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# APPROXIMATION THEOREMS IN $L^p$ SPACES OF FUNCTIONS OF SEVERAL VARIABLES

**Ali Guven** (Balikesir, Turkey)

**Hasan Yurt** (Canakkale, Turkey)

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**Abstract.** The direct and converse trigonometric approximation theorems in Lebesgue spaces of functions of several variables are proved by using moduli of smoothness of fractional order. Also, a constructive characterization of the generalized Lipschitz classes is obtained.

## 1. Introduction

Let $\mathbb{T} := [0, 2\pi]$ and $L^p(\mathbb{T})$ $(1 \leq p \leq \infty)$ be the space of $2\pi$-periodic Lebesgue measurable functions such that $\|f\|_{L^p(\mathbb{T})} < \infty$, where

$$\|f\|_{L^p(\mathbb{T})} := \begin{cases} \left( \int\limits_{\mathbb{T}} |f(x)|^p \, dx \right)^{1/p}, & 1 \leq p < \infty \\ ess \sup\limits_{x \in \mathbb{T}} |f(x)|, & p = \infty. \end{cases}$$

For $r = 1, 2, \ldots$ the $r-$modulus of smoothness of the function $f \in L^p(\mathbb{T})$ is defined by

$$\omega_r(f, \delta)_p := \sup_{0 < h \leq \delta} \|\Delta_h^r f(x)\|_{L^p(\mathbb{T})}, \ \ (\delta > 0)$$

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where

$$\Delta_h^r f(x) := \sum_{k=0}^{r} (-1)^k \binom{r}{k} f(x + (r-k)h).$$

Let $\mathcal{T}_n$ $(n = 0, 1, ...)$ be the set of trigonometric polynomials of degree at most $n$ and let $E_n(f)_p$ be the best approximation of $f \in L^p(\mathbb{T})$ by elements of $\mathcal{T}_n$, i.e.

$$E_n(f)_p = \inf_{t_n \in \mathcal{T}_n} \|f - t_n\|_{L^p(\mathbb{T})}.$$

There are many results on approximation of functions belong to $L^p(\mathbb{T})$ spaces $(1 \leq p \leq \infty)$. Especially, the classical Jackson theorem

$$E_n(f)_p \leq c\omega_r\left(f, \frac{1}{n}\right)_p, \quad n = 1, 2, ... \tag{1.1}$$

and its weak converse

$$\omega_r\left(f, \frac{1}{n}\right)_p \leq \frac{c}{n^r} \sum_{k=0}^{n} (k+1)^{r-1} E_k(f)_p, \quad n = 1, 2, ... \tag{1.2}$$

are very important in trigonometric approximation theory. We refer to monographs [7] and [16] for these theorems and other results of trigonometric approximation.

Approximation problems of functions in $L^p(\mathbb{T})$ spaces by trigonometric polynomials are also studied by using modulus of smoothness of any positive order, called fractional modulus of smoothness.

Let $f \in L^1(\mathbb{T})$ has the Fourier series

$$\sum_{k=-\infty}^{\infty} c_k(f) e^{ikx}, \tag{1.3}$$

where

$$c_k(f) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} dx \quad (k \in \mathbb{Z}).$$

We denote by $L_0^1(\mathbb{T})$ the class of functions $f \in L^1(\mathbb{T})$ for which $c_0 = 0$ in (1.3). For $\alpha > 0$,the *$\alpha$−th fractional integral* of $f \in L_0^1(\mathbb{T})$ is defined as

$$I_\alpha(x, f) := \sum_{k \in \mathbb{Z}^*} c_k(f) (ik)^{-\alpha} e^{ikx},$$

where $(ik)^{-\alpha} := |k|^{-\alpha} e^{(-1/2)\pi i \alpha \operatorname{sign} k}$ and $\mathbb{Z}^* := \{\pm 1, \pm 2, \pm 3, ...\}$ (see, for example [19, Vol. II., p. 134]).

For $\alpha \in (0,1)$ and $r = 1,2,...$ we set

$$f^{(\alpha)}(x) \; : \; = \frac{d}{dx} I_{1-\alpha}(x,f),$$
$$f^{(\alpha+r)}(x) \; : \; = \left(f^{(\alpha)}(x)\right)^{(r)} = \frac{d^{r+1}}{dx^{r+1}} I_{1-\alpha}(x,f)$$

when the right sides exist ([14, p. 347]).

Let $x,t \in \mathbb{R}$, $\alpha \in \mathbb{R}^+ := (0,\infty)$ and let

$$\Delta_h^\alpha f(x) := \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x + (\alpha - k) h), \; f \in L^1(\mathbb{T})$$

converges in $L^1(\mathbb{T})$ and $\Delta_h^\alpha f(\cdot)$ is a measurable function ([15]). Since the Binomial coefficients satisfy ([14, p.14])

$$\left|\binom{\alpha}{k}\right| := \left|\frac{\alpha(\alpha-1)...(\alpha-k+1)}{k!}\right| \leq \frac{c}{k^{\alpha+1}} \quad (k \in \mathbb{N})$$

we have

$$c(\alpha) := \sum_{k=0}^{\infty} \left|\binom{\alpha}{k}\right| < \infty$$

and therefore $\Delta_h^\alpha f(\cdot)$ is defined almost everywhere on $\mathbb{R}$ and

(1.4) $$\left\|\Delta_h^\alpha f\right\|_{L^p(\mathbb{T})} \leq c(\alpha) \left\|f\right\|_{L^p(\mathbb{T})}$$

for $\alpha \in \mathbb{R}^+$ ([15]).

If $\alpha \in \mathbb{N}$, then the fractional difference $\Delta_h^\alpha f(\cdot)$ coincides with usual forward difference.

