishing under which conditions we have that

$$\frac{h_f(n)}{n} \to 0 \quad \text{as } n \to \infty \qquad \text{for almost all } n.$$

Key words and phrases: Completely multiplicative functions.
2010 Mathematics Subject Classification: 11N37.

[1] Research supported in part by a grant from NSERC.

https://doi.org/10.71352/ac.45.075

Letting f be as above and $\alpha \in \mathbb{R}$, consider the exponential sum

$$S_f(N,\alpha) := \sum_{n=1}^{N-1} f(n)e(n\alpha),$$

where we used the classical notation $e(y) = e^{2\pi iy}$. Then it is clear that

$$S_f^2(N,\alpha) = \sum_{n=1}^{2N-2} h_{f,N}(n)e(n\alpha),$$

where

$$h_{f,N}(n) = \begin{cases} h_f(n) & \text{if} \quad n \le N, \\ \displaystyle\sum_{\max(\nu,n-\nu)\le N-1} f(\nu)f(n-\nu) & \text{if} \quad n > N. \end{cases}$$

From this, it follows that

$$\begin{aligned} \sum_{n=1}^{N} |h_f(n)|^2 &\le \sum_{n\le 2N} |h_{f,N}(n)|^2 = \int_{-1/2}^{1/2} |S_f(N,\alpha)|^4 \, d\alpha \le \\ &\le \max_{\alpha\in[0,1)} |S_f(N,\alpha)|^2 \cdot \int_{-1/2}^{1/2} |S_f(N,\alpha)|^2 \, d\alpha, \end{aligned}$$

from which it follows, in light of the fact that this very last integral is equal to N, that

$$\sum_{N/2\le n\le N} \frac{|h_f(n)|^2}{n^2} \le \max_{\alpha\in[0,1)} \left| \frac{S_f(N,\alpha)}{N} \right|^2. \tag{1.1}$$

Hence, using (1.1), it follows that if

$$\max_{\alpha\in[0,1)} \left| \frac{S_f(N,\alpha)}{N} \right| \to 0 \qquad (N \to \infty), \tag{A}$$

then

$$\frac{h_f(n)}{n} \to 0 \qquad (n \to \infty) \qquad \text{for almost all } n. \tag{B}$$

This raises the following question: “Under which condition does (A) hold ?”

Now, an obvious necessary condition for (A) to hold is that for every integers $q \ge 1$ and $\ell \ge 0$, we have that

$$\frac{1}{N} \sum_{\substack{n\le N \\ n\equiv \ell \pmod q}} f(n) \to 0 \qquad (N \to \infty). \tag{C}$$

Indeed, this follows from the fact that

$$\sum_{\substack{n\le N\\ n\equiv \ell \pmod q}} f(n) = \sum_{n\le N} f(n)\cdot \frac{1}{q}\sum_{a=0}^{q-1} e\left(\frac{a(n-\ell)}{q}\right) = \frac{1}{q}\sum_{a=0}^{q-1} e\left(-\frac{a\ell}{q}\right) S_f(N,a/q).$$

Here, we investigate the particular cases when f is a completely multiplicative function and when f is q-multiplicative.

2. Main results

Let $\mathcal{M}_1^*$ stand for those functions $f\in\mathcal{M}_1$ which are completely multiplicative.

Theorem 1. *Let $f\in\mathcal{M}_1^*$. Statement (A) holds if and only if for every $q\in\mathbb{N}$ and every corresponding Dirichlet character χ_q and every $\tau\in\mathbb{R}$ we have*

$$\sum_p \frac{\Re(1-\chi_q(p)p^{i\tau}f(p))}{p} = \infty$$

and

$$\sum_p \frac{\Re(1-p^{i\tau}f(p))}{p} = \infty.$$

Let $\mathcal{M}_q^{(1)}$ stand for the set of all q-multiplicative functions $f:\mathbb{N}_0\to\mathbb{C}$ satisfying $f(0)=1$ and $|f(n)$=1 for each positive integer n. To each $f\in\mathcal{M}_q^{(1)}$, we associate the sum

$$S_x(z) := \sum_{n<x} f(n)z^n. \tag{2.1}$$

Theorem 2. *Let $f\in\mathcal{M}_q^{(1)}$ with corresponding sum $S_x(z)$ defined in* (2.1). *Further set*

$$G_m(z) = \sum_{a=0}^{q-1} f(aq^m)z^a \qquad (m=1,2,\ldots)$$

and assume that

$$\sum_{m=1}^{\infty}\left(1-\max_{|z|=1}\left|\frac{G_m(z)}{q}\right|^2\right) = \infty. \tag{2.2}$$