For $\alpha \in \mathbb{R}^+$, the $\alpha-$th fractional modulus of smoothness of $f \in L^p(\mathbb{T})$ is defined as

(1.5) $$\omega_\alpha(f,\delta)_p := \sup_{0<h\leq\delta} \left\|\Delta_h^\alpha f(x)\right\|_{L^p(\mathbb{T})}.$$

The modulus of smoothness $\omega_\alpha(f,\cdot)_p$ is a non-decreasing function of $\delta \geq 0$, $\omega_\alpha(f,0)_p = 0$ and satisfies

$$\omega_\alpha(f_1+f_2,\delta)_p \leq \omega_\alpha(f_1,\delta)_p + \omega_\alpha(f_2,\delta)_p$$

for $\delta \geq 0$ ([15]).

More general information about fractional moduli of smoothness can be found in [6] and [15].

Throughout the paper, $c$ will denote positive constants which are not important for the questions involve in the paper and can be different at each occurrence.

Analogues of the inequalities (1.1) and (1.2) were proved in [15] for $\alpha > 0$:

**Theorem A.** *Let* $1 \leq p \leq \infty$ *and* $\alpha \in \mathbb{R}^+$. *If* $f \in L^p(\mathbb{T})$, *then the estimates*

$$E_n(f)_p \leq c\omega_\alpha\left(f, \frac{1}{n}\right)_p, \quad n = 1, 2, ... \tag{1.6}$$

and

$$\omega_\alpha\left(f, \frac{1}{n}\right)_p \leq \frac{c}{n^\alpha}\sum_{k=0}^{n}(k+1)^{\alpha-1}E_k(f)_p, n = 1, 2, .... \tag{1.7}$$

*holds.*

Approximation problems concerning fractional moduli of smoothness was recently studied in various spaces of $2\pi$-periodic functions. For example, in the papers[1], [2], [3], [5], [4], [17] and [18] the idea of fractional moduli of smoothness is used.

Approximation problems for functions of several variables were also studied by many mathematicians. Some of these results can be found in [8], [10] and [13]. They are also summarized in [9]. In all of these studies authors considered the moduli of smoothness of integer order.

In this work, analogues (1.6) and (1.7) are obtained in Lebesgue spaces of functions of several variables.

## 2. Main results

Let $\mathbb{T}^m$ $(m \geq 1)$ denote the $m-$dimensional cube $[-\pi, \pi]^m$. We denote by $L^p(\mathbb{T}^m)$ $(1 \leq p \leq \infty)$ the space of all measurable functions $F$ of $m-$variables, which are $2\pi-$periodic in each variable and satisfy $\|F\|_{L^p(\mathbb{T}^m)} < \infty$, where

$$\|F\|_{L^p(\mathbb{T}^m)} = \begin{cases} \left(\int\limits_{\mathbb{T}^m}|F(x_1, ..., x_m)|^p dx_1...dx_m\right)^{\frac{1}{p}}, 1 \leq p < \infty \\ ess\sup\limits_{(x_1,...,x_m)\in\mathbb{T}^m}|F(x_1, ..., x_m)|, p = \infty. \end{cases}$$

For points in $\mathbb{R}^m$ we will use the notations $\underline{x} = (x_1, ..., x_m)$ and

$$\underline{dx_i} := dx_1...dx_{i-1}dx_{i+1}...dx_m \ (1 \leq i \leq m).$$

Also, for functions $F(\underline{x})$, defined on $\mathbb{R}^m$, we consider the sections

$$F_{i,\underline{x}}(t) := F(x_1, ..., x_{i-1}, t, x_{i+1}, ..., x_m) \quad (1 \le i \le m).$$

Our approximation tools are $m-$variable trigonometric polynomials. It is appropriate to describe in term of multivariate complex polynomials. So, let

$$\mathcal{T}_n^m := \left\{T(\underline{x}) = ReP\left(e^{ix_1}, ..., e^{ix_m}\right) : P \in \mathcal{P}_n^m\right\},$$

where $\mathcal{P}_n^m$ be the set of all are $m-$variable complex polynomials of degree at most $n$, i. e.

$$\mathcal{P}_n^m := \left\{P(Z_1, ..., Z_m) = \sum_{\substack{0 \le k_i \le n \\ (i=1,...,m)}} c_{k_1,..,k_m} Z_1^{k_1} ... Z_m^{k_m} : c_{k_1,..,k_m} \in \mathbb{C}\right\}.$$

Note that, $\mathcal{T}_n^m$ consists of all $m-$variable trigonometric polynomials of order at most $n$ (in each variable) ([10]).

The best approximations of $F \in L^p(\mathbb{T}^m)$ in the class $\mathcal{T}_n^m$ are

$$E_n(F)_{L^p(\mathbb{T}^m)} := \inf_{T_n \in \mathcal{T}_n^m} \|F - T_n\|_{L^p(\mathbb{T}^m)} \quad (n \in \mathbb{N}). \tag{2.1}$$

Let $F \in L^p(\mathbb{T}^m)$. For $\alpha \in \mathbb{R}^+$, we define the partial differences

$$\begin{aligned} \Delta_h^{\alpha,i} F(\underline{x}) \quad : &= \Delta_h^{\alpha,i} F_{i,\underline{x}}(x_i) \\ : &= \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} F(x_1, ..., x_{i-1}, x_i + (\alpha - k) h, x_{i+1}, ..., x_m). \end{aligned}$$

For $\alpha \in \mathbb{R}^+$ we define the $\alpha-th$ partial fractional modulus of smoothness of $F \in L^p(\mathbb{T}^m)$ $(1 \le p \le \infty)$ as

$$\Omega_{\alpha,i}(F, \delta)_{L^p(\mathbb{T}^m)} := \sup_{0 < h \le \delta} \left\|\Delta_h^{\alpha,i} F\right\|_{L^p(\mathbb{T}^m)} \quad (1 \le i \le m).$$

The fractional modulus of smoothness of $F \in L^p(\mathbb{T}^m)$ is defined by

$$\Omega_\alpha(F, \delta)_{L^p(\mathbb{T}^m)} := \max_{1 \le i \le m} \Omega_{\alpha,i}(F, \delta)_{L^p(\mathbb{T}^m)} \quad (\delta > 0).$$

It follows from (1.4) and Fubini theorem that

$$\Omega_{\alpha,i}(F, \delta)_{L^p(\mathbb{T}^m)} \le c \|F\|_{L^p(\mathbb{T}^m)} \quad (\delta > 0), \tag{2.2}$$

and hence the modulus of smoothness $\Omega_{\alpha,i}(F, \cdot)_{L^p(\mathbb{T}^m)}$ exists for $F \in L^p(\mathbb{T}^m)$.

By using definition of $\Omega_{\alpha,i}(F,\cdot)_{L^p(\mathbb{T}^m)}$, one easily get

$$\Omega_{\alpha,i}(F+G,\delta)_{L^p(\mathbb{T}^m)} \le \Omega_{\alpha,i}(F,\delta)_{L^p(\mathbb{T}^m)} + \Omega_{\alpha,i}(G,\delta)_{L^p(\mathbb{T}^m)} \tag{2.3}$$

for $F,G \in L^p(\mathbb{T}^m)$ and $\alpha \in \mathbb{R}^+$.

Now we can give multivariate direct and converse approximation theorems for functions of several variables, which are our main results.

**Theorem 2.1.** *Let $m \ge 1$ , $1 \le p \le \infty$ and $\alpha \in \mathbb{R}^+$. Then the Jackson type inequality*

$$E_n(F)_{L^p(\mathbb{T}^m)} \le c\,\Omega_\alpha\left(F,\frac{1}{n}\right)_{L^p(\mathbb{T}^m)}, \ n=1,2,.. \tag{2.4}$$

*holds for $F \in L^p(\mathbb{T}^m)$.*

**Theorem 2.2.** *Let $1 \le p \le \infty$ and $m \ge 1$. Then for $F \in L^p(\mathbb{T}^m)$ and $\alpha \in \mathbb{R}^+$, we have*

$$\Omega_\alpha\left(F,\frac{1}{n}\right)_{L^p(\mathbb{T}^m)} \le \frac{c}{n^\alpha}\sum_{k=0}^{n}(k+1)^{\alpha-1}E_k(F)_{L^p(\mathbb{T}^m)}. \tag{2.5}$$

**Corollary 2.1.** *Let $m \ge 1$ , $1 \le p \le \infty$, $\beta > 0$, and $F \in L^p(\mathbb{T}^m)$. If*

$$E_n(F)_{L^p(\mathbb{T}^m)} = O\left(n^{-\beta}\right), \ n=1,2,...$$

*then*

$$\Omega_\alpha(F,\delta)_{L^p(\mathbb{T}^m)} = \begin{cases} O\left(\delta^\beta\right), & \alpha > \beta, \\ O\left(\delta^\beta \log\left(\frac{1}{\delta}\right)\right), & \alpha = \beta, \\ O\left(\delta^\alpha\right), & \alpha < \beta, \end{cases}$$

*for every $\delta > 0$ and $\alpha \in \mathbb{R}^+$.*

Let $\beta > 0$. The generalized Lipschitz class $Lip^*(\beta,p)$ is defined by

$$Lip^*(\beta,p) := \left\{F \in L^p(\mathbb{T}^m) : \Omega_\alpha(F,\delta)_{L^p(\mathbb{T}^m)} = O\left(\delta^\beta\right), \ \delta > 0\right\},$$

where $\alpha > \beta$.

**Corollary 2.2.** *Let $m \ge 1$, $1 \le p \le \infty, \beta > 0$ and $F \in L^p(\mathbb{T}^m)$. If*

$$E_n(F)_{L^p(\mathbb{T}^m)} = O\left(n^{-\beta}\right), \ n=1,2,...$$

*for some $\beta > 0$, then $F \in Lip^*(\beta,p)$.*

Note that the analogue of the class $Lip^*(\beta, p)$ for periodic functions on the real line was defined in ([6]).

Combining Theorem 2.1 and Corollary 2.2 yields the following constructive characterization of the classes $Lip^*(\beta, p)$.

**Theorem 2.3.** *Let $1 \le p \le \infty$, $m \ge 1$, $F \in L^p(\mathbb{T}^m)$ and $\beta > 0$. The following assertions are equivalent:*

(i) $F \in Lip^*(\beta, p)$

(ii) $E_n(F)_{L^p(\mathbb{T}^m)} = O\left(n^{-\beta}\right)$, $n = 1, 2, \ldots$ .

## 3. Auxiliary results

Let $W^p_\alpha(\mathbb{T})$ ,$(\alpha = 1, 2, \ldots)$, be the linear space of functions for which $f^{(\alpha-1)}$ is absolutely continuous and $f^{(\alpha)} \in L^p(\mathbb{T})$. $W^p_\alpha(\mathbb{T})$ becomes a Banach space with respect to the norm

$$\|f\|_{W^p_\alpha(\mathbb{T})} := \|f\|_{L^p(\mathbb{T})} + \left\|f^{(\alpha)}\right\|_{L^p(\mathbb{T})}.$$

Let $1 \le p \le \infty$. For $f \in L^p(\mathbb{T})$ and $\delta > 0$ , the $K-$functional is defined as

$$\mathcal{K}_\alpha(f, \delta) := \inf\left\{\|f - g\|_{L^p(\mathbb{T})} + \delta^\alpha \left\|g^{(\alpha)}\right\|_{L^p(\mathbb{T})} : g \in W^p_\alpha(\mathbb{T})\right\}. \tag{3.1}$$

It is known that the $K-$functional (3.1) and the modulus of smoothness (1.5) are equivalent, i.e.,

$$\omega_\alpha(f, \delta)_p \sim \mathcal{K}_\alpha(f, \delta) \quad (\delta > 0) \tag{3.2}$$

for $f \in L^p(\mathbb{T})$ and $\alpha = 1, 2, \ldots$ ([12, pp. 41-50]).