Then,

$$\lim_{x\to\infty}\max_{|z|=1}\frac{|S_x(z)|}{x}=0 \tag{2.3}$$

and

$$\frac{h_f(n)}{n}\to 0 \qquad \text{as } n\to\infty \qquad \text{for almost all } n. \tag{2.4}$$

3. Preliminary results

In 1974, Daboussi [3] proved that given $\alpha \in [0,1)$ and assuming that $\left|\alpha - \frac{a}{q}\right| \le \frac{1}{q^2}$, where a and q are positive integers satisfying $(a,q)=1$ and $3\le q\le \sqrt{N/\log N}$, there exists an absolute constant $c_1>0$ such that

$$\max_{f\in\mathcal{M}_1^*}|S_N(f,\alpha)|\le \frac{c_1 N}{\sqrt{\log\log q}}.$$

In 1977, Montgomery and Vaughan [7] proved that letting a and q be two positive integers such that $(a,q)=1$ and $q\le N$, there exists an absolute constant $c_2>0$ such that

$$\max_{f\in\mathcal{M}_1^*}|S_N(f,a/q)|\le \frac{c_2 N}{\sqrt{\log 2N}}+\frac{c_2}{\phi(q)}+c_2\left(\frac{q}{N}\right)^{1/2}\left(\log\frac{2N}{q}\right)^{3/2},$$

where ϕ stands for the Euler totient function.

As an immediate consequence of Montgomery and Vaughan's inequality, we have the following result.

Let $\alpha\in[0,1)$ and assume that $\left|\alpha-\frac{a}{q}\right|\le\frac{1}{q^2}$, where a, q and R are positive integers satisfying $(a,q)=1$ and $2\le R\le q\le N/R$. Then, there exists an absolute constant $c_3>0$ such that

$$\max_{f\in\mathcal{M}_1^*}|S_N(f,\alpha)|\le\frac{c_3}{\log N}+c_3\frac{(\log R)^{3/2}}{\sqrt{R}}. \tag{3.1}$$

The following is a variant of a theorem of Kátai (see Kátai [6]).

Lemma 1. *Let $\wp_N$ be an arbitrary subset of the primes, each element of which does not exceed N. Further set $A_N := \sum_{p\in\wp_N} 1/p$ and, given any arbitrary real number $\alpha \in [0,1)$, let*

$$h_N(\alpha) := \sum_{\substack{p,q\in\wp_N\\ p\neq q}} \frac{1}{\|\alpha(p-q)\|}, \tag{3.2}$$

where $\|y\|$ stands for the distance from y to the nearest integer. Then there exist absolute constants $C_4 > 0$ and $C_5 > 0$ such that

$$\max_{f\in\mathcal{M}_1^*} \frac{|S_N(f,\alpha)|}{N} \le \frac{C_4}{\sqrt{A_N}} + \frac{C_5}{A_N}\sqrt{\frac{h_N(\alpha)}{N}}. \tag{3.3}$$

Proof. Let $\omega_{\wp_N}(n) := \sum_{\substack{p|n\\ p\in\wp_N}} 1$. Then, by using the Turán-Kubilius inequality, it is clear that there exists an absolute constant $C_1 > 0$ such that