We define another modulus of smoothness for $f \in L^p(\mathbb{T})$ and $\alpha \in \mathbb{R}^+$ by

$$\widetilde{\omega}_\alpha(f, \delta)_p := \begin{cases} \left(\frac{1}{\delta}\int\limits_0^\delta \|\Delta^\alpha_h f(x)\|^p_{L^p(\mathbb{T})}\, dh\right)^{\frac{1}{p}}, & 1 \le p < \infty \\ \frac{1}{\delta}\int\limits_0^\delta \|\Delta^\alpha_h f(x)\|_{L^\infty(\mathbb{T})}\, dh, & p = \infty. \end{cases}$$

**Lemma 3.1.** *Let $1 \le p \le \infty$ and $f \in L^p(\mathbb{T})$, then for $\alpha = 1, 2, \ldots$ the equivalence*

$$\omega_\alpha(f, \delta)_p \sim \widetilde{\omega}_\alpha(f, \delta)_p \quad (\delta \ge 0) \tag{3.3}$$

*holds.*

**Proof.** For $1 < p < \infty$, this lemma was proved in ([10]).

Now, we will prove the lemma for $p = 1$ or $p = \infty$. From definition of new moduli, we can obtain easily

$$\widetilde{\omega}_\alpha (f, \delta)_p \leq c\omega_\alpha (f, \delta)_p . \tag{3.4}$$

To prove the converse inequality, we consider following Steklov transform given in ([12, p. 50]):

$$f_{\alpha,h} (x) := \frac{1}{h^r} \int_0^h \cdots \int_0^h \sum_{s=0}^{\alpha-1} (-1)^{\alpha+s+1} \binom{\alpha}{s} f\left(x + \tfrac{\alpha-s}{\alpha} (t_1 + \ldots + t_\alpha)\right) dt_1 \ldots dt_\alpha .$$

Let $p = 1$ or $p = \infty$.

$$\begin{aligned} \|f_{\alpha,h} (x) - f (x)\|_{L^p(\mathbb{T})} &\leq \left\| \frac{1}{h^r} \int_0^h \cdots \int_0^h \Delta^\alpha_{\frac{t_1+\ldots+t_\alpha}{\alpha}} f (x)\, dt_1 \ldots dt_\alpha \right\|_{L^p(\mathbb{T})} \leq \\ &\leq \frac{1}{h^r} \int_0^h \cdots \int_0^h \left\| \Delta^\alpha_{\frac{t_1+\ldots+t_\alpha}{\alpha}} f (x) \right\|_{L^p(\mathbb{T})} dt_1 \ldots dt_\alpha \end{aligned} \tag{3.5}$$

Using a known method in the approximation theory we define another suitable transform

$$g_\delta (x) := \frac{2}{\delta} \int_{\delta/2}^{\delta} f_{\alpha,h} (x)\, dh \quad (0 < \delta \leq 1) . \tag{3.6}$$

By (3.5) and Fubini theorem, we can easily obtain

$$\|g_\delta (x) - f (x)\|_{L^p(\mathbb{T})} \leq c\widetilde{\omega}_\alpha (f, \delta)_p . \tag{3.7}$$

From (3.6) we get

$$g^{(\alpha)}_\delta (x) = \frac{d^\alpha}{dx^\alpha} g_\delta (x) = \frac{2}{\delta} \int_{\delta/2}^{\delta} f^{(\alpha)}_{\alpha,h} (x)\, dh$$

and hence $g_\delta \in W^p_\alpha (\mathbb{T})$.

We'll use the fact

$$f_{\alpha,h}^{(\alpha)}(x) = h^{-r}\sum_{s=0}^{\alpha-1}(-1)^{\alpha+s+1}\binom{\alpha}{s}\left(\frac{\alpha}{\alpha-s}\right)^{\alpha}\Delta^{\alpha}_{\frac{\alpha-s}{\alpha}h}f(x). \tag{3.8}$$

For $p=1$, by using (3.8) and Fubini theorem, and considering the definition of $\widetilde{\omega}_r(f,\delta)$, we get

$$\begin{aligned}
\left\|g_{\delta}^{(\alpha)}(x)\right\|_{L^1(\mathbb{T})} &\leq \frac{2}{\delta}\int_{\delta/2}^{\delta}\left\|f_{\alpha,h}^{(\alpha)}(x)\right\|_{L^1(\mathbb{T})}dh \leq \\
&\leq \frac{c}{\delta^{\alpha+1}}\int_0^{\delta}\left(\sum_{s=0}^{\alpha-1}\binom{\alpha}{s}\left(\frac{\alpha}{\alpha-s}\right)^{\alpha}\int_0^{2\pi}\left|\Delta^{\alpha}_{\frac{\alpha-s}{\alpha}h}f(x)\right|dx\right)dh \leq \\
&\leq \frac{c}{\delta^{\alpha+1}}\sum_{s=0}^{\alpha-1}\binom{\alpha}{s}\left(\frac{\alpha}{\alpha-s}\right)^{\alpha}\int_0^{\delta}\left(\int_0^{2\pi}\left|\Delta^{\alpha}_{\frac{\alpha-s}{\alpha}h}f(x)\right|dx\right)dh \leq \\
&\leq \frac{c}{\delta^{\alpha+1}}\sum_{s=0}^{\alpha-1}\binom{\alpha}{s}\left(\frac{\alpha}{\alpha-s}\right)^{\alpha}\int_0^{2\pi}\left(\int_0^{\delta}\left|\Delta^{\alpha}_{\frac{\alpha-s}{\alpha}h}f(x)\right|dh\right)dx = \\
&= \frac{c}{\delta^{\alpha+1}}\sum_{s=0}^{\alpha-1}\binom{\alpha}{s}\left(\frac{\alpha}{\alpha-s}\right)^{\alpha}\int_0^{2\pi}\left(\int_0^{\frac{\alpha-s}{\alpha}\delta}\left|\Delta^{\alpha}_{t}f(x)\right|dh\right)dx \leq \\
&\leq \frac{c}{\delta^{\alpha+1}}\int_0^{\delta}\left(\int_0^{2\pi}\left|\Delta^{\alpha}_{t}f(x)\right|dx\right)dt = \\
&= \frac{c}{\delta^{\alpha+1}}\int_0^{\delta}\left\|\Delta^{\alpha}_{t}f(x)\right\|_{L^1(\mathbb{T})}dt = \\
&= c\delta^{-\alpha}\widetilde{\omega}_r(f,\delta)_p.
\end{aligned}$$

Hence we obtain

$$\left\|g_{\delta}^{(\alpha)}(x)\right\|_{L^1(\mathbb{T})} \leq c\widetilde{\omega}_{\alpha}(f,\delta)_p. \tag{3.9}$$

For $p=\infty$, by (3.8), Fubini theorem and definition of $\widetilde{\omega}_r(f,\delta)$, we have

$$\begin{aligned}
\left|g_\delta^{(\alpha)}(x)\right| &\leq \frac{2}{\delta}\int\limits_{\delta/2}^{\delta}\left|f_{\alpha,h}^{(\alpha)}(x)\right|dh \leq \\
&\leq \frac{c}{\delta^{\alpha+1}}\sum_{s=0}^{\alpha-1}\binom{\alpha}{s}\left(\frac{\alpha}{\alpha-s}\right)^{\alpha}\int\limits_0^{\delta}\left|\Delta_{\frac{\alpha-s}{\alpha}h}^{\alpha}f(x)\right|dh = \\
&= \frac{c}{\delta^{\alpha+1}}\sum_{s=0}^{\alpha-1}\binom{\alpha}{s}\left(\frac{\alpha}{\alpha-s}\right)^{\alpha}\int\limits_0^{\frac{\alpha-s}{\alpha}\delta}\left|\Delta_t^{\alpha}f(x)\right|dt \leq \\
&\leq \frac{c}{\delta^{\alpha+1}}\sum_{s=0}^{\alpha-1}\binom{\alpha}{s}\left(\frac{\alpha}{\alpha-s}\right)^{\alpha}\int\limits_0^{\delta}\left|\Delta_t^{\alpha}f(x)\right|dt = \\
&= \frac{c}{\delta^{\alpha+1}}\int\limits_0^{\delta}\left|\Delta_t^{\alpha}f(x)\right|dt \leq \\
&\leq \frac{c}{\delta^{\alpha+1}}\int\limits_0^{\delta}\left\|\Delta_h^{\alpha}f(x)\right\|_{L^\infty(\mathbb{T})}dh.
\end{aligned}$$

So, the inequality

$$\left\|g_\delta^{(\alpha)}(x)\right\|_{L^\infty(\mathbb{T})} \leq c\delta^{-\alpha}\widetilde{\omega}_r(f,\delta)_p \tag{3.10}$$

holds.

Therefore, for $p=1$ or $p=\infty$, we get from (3.7), (3.9) and (3.10)

$$\begin{aligned}
\mathcal{K}_\alpha(f,\delta)_p &\leq \|f-g_\delta\|_{L^p(\mathbb{T})}+\delta^{\alpha}\left\|g_\delta^{(\alpha)}(x)\right\|_{L^p(\mathbb{T})} \\
&\leq c\widetilde{\omega}_\alpha(f,\delta)_p
\end{aligned}$$

and by (3.2)

$$\omega_\alpha(f,\delta)_p \leq c\mathcal{K}_\alpha(f,\delta)_p \leq c\widetilde{\omega}_\alpha(f,\delta)_p.$$

The last inequality and (3.4) yield (3.3). ■

**Lemma 3.2.** *Let* $1 \le p \le \infty$ *and* $f \in L^p(\mathbb{T})$*, then the inequality*

$$\widetilde{\omega}_\beta (f,\delta)_p \le c \widetilde{\omega}_\alpha (f,\delta)_p, \ \delta \ge 0$$

*holds for* $0 < \alpha < \beta$.

**Proof.** By Theorem 5 in [15] we have

$$\left\| \Delta_h^\beta f(\cdot) \right\|_{L^p(\mathbb{T})} \le c \left\| \Delta_h^\alpha f(\cdot) \right\|_{L^p(\mathbb{T})}.$$

From this inequality we get

$$\left\| \Delta_h^\beta f(\cdot) \right\|_{L^p(\mathbb{T})}^p \le c \left\| \Delta_h^\alpha f(\cdot) \right\|_{L^p(\mathbb{T})}^p$$

for $1 \le p \le \infty$. Now by integrating the last inequality with respect to $dh$ we get

$$\int_0^\delta \left\| \Delta_h^\beta f(\cdot) \right\|_{L^p(\mathbb{T})}^p dh \le c \int_0^\delta \left\| \Delta_h^\alpha f(\cdot) \right\|_{L^p(\mathbb{T})}^p dh.$$

Since $\delta > 0$ we can write

$$\frac{1}{\delta}\int_0^\delta \left\| \Delta_h^\beta f(\cdot) \right\|_{L^p(\mathbb{T})}^p dh \le c \frac{1}{\delta}\int_0^\delta \left\| \Delta_h^\alpha f(\cdot) \right\|_{L^p(\mathbb{T})}^p dh.$$

From the last inequality we obtain

$$\left( \frac{1}{\delta}\int_0^\delta \left\| \Delta_h^\beta f(x) \right\|_{L^p(\mathbb{T})}^p dh \right)^{\frac{1}{p}} \le c \left( \frac{1}{\delta}\int_0^\delta \left\| \Delta_h^\alpha f(x) \right\|_{L^p(\mathbb{T})}^p dh \right)^{\frac{1}{p}}$$

which completes the proof with usual modification $p = \infty$. ■

For fractional moduli $\omega_\alpha (f,\cdot)_p, \ \alpha \in \mathbb{R}^+$, this lemma was proved in [15].