$$\sum_{n\le N} |\omega_{\wp_N}(n) - A_N|^2 \le C_1 N A_N$$

and therefore that, for some absolute constant $C_2 > 0$, we have

$$\sum_{n\le N} |\omega_{\wp_N}(n) - A_N| \le C_2 N\sqrt{A_N}. \tag{3.4}$$

Let

$$U_N(f,\alpha) := \sum_{n\le N} f(n)e(n\alpha)\omega_{\wp_N}(\alpha).$$

Since

$$U_N(f,\alpha) = \sum_{\substack{pm\le N\\ p\in\wp_N}} f(p)f(m)e(pm\alpha),$$

it follows from (3.4) that

$$|S_N(f,\alpha)|A_N \le c_2 N\sqrt{A_N} + |U_N(f,\alpha)|. \tag{3.5}$$

Now, observe that

$$\begin{aligned}
|U_N(f,\alpha)|^2 &\le \left\{\sum_{m\le N} 1\right\}\left\{\sum_{m\le N}\left|\sum_{\substack{p\le N/m\\ p\in\wp_N}} f(p)e(pm\alpha)\right|^2\right\} \le \\
&\le N\left\{\sum_{\substack{pm\le N\\ p\in\wp_N}} 1 + \sum_{\substack{p_1,p_2\in\wp_N\\ p_1\ne p_2}}\ \sum_{m\le\min(N/p_1,N/p_2)} e((p_1-p_2)m\alpha)\right\} \le \\
&\le N\left\{c_3 N A_N + \sum_{\substack{p_1,p_2\in\wp_N\\ p_1\ne p_2}} \min\left(\frac{1}{\|\alpha(p_1-p_2)\|},\frac{N}{p_1},\frac{N}{p_2}\right)\right\},
\end{aligned}$$

thereby implying that

$$\left|\frac{1}{N}U_N(f,\alpha)\right|^2 \le C_3 A_N + \frac{h_N(\alpha)}{N},$$

that is,

$$\left|\frac{1}{N}U_N(f,\alpha)\right| \le C_4\sqrt{A_N} + \frac{C_5}{\sqrt{N}}\sqrt{h_N(\alpha)},$$

which, with (3.5), proves (3.3), thereby completing the proof of Lemma 1. ■

As an immediate consequence of Lemma 1, we have the following.

Lemma 2. *Let $f\in\mathcal{M}_1^*$. Let $\varepsilon>0$ be an arbitrarily small number. Let $p_1<\cdots<p_k$ be a sequence of prime numbers satisfying $\sum_{j=1}^k 1/p_j > 1/\varepsilon$. Then, provided $N>p_k$, we have*

$$\frac{|S_N(f,\alpha)|}{N} \le C_4\sqrt{\varepsilon} + C_5\varepsilon\sqrt{\frac{h_N(\alpha)}{N}}.$$

The following is a consequence of a theorem of G. Halász [5].

Lemma 3. *If statement (A) holds for some $f\in\mathcal{M}_1^*$, then for every positive integer q and every corresponding Dirichlet character χ_q, we have for each $\tau\in\mathbb{R}$,*

$$\sum_p \frac{\Re(1-\chi_q(p)p^{i\tau}f(p))}{p} = \infty \tag{3.6}$$

and

$$\sum_p \frac{\Re(1-p^{i\tau}f(p))}{p} = \infty. \tag{3.7}$$

Lemma 4. *Let $f \in \mathcal{M}_1^*$. If estimate* (3.6) *holds for every q and χ_q and if relation* (3.7) *holds as well, then*

$$\frac{S_N(f, a/q)}{N} \to 0 \qquad (N \to \infty) \tag{3.8}$$

for every congruence class $a \pmod q$, $a = 0, 1, \dots, q-1$.

Proof. Assume first that $(a, q) = 1$ and let

$$\ell(n) := \begin{cases} 1 & \text{if } n \equiv a \pmod q, \\ 0 & \text{otherwise.} \end{cases}$$

It is known that

$$\ell(n) = \sum_{\chi} d_\chi \cdot \chi(n),$$

where χ runs over the characters mod q and d_χ are suitable constants. Hence, it follows that

$$\sum_{\substack{n \le N \\ n \equiv a \pmod q}} f(n) = \sum_{\chi} d_\chi \cdot S_N(f, \chi_q) = o(N) \qquad (N \to \infty),$$

so that

$$\frac{1}{N} S_N(f, a/q) = \frac{1}{N} \sum_{n \le N} f(n) e\left(\frac{an}{q}\right) \to 0 \qquad (N \to \infty),$$

thus completing the proof of Lemma 4 in the case where $(a, q) = 1$.

In the more general case, that is when $(a, q) = \Delta \ge 1$, let $a_1 = a/\Delta$ and $q_1 = q/\Delta$. We then have that

$$S_N(f, a/q) = S_N(f, a_1/q_1),$$

allowing one to easily establish that (3.8) holds in the general case as well. ■

Let $f \in \mathcal{M}_q^{(1)}$ with corresponding sum $S_x(z)$ defined in (2.1). We then have

$$S_{q^N}(z) = \prod_{j=0}^{N-1} \left\{ \sum_{a=0}^{q-1} f(aq^j) z^{aq^j} \right\}. \tag{3.9}$$

The following result will be used in the proof of Theorem 2.