**Corollary 3.1.** *Let* $\alpha \in \mathbb{R}^+$ *and* $1 \le p \le \infty$*. Then, for* $f \in L^p(\mathbb{T})$ *the estimate*

$$E_n(f)_p \le c \widetilde{\omega}_\alpha \left( f, \frac{1}{n} \right)_p, \ n = 1, 2, ... \tag{3.11}$$

*holds.*

It is known that ([16]) for $f\in L^{p}(\mathbb{T})$ and $\alpha\in\mathbb{R}^{+}$,

$$E_{n}(f)_{p}\leq c\omega_{[\alpha]+2}\left(f,\frac{1}{n}\right)_{p},$$

where $[\alpha]$ denotes the integer part of the real number $\alpha$. Then by the last inequality, Lemma 3.1 and Lemma 3.2, we get

$$E_{n}(f)_{p}\leq c\omega_{[\alpha]+2}\left(f,\frac{1}{n}\right)_{p}\leq c\widetilde{\omega}_{[\alpha]+2}\left(f,\frac{1}{n}\right)_{p}\leq c\widetilde{\omega}_{\alpha}\left(f,\frac{1}{n}\right)_{p}.$$

The last estimate is important for proof of Theorem 2.1.

**Lemma 3.3.** *Let $1\leq p\leq\infty$ and $m\geq 1,\alpha\in\mathbb{R}^{+}$. If $T_{n}\in\mathcal{T}_{n}^{m},n\geq 1,$ then there exist a constant $c>0$ depending only $r$ and $p$ such that*

$$\left\Vert\frac{\partial^{\alpha}T_{n}}{\partial x_{i}^{\alpha}}\right\Vert_{L^{p}(\mathbb{T}^{m})}\leq cn^{\alpha}\left\Vert T_{n}\right\Vert_{L^{p}(\mathbb{T}^{m})}.$$

**Proof.** It is known that ([15]) for $t_{n}\in\mathcal{T}_{n}$ and $1\leq p\leq\infty$

$$\left\Vert t_{n}^{(\alpha)}\right\Vert_{L^{p}(\mathbb{T})}\leq cn^{\alpha}\left\Vert t_{n}\right\Vert_{L^{p}(\mathbb{T})},\ n=1,2,...,\tag{3.12}$$

holds. From this inequality, we obtain

$$\begin{aligned}\left\Vert\frac{\partial^{\alpha}T_{n}(\underline{x})}{\partial x_{i}^{\alpha}}\right\Vert_{L^{p}(\mathbb{T}^{m})} &= \left\{\int\limits_{\mathbb{T}^{m-1}}\left(\int\limits_{\mathbb{T}}\left|\frac{\partial^{\alpha}T_{n,i,\underline{x}}(x_{i})}{\partial x_{i}^{\alpha}}\right|^{p}dx_{i}\right)^{\frac{1}{p}.p}\underline{dx_{i}}\right\}^{\frac{1}{p}}\leq\\ &\leq cn^{\alpha}\left\{\int\limits_{\mathbb{T}^{m-1}}\left\Vert T_{n,i,\underline{x}}(x_{i})\right\Vert_{L^{p}(\mathbb{T})}^{p}\underline{dx_{i}}\right\}^{\frac{1}{p}}=\\ &= cn^{\alpha}\left\{\int\limits_{\mathbb{T}^{m-1}}\left(\int\limits_{\mathbb{T}}\left|T_{n,i,\underline{x}}(x_{i})\right|^{p}dx_{i}\right)\underline{dx_{i}}\right\}^{\frac{1}{p}}=\\ &= cn^{\alpha}\left\{\int\limits_{\mathbb{T}^{m}}\left|T_{n}(\underline{x})\right|^{p}dx_{1}...dx_{m}\right\}^{\frac{1}{p}}=\\ &= cn^{\alpha}\left\Vert T_{n}(\underline{x})\right\Vert_{L^{p}(\mathbb{T}^{m})}.\end{aligned}$$

■

The following lemma can be proved by the method used in proof of Lemma 3.3 and the one variable case in [15].

**Lemma 3.4.** *Let* $m \geq 1$ $, 1 \leq p \leq \infty$ *and* $\alpha \in \mathbb{R}^+$. *If* $T_n \in T_n^m, n \geq 1,$ *then there exists a constant* $c > 0$ *depending only* $\alpha$ *and* $p$ *such that*

$$\Omega_{\alpha,i}\left(T_n, \delta\right)_{L^p(\mathbb{T}^m)} \leq c\delta^\alpha \left\| \frac{\partial^\alpha T_n}{\partial x_i^\alpha} \right\|_{L^p(\mathbb{T}^m)}.$$

## 4. Proofs of main results

**Proof of Theorem 2.1.** Let $f \in L^p(\mathbb{T})$ has the Fourier series $(1.3)$, and let $(s_n(f))$ be the sequence of partial sums of $(1.3)$, i .e.

$$s_n(f)(x) = \sum_{|k| \leq n} c_k(f) e^{ikx} \quad (n \in \mathbb{N}).$$

The sequence of de la Vallée-Poussin means of (1.3) is defined by

$$v_n(f)(x) := \frac{1}{n+1} \sum_{k=n}^{2n} s_k(f)(x) \quad (n \in \mathbb{N}).$$

Note that $v_n \in \mathcal{T}_{2n}$ for $n \in \mathbb{N}$.

For $f \in L^p(\mathbb{T})$ $(1 \leq p \leq \infty)$ the estimates

$$\|f - v_n(f)\|_{L^p(\mathbb{T})} \leq cE_n(f)_p \tag{4.1}$$

$$\|v_n(f)\|_{L^p(\mathbb{T})} \leq c\|f\|_{L^p(\mathbb{T})} \tag{4.2}$$

hold (see, for example [11, p. 196]).

For $F \in L^p(\mathbb{T}^m)$ we set

$$V_{n,i}F(\underline{x}) := v_n\left(F_{i,\underline{x}}\right)(x_i) \quad (1 \leq i \leq m).$$

For $1 \leq p < \infty$, we have from (4.1), (3.11) and Fubini theorem

$$\begin{aligned}
\|F - V_{n,i}F\|_{L^p(\mathbb{T}^m)} &= \left\{ \int\limits_{\mathbb{T}^m} |(F - V_{n,i}F)(\underline{x})|^p \, dx_1 ... dx_m \right\}^{\frac{1}{p}} = \\
&= \left\{ \int\limits_{\mathbb{T}^{m-1}} \left( \int\limits_{\mathbb{T}} \left| \left( F_{i,\underline{x}} - v_n F_{i,\underline{x}} \right)(x_i) \right|^p dx_i \right)^{\frac{1}{p}p} \underline{dx_i} \right\}^{\frac{1}{p}} = \\
&= \left\{ \int\limits_{\mathbb{T}^{m-1}} \left\| \left( F_{i,\underline{x}} - v_n F_{i,\underline{x}} \right)(x_i) \right\|_{L^p(\mathbb{T})}^p \underline{dx_i} \right\}^{\frac{1}{p}} \leq \\
&\leq c \left\{ \int\limits_{\mathbb{T}^{m-1}} \left( E_n \left( F_{i,\underline{x}} \right)_p \right)^p \underline{dx_i} \right\}^{\frac{1}{p}} \leq \\
&\leq c \left\{ \int\limits_{\mathbb{T}^{m-1}} \left( \widetilde{\omega}_\alpha \left( F_{i,\underline{x}}, \frac{1}{n} \right)_p \right)^p \underline{dx_i} \right\}^{\frac{1}{p}} = \\
&= c \left\{ \int\limits_{\mathbb{T}^{m-1}} \left( \frac{1}{n^{-1}} \int\limits_0^{n^{-1}} \left\| \Delta_h^\alpha F_{i,\underline{x}} \right\|_{L^p(\mathbb{T})}^p dh \right) \underline{dx_i} \right\}^{\frac{1}{p}} = \\
&= c \left\{ \int\limits_{\mathbb{T}^{m-1}} \left( \frac{1}{n^{-1}} \int\limits_0^{n^{-1}} \left( \int\limits_{\mathbb{T}} \left| \Delta_h^\alpha F_{i,\underline{x}}(x_i) \right|^p dx_i \right) dh \right) \underline{dx_i} \right\}^{\frac{1}{p}} = \\
&= c \left\{ \frac{1}{n^{-1}} \int\limits_0^{n^{-1}} \left( \int\limits_{\mathbb{T}^m} \left| \Delta_h^{\alpha,i} F(\underline{x}) \right|^p dx_1 ... dx_m \right) dh \right\}^{\frac{1}{p}} = \\
&= c \left\{ \frac{1}{n^{-1}} \int\limits_0^{n^{-1}} \left( \left\| \Delta_h^{\alpha,i} F \right\|_{L^p(\mathbb{T}^m)}^p \right) dh \right\}^{\frac{1}{p}} \leq \\
&\leq c \left\{ \frac{1}{n^{-1}} \int\limits_0^{n^{-1}} \left( \Omega_{\alpha,i} \left( F, \frac{1}{n} \right)_{L^p(\mathbb{T}^m)} \right)^p dh \right\}^{\frac{1}{p}} = \\
&= c\Omega_{\alpha,i} \left( F, \frac{1}{n} \right)_{L^p(\mathbb{T}^m)}.
\end{aligned}$$

Hence,

$$\left\| F - V_{n,i}F \right\|_{L^p(\mathbb{T}^m)} \le c\Omega_{\alpha,i}\left(F, \frac{1}{n}\right)_{L^p(\mathbb{T}^m)} \quad \left(\alpha \in \mathbb{R}^+\right) \tag{4.3}$$

holds for all partial fractional moduli. Let us note that $V_{n,i}F\left(\underline{x}\right)$ are not trigonometric polynomials of several variables (namely, of $\underline{x}$), therefore the inequality in (4.3) does not give the degree of approximations in each variable. However based on (4.3) we can get the inequality (2.4) by a well known technique of the approximation theory (see [8, p. 200-201 ]). In the second step of the proof, we will construct multivariate trigonometric polynomials as

$$\widetilde{V}_n F\left(\underline{x}\right) := V_{n,1}V_{n,2}...V_{n,m}F\left(\underline{x}\right).$$

By (4.2), (4.3) and by definition of $\Omega_\alpha\left(F, .\right)_{L^p(\mathbb{T}^m)}$, we obtain

$$\begin{aligned} E_n\left(F\right)_{L^p(\mathbb{T}^m)} &\le \left\| F - \widetilde{V}_n F \right\|_{L^p(\mathbb{T}^m)} = \\ &= \left\| \left(F - V_{n,1}F\right) + \left(V_{n,1}F - V_{n,1}V_{n,2}F\right) + \right. \\ &\quad \left. +... + \left(V_{n,1}V_{n,2}...V_{n,m-1}F - V_{n,1}V_{n,2}...V_{n,m}F\right) \right\|_{L^p(\mathbb{T}^m)} \le \\ &\le c\sum_{i=1}^{m} \left\| F - V_{n,i}F \right\|_{L^p(\mathbb{T}^m)} \le \\ &\le c\sum_{i=1}^{m} \Omega_{\alpha,i}\left(F, \frac{1}{n}\right)_{L^p(\mathbb{T}^m)} \le c\Omega_\alpha\left(F, \frac{1}{n}\right)_{L^p(\mathbb{T}^m)}. \end{aligned}$$

For $p = \infty$, Theorem 2.1 can be similarly proved by using (4.1), (4.2) and (3.11). ■

Note that, for $1 < p < \infty$, Theorem 2.1 can be also proved by using the sequence of partial sums instead of de la Vallée-Poussin means.

**Proof of Theorem 2.2.** Let $F \in L^p\left(\mathbb{T}^m\right)$ and $T_n$ $\left(n \in \mathbb{N}\right)$ be the polynomials of best approximation to $F$ in the class $\mathcal{T}_n^m$. Also, let $n \in \mathbb{N}$ and $\delta := 1/n$.