Lemma 5. *Let $f \in \mathcal{M}_q^{(1)}$ with corresponding function $S_x(z)$ defined in* (2.1). *If*

$$\max_{|z|=1} \frac{|S_{q^N}(z)|}{q^N} \to 0 \qquad (N \to \infty),$$

then

$$\max_{|z|=1} \frac{|S_x(z)|}{x} \to 0 \qquad (x \to \infty).$$

Proof. Let

$$\Lambda_N := \sup_{|z|=1} \frac{\left|S_{q^N}(z)\right|}{q^N}$$

and write successively

$$\begin{aligned} x &= b_0 q^N + x_1, \\ \text{where } x_1 &= b_1 q^{N-1} + x_2, \\ \text{where } x_2 &= b_2 q^{N-2} + x_3, \\ &\vdots \end{aligned}$$

We then have

$$S_x(z) = \left(\sum_{a=0}^{b_0} f(aq^N) z^{aq^N} \right) S_{q^N}(z) + f(b_0 q^N) z^{b_0 q^N} S_{x_1}(z),$$

where

$$S_{x_1}(z) = \left(\sum_{a=0}^{b_1} f(aq^{N-1}) z^{aq^{N-1}} \right) S_{q^{N-1}}(z) + f(b_1 q^{N-1}) z^{b_1 q^{N-1}} S_{x_2}(z),$$

and so on, at each step introducing successively the definitions of $S_{x_2}(z)$, $S_{x_3}(z)$, ...

It follows from this representation of $S_x(z)$ that

$$\begin{aligned} \frac{|S_x(z)|}{x} &\leq (b_0+1)\frac{q^N \Lambda_N}{x} + \frac{(b_1+1)q^{N-1}\Lambda_{N-1}}{x} + \cdots \leq \\ &\leq q\left(\Lambda_N + \frac{\Lambda_{N-1}}{q} + \frac{\Lambda_{N-2}}{q^2} + \cdots \right), \end{aligned}$$

which clearly implies (3.9), thus completing the proof of Lemma 5. ■

4. The proof of Theorem 1

It is sufficient to prove that if (3.6) and (3.7) hold, then (A) holds. To do so, we first observe that Lemmas 3 and 4 imply that

$$\frac{1}{N} S_N(f, a/q) \to 0 \qquad \text{as } N \to \infty \text{ for every } q \text{ and every } a \pmod q.$$

Consequently, there exists a suitable sequence $(K_N)_{N\ge 1}$ which tends to infinity with N such that

$$\max_{\substack{q\le K_N\\ a=0,1,\dots,q-1}} \frac{1}{N}\left|S_N(f,a/q)\right| \to 0 \qquad (N\to\infty). \tag{4.1}$$

Now, let $\delta>0$ be an arbitrarily small number. Then, for every pair of positive integers $N_1<N_2$ such that $N_2(1-\delta)\ge N_1$, we have

$$\max_{\substack{q\le K_{N_1}\\ a=0,1,\dots,q-1}} \frac{1}{N_2-N_1}\sum_{N_1\le n\le N_2} f(n)e\left(\frac{an}{q}\right) \to 0 \qquad (N_1\to\infty). \tag{4.2}$$

Assume that $|\alpha|\le \varepsilon/N$. We then have

$$|S_N(f,\alpha)-S_N(f,0)|\le \varepsilon N, \tag{4.3}$$

so that

$$\max_{|\alpha|\le\varepsilon/N}\frac{1}{N}|S_N(f,\alpha)|\le \varepsilon+\frac{1}{N}|S_N(f,0)|. \tag{4.4}$$

If, on the other hand, we have

$$0\le \frac{\varepsilon}{N}<\alpha\le\frac{K}{N},$$

we then create the sequence

$$N_0=\delta N,\qquad N_j=(j+1)\delta N\quad \text{for } j=1,2,\dots, \tag{4.5}$$

so that

$$S_{N_{j+1}}(f,\alpha)-S_{N_j}(f,\alpha)=e(N_j\alpha)\sum_{\ell=0}^{N_{j+1}-N_j} f(N_j+\ell)e(\ell\alpha). \tag{4.6}$$