By subadditivity of the modulus,

$$\Omega_{\alpha,i}\left(F, \delta\right)_{L^p(\mathbb{T}^m)} \le \Omega_{\alpha,i}\left(F - T_{2^{v+1}}, \delta\right)_{L^p(\mathbb{T}^m)} + \Omega_{\alpha,i}\left(T_{2^{v+1}}, \delta\right)_{L^p(\mathbb{T}^m)} \tag{4.4}$$

for every $v = 0, 1, ...$ . The inequality (2.2) gives

$$\Omega_{\alpha,i}\left(F - T_{2^{v+1}}, \delta\right)_{L^p(\mathbb{T}^m)} \le c\left\| F - T_{2^{v+1}} \right\|_{L^p(\mathbb{T}^m)} = cE_{2^{v+1}}\left(F\right)_{L^p(\mathbb{T}^m)}. \tag{4.5}$$

Since the sequence of best approximations is decreasing, by Lemma 3.4 we get

$$\begin{aligned} & \Omega_{\alpha,i}\left(T_{2^{v+1}},\delta\right)_{L^p(\mathbb{T}^m)} \le c\delta^{\alpha}\left\|\frac{\partial^{\alpha}T_{2^{v+1}}}{\partial x_i^{\alpha}}\right\|_{L^p(\mathbb{T}^m)} \le \\ \le\ & c\delta^{\alpha}\left\{\left\|\frac{\partial^{\alpha}T_1}{\partial x_i^{\alpha}}\right\|_{L^p(\mathbb{T}^m)} + \sum_{s=0}^{v}\left\|\frac{\partial^{\alpha}T_{2^{s+1}}}{\partial x_i^{\alpha}} - \frac{\partial^{\alpha}T_{2^{s}}}{\partial x_i^{\alpha}}\right\|_{L^p(\mathbb{T}^m)}\right\} \le \\ \le\ & c\delta^{\alpha}\left\{E_0\left(F\right)_{L^p(\mathbb{T}^m)} + \sum_{s=0}^{v}2^{(s+1)\alpha}\left\|T_{2^{s+1}} - T_{2^s}\right\|_{L^p(\mathbb{T}^m)}\right\} \le \\ \le\ & c\delta^{\alpha}\left\{E_0\left(F\right)_{L^p(\mathbb{T}^m)} + \sum_{s=0}^{v}2^{(s+1)\alpha}E_{2^s}\left(F\right)_{L^p(\mathbb{T}^m)}\right\}. \end{aligned}$$

Furthermore, for $s \ge 1$, by considering the inequality

$$2^{(s+1)\alpha}E_{2^s}\left(F\right)_{L^p(\mathbb{T}^m)} \le c^{*}\sum_{k=2^{s-1}+1}^{2^s}k^{\alpha-1}E_k\left(F\right)_{L^p(\mathbb{T}^m)}, \tag{4.6}$$

where $c^* = 2^{\alpha+1}$ if $0 < \alpha < 1$ and $c^* = 2^{2\alpha}$ if $\alpha \ge 1$, we obtain

$$\Omega_{\alpha,i}\left(T_{2^{v+1}},\delta\right)_{L^p(\mathbb{T}^m)} \le c\delta^{\alpha}\left\{E_0\left(F\right)_{L_W^{p,q}} + \sum_{k=1}^{2^v}k^{\alpha-1}E_k\left(F\right)_{L^p(\mathbb{T}^m)}\right\}. \tag{4.7}$$

If we choose $v$ such that $2^v \le n < 2^{v+1}$, and by (4.6),

$$E_{2^{v+1}}\left(F\right)_{L^p(\mathbb{T}^m)} \le \frac{2^{(v+1)\alpha}E_{2^{v+1}}\left(F\right)_{L^p(\mathbb{T}^m)}}{2^{(v+1)\alpha}} \le c\delta^{\alpha}\sum_{k=1}^{2^v}k^{\alpha-1}E_k\left(F\right)_{L^p(\mathbb{T}^m)}.$$

Now combining (4.4), (4.5), (4.7) and the last inequality we have

$$\begin{aligned} & \Omega_{\alpha,i}\left(F,\delta\right)_{L^p(\mathbb{T}^m)} \le \\ \le\ & cE_{2^{v+1}}\left(F\right)_{L^p(\mathbb{T}^m)} + c\delta^{\alpha}\left\{E_0\left(F\right)_{L^p(\mathbb{T}^m)} + \sum_{k=1}^{2^v}k^{\alpha-1}E_k\left(F\right)_{L^p(\mathbb{T}^m)}\right\} \le \\ \le\ & c\delta^{\alpha}\sum_{k=1}^{2^v}k^{\alpha-1}E_k\left(F\right)_{L^p(\mathbb{T}^m)} + \\ & \quad + c\delta^{\alpha}\left\{\sum_{k=1}^{2^v}k^{\alpha-1}E_k\left(F\right)_{L^p(\mathbb{T}^m)} + E_0\left(F\right)_{L^p(\mathbb{T}^m)}\right\} \le \\ \le\ & c\delta^{\alpha}\left\{\sum_{m=1}^{2^v}k^{\alpha-1}E_k\left(F\right)_{L^p(\mathbb{T}^m)} + E_0\left(F\right)_{L^p(\mathbb{T}^m)}\right\} \le \end{aligned}$$

$$\leq c\delta^{\alpha}\left\{\sum_{k=1}^{n}k^{\alpha-1}E_k\left(F\right)_{L^p(\mathbb{T}^m)}+E_0\left(F\right)_{L^p(\mathbb{T}^m)}\right\}\leq$$

$$\leq \frac{c}{n^{\alpha}}\sum_{k=0}^{n}\left(k+1\right)^{\alpha-1}E_k\left(F\right)_{L^p(\mathbb{T}^m)}.$$

Thus,

$$\Omega_{\alpha}\left(F,\delta\right)_{L^p(\mathbb{T}^m)}=\max_{1\leq i\leq m}\Omega_{\alpha,i}\left(F,\delta\right)_{L^p(\mathbb{T}^m)}\leq c\frac{1}{n^{\alpha}}\sum_{k=0}^{n}\left(k+1\right)^{\alpha-1}E_k\left(F\right)_{L^p(\mathbb{T}^m)}$$

and Theorem 2.2 is proved. ■

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**Ali Guven**
Department of Mathematics
Faculty of Arts and Sciences
Balikesir University
10145 Balikesir
Turkey
guvennali@gmail.com

**Hasan Yurt**
Department of Mathematics
Canakkale Onsekiz Mart University
Canakkale
Turkey
hyurt@comu.edu.tr