Using the fact that $|e(\ell\alpha)-1|\le \ell K/N$, it follows from relations (4.3) to (4.6) that

$$\begin{aligned}\left|S_{N_{j+1}}(f,\alpha)-S_{N_j}(f,\alpha)\right| &\le \left|S_{N_{j+1}}(f,0)-S_{N_j}(f,0)\right|+\frac{K}{N}\sum_{\ell=0}^{\delta N}\ell=\\ &=O(1)\delta N+K\delta^2 N.\end{aligned}$$

Summing the above inequality for all $j\le 1/\delta$, we obtain that

$$|S_N(f,\alpha)|\le o(1)\frac{\delta}{\delta}N+K\delta N \qquad (N\to\infty)$$

and therefore that

$$\left|\frac{1}{N}S_N(f,\alpha)\right| \le o(1) + K\delta \qquad (N\to\infty). \tag{4.7}$$

Let us now assume that we have chosen $\delta = \varepsilon$. It then follows from (4.7) that

$$\max_{|\alpha|\le K/N} \frac{1}{N}|S_N(f,\alpha)| \le K\varepsilon + o(1) \qquad (N\to\infty). \tag{4.8}$$

Assume that $p_1 < \cdots < p_k$ are fixed primes and further assume that $\sum_{j=1}^k 1/p_j >$ $> 1/\varepsilon$ and that $N > p_k$. Moreover, let T be a large number and let $\mathcal{B}_N$ be the set of those $\alpha \in [0,1)$ for which the inequality

$$\|(p_i - p_j)\alpha\| > T/N \tag{4.9}$$

holds for every prime pair p_i, p_j with $1 \le j < i \le k$. In this case, we obtain that

$$\left|\alpha - \frac{R}{p_i - p_j}\right| > \frac{T}{N(p_i-p_j)} \qquad (R\in\mathbb{Z}).$$

Using this and the representation of $h_N(\alpha)$ provided by (3.2), it follows that

$$h_N(\alpha) \le \frac{Np_k k^2}{T}.$$

Choosing $T = p_k k^2$, it follows, using Lemma 2, that

$$\sup_{\alpha} \frac{1}{N}|S_N(f,\alpha)| \le \frac{C_6}{\sqrt{A_N}} < C_6\sqrt{\varepsilon}.$$

On the other hand, if (4.9) does not hold, then

$$\left|\alpha - \frac{R}{p_i - p_j}\right| \le \frac{T}{N(p_i-p_j)} \qquad (R\in\mathbb{Z}),$$

we may first write that

$$\alpha = \frac{R}{p_i-p_j} + \beta, \qquad \text{where } |\beta| < \frac{T}{N(p_i-p_j)}.$$

Repeating the argument used above, namely by first defining the sequence $(N_j)_{j\ge 0}$ as in (4.5), we obtain that

$$\begin{aligned} &S_{N_{j+1}}(f,\alpha) - S_{N_j}(f,\alpha) = \\ &\quad = e(N_j\alpha)\sum_{\ell=0}^{N_{j+1}-N_j} f(N_j+\ell)e\left(\frac{R}{p_i-p_j}(N_j-\ell)\right)\cdot e(\beta\ell) = \\ &\quad = O(\beta\delta^2 N) + e(N_j\beta)\left(S_{N_{j+1}}\left(f,\frac{R}{p_i-p_j}\right) - S_{N_j}\left(f,\frac{R}{p_i-p_j}\right)\right). \end{aligned}$$

Observing that

$$S_{N_{j+1}}\left(f, \frac{R}{p_i - p_j}\right) - S_{N_j}\left(f, \frac{R}{p_i - p_j}\right) = o(N_{j+1} - N_j)$$

for every j uniformly as $j \leq 1/\delta$, it follows that

$$\sup_{\alpha \in \overline{\mathcal{B}_N}} \frac{1}{N} |S_N(f, \alpha)| \to 0 \qquad (N \to \infty),$$

thus completing the proof of Theorem 1. ∎

5. The proof of Theorem 2

Let us separate the real and imaginary parts of $G_m(z) - 1$ by writing

$$G_m(z) - 1 = \sum_{a=1}^{q-1} g(aq^m) z^a = U + iV, \quad U, V \in \mathbb{R},$$

where U and V depend on z. Since $U^2 + V^2 \leq (q-1)^2$, it follows that, for each $m \geq 1$, there exists some $\rho_m > 0$ such that

$$\max_{|z|=1} \Re(G_m(z) - q) = -\rho_m.$$

Now, $1 + U - q \leq -\rho_m$ implies that $U \leq (q-1) - \rho_m$, from which it follows that

$$\begin{aligned} |G_m(z)|^2 = (1+U)^2 + V^2 = U^2 + V^2 + 2U + 1 \leq \\ \leq (q-1)^2 + 1 + 2(q-1) - 2\rho_m = q^2 - 2\rho_m. \end{aligned}$$

From this, we obtain that

$$\left| \frac{G_m(z)}{q} \right|^2 \leq 1 - \frac{2\rho_m}{q^2}$$

and therefore that

$$\frac{2\rho_m}{q^2} \leq 1 - \left| \frac{G_m(z)}{q} \right|^2.$$

Using this, we get that

$$\left| \frac{S_{q^N}(z)}{q^N} \right|^2 = \prod_{m=0}^{N-1} \left| \frac{G_m(z)}{q} \right|^2 \leq \prod_{m=0}^{N-1} \left(1 - \max_{|z|=1} \left(1 - \left| \frac{G_m(z)}{q} \right|^2 \right) \right).$$

In light of hypotheses (2.2), it follows from the above inequality that

$$\max_{|z|=1} \frac{S_{q^N}(z)}{q^N} \to 0 \qquad (N \to \infty),$$

thus, in light of Lemma 5, establishing (2.3). Finally, since (2.4) is an immediate consequence of (2.3), the proof of Theorem 2 is complete. ■

6. Final remarks

For the general case, that is when we do not assume that the arithmetic function f belongs to $\mathcal{M}_1^*$ or to $\mathcal{M}_q^{(1)}$, we are unable to prove results similar to those stated in Theorems 1 or 2. However, it is interesting to observe that if the arithmetic function f is such that $|f(n)| \leq 1$, one can prove that, given any $\varepsilon > 0$,

$$S_f(N,\alpha) := \sum_{n\leq N} f(n)e(n\alpha) = O(\sqrt{N}\cdot(\log N)^{\frac{1}{2}+\varepsilon}) \quad \text{for almost all } \alpha \in \mathbb{R}. \tag{6.1}$$

This can be deduced from the famous result of Carleson [1] which states that if $\sum_{k=0}^{\infty} |c_k|^2 < \infty$, then the corresponding Fourier series $\sum_{k=0}^{\infty} c_k e(k\theta)$ converges for almost all $\theta \in \mathbb{R}$. A deduction of (6.1) from the Carleson result can be found in the paper of Murty and Sankaranarayanan [8].

References

[1] **Carleson, L.,** On convergence and growth of partial sums of Fourier series, *Acta Math.*, **116** (1966), 135–157.

[2] **Corrádi, C.A. and I. Kátai,** Some problems concerning the convolutions of number-theoretical functions, *Archiv der Mathematik*, **20** (1969), 24–29.

[3] **Daboussi, H.,** *Fonctions multiplicatives presque périodiques*, B. D'après un travail commun avec Hubert Delange. Journées Arithmétiques de Bordeaux (Conf., Univ. Bordeaux, Bordeaux, 1974), pp. 321–324. Astérisque, No. 24–25, Soc. Math. France, Paris, 1975.

[4] **De Koninck, J.-M., L. Germán and I. Kátai,** On the convolution of the Liouville function under the existence of Siegel zeros, *Lith. Math. J.*, **55(3)** (2015), 331–342.

[5] **Halász, G.,** Über die Mittelwerte multiplikativer zahlentheoretischer Funktionen, *Acta Math. Acad. Sci. Hungar.*, **19** (1968), 365–403.

[6] **Kátai, I.,** A remark on a theorem of H. Daboussi, *Acta Math. Hungar.*, **47(1–2)** (1986), 223–225.

[7] **Montgomery, H.L. and R. Vaughan,** Exponential sums with multiplicative coefficients, *Inventiones Mathematicae*, **43** (1977), 69–82.

[8] **Murty, M. R. and A. Sankaranarayanan,** Averages of exponential twists of the Liouville function, *Forum Mathematicum*, **14** (2002), 273–291.

J.-M. De Koninck
Dép. math. et statistique
Université Laval
Québec
Québec G1V 0A6
Canada
jmdk@mat.ulaval.ca

I. Kátai
Computer Algebra Department
Eötvös Loránd University
1117 Budapest
Pázmány Péter Sétány I/C
Hungary
katai@inf.elte.hu