

## ON THE $n$ -TH ELEMENT OF A SET OF POSITIVE INTEGERS

Jean-Marie De Koninck<sup>1</sup> and Vincent Ouellet  
(Québec, Canada)

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**Abstract.** Given a set  $A$  of positive integers and its counting function  $A(x) := \#\{n \leq x : n \in A\}$ , we examine the size of the  $n$ -th element of  $A$  using the size of  $A(x)$ .

### 1. Introduction and notation

Determining the size of the  $n$ -th element of a set of positive integers using the known size of the counting function of that set is a classical problem in analytic number theory. For example, letting  $\pi(x)$  stand for the number of prime numbers  $p \leq x$ , by using the Prime Number Theorem in the form  $\pi(x) \sim x/\log x$  as  $x \rightarrow \infty$ , one can easily show that the  $n$ -th prime number  $p_n$  satisfies

$$p_n = (1 + o(1)) n \log n \quad (n \rightarrow \infty).$$

In fact, in 1902, by using the logarithmic integral function, Cipolla [3] improved this estimate by showing that there exists a unique sequence of polynomials

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$(Q_j)_{j \geq 1}$  with rational coefficients such that, for any given positive integer  $m$ ,

$$(1.1) \quad p_n = n \left( \log n + \log_2 n - 1 + \sum_{j=1}^m \frac{(-1)^{j-1} Q_j(\log_2 n)}{\log^j n} + o\left(\frac{1}{\log^m n}\right) \right) \\ (n \rightarrow \infty).$$

Here and in what follows, we write  $\log_2 x$  for  $\max(1, \log \log x)$ .

Another example is given by the search of an estimate for  $a_n$ , the  $n$ -th composite number. Bojarincev [2] and Shiu [12] showed that, for any given positive integer  $m$ ,

$$(1.2) \quad a_n = n \left( 1 + \frac{\beta_1}{\log n} + \frac{\beta_2}{\log^2 n} + \cdots + \frac{\beta_m}{\log^m n} + o\left(\frac{1}{\log^m n}\right) \right) \\ (n \rightarrow \infty),$$

where the  $\beta_i$  are computable constants.

Finally, recall that we say that a number is a *powerful number* (or a *square-full number*) if  $p \mid n$  implies that  $p^2 \mid n$ . Let  $\wp_n$  denote the  $n$ -th powerful number. In 1982, Ivić and Shiu [7] showed that

$$(1.3) \quad \wp_n = \left( \frac{\zeta(3)}{\zeta(3/2)} \right)^2 n^2 + O\left(n^{5/3}\right) \quad (n \rightarrow \infty).$$

Here, we examine the problem of estimating the size of the  $n$ -th element of a given set  $A$  of positive integers using the size of  $A(x) := \#\{n \leq x : n \in A\}$ , often called the counting function of  $A$ . We will do so in two particular cases. The first one is when  $A(x) = b_1 x^{\lambda_1} + b_2 x^{\lambda_2} + R(x)$ , where  $R(x) = o(x^{\lambda_3})$ , for some real constants  $b_1 > 0$  and  $b_2$ , with  $1 > \lambda_1 > \lambda_2 > \lambda_3 > 0$ , from which we will then deduce an improvement of the estimate (1.3).

The second case is when  $A(x) = \frac{x}{L(x)} \left( 1 + O\left(\frac{1}{\varphi(x)}\right) \right)$  where  $\varphi$  is an increasing function which tends to  $+\infty$  as  $x \rightarrow \infty$  and  $L$  is a differentiable increasing slowly oscillating function. Recall that a function  $L : [M, +\infty) \rightarrow \mathbb{R}$  continuous on  $[M, +\infty)$ , where  $M$  is a positive real number, is said to be a *slowly oscillating function* if for each positive real number  $c > 0$ ,

$$(1.4) \quad \lim_{x \rightarrow \infty} \frac{L(cx)}{L(x)} = 1.$$

This class of functions was introduced by Karamata [8] in 1930. His paper, along with [9] as well as the book of Seneta [11], provide some interesting

properties of slowly oscillating functions. In particular, it is possible to show that a differentiable function  $L$  is slowly oscillating if and only if

$$(1.5) \quad \frac{xL'(x)}{L(x)} = o(1) \quad (x \rightarrow \infty)$$

and, in fact, that  $L$  is slowly oscillating if and only if there exists  $x_0 > 0$  such that

$$(1.6) \quad L(x) = C(x) \exp \left\{ \int_{x_0}^x \frac{\eta(t)}{t} dt \right\},$$

where  $\lim_{x \rightarrow \infty} C(x) = C$ , for a certain constant  $C \neq 0$ , and  $\eta(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

We shall denote by  $\mathcal{L}$  the set of increasing and differentiable slowly oscillating functions.

From here on, the letter  $c$ , with or without subscript, stands for an absolute positive constant, but not necessarily the same at each occurrence, while the letter  $p$ , with or without subscript, will always denote a prime number.

## 2. Main results

**Theorem 1.** *Given a sequence of positive integers  $a_1 < a_2 < \dots$ , let  $A = \{a_1, a_2, \dots\}$  with counting function  $A(x)$  satisfying*

$$(2.1) \quad A(x) = b_1 x^{\lambda_1} + b_2 x^{\lambda_2} + R(x),$$

where

$$R(x) = o(x^{\lambda_3}) \quad (x \rightarrow \infty)$$

and where  $b_1 > 0$  and  $b_2$  are real constants, with  $1 > \lambda_1 > \lambda_2 > \lambda_3 > 0$  which satisfy

$$\frac{3\lambda_2}{\lambda_1} - 3 < \frac{\lambda_3}{\lambda_1} - 1 \leq \frac{2\lambda_2}{\lambda_1} - 2.$$

Then

$$(2.2) \quad a_n = \frac{n^{\frac{1}{\lambda_1}}}{b_1^{\frac{1}{\lambda_1}}} - \frac{1}{\lambda_1} \frac{b_2}{b_1^{\frac{\lambda_2+1}{\lambda_1}}} n^{\frac{\lambda_2+1}{\lambda_1}-1} + \frac{1}{2} \left( \frac{2\lambda_2+1}{\lambda_1^2} - \frac{1}{\lambda_1} \right) \frac{b_2^2}{b_1^{\frac{2\lambda_2+1}{\lambda_1}}} n^{\frac{2\lambda_2+1}{\lambda_1}-2} + o\left(n^{\frac{\lambda_3+1}{\lambda_1}-1}\right).$$

**Theorem 2.** *Given a sequence of positive integers  $a_1 < a_2 < \dots$ , let  $A = \{a_1, a_2, \dots\}$  with counting function  $A(x)$  satisfying*

$$(2.3) \quad A(x) = \frac{x}{L(x)} \left( 1 + O \left( \frac{1}{\varphi(x)} \right) \right) \quad (x \rightarrow \infty),$$

*where  $\varphi$  is an increasing function which tends to  $+\infty$  as  $x \rightarrow \infty$  and where  $L \in \mathcal{L}$  with corresponding function  $\eta(t)$  defined implicitly by (1.6). Moreover, assume that  $\eta(t)$  is a decreasing function and that*

$$(2.4) \quad C(x) = C + O \left( \frac{1}{\psi(x)} \right) \quad (x \rightarrow \infty),$$

*where  $\psi(x)$  is an increasing function which tends to  $+\infty$  as  $x \rightarrow \infty$ . Then,*

$$(2.5) \quad a_n = n \frac{C(a_n)}{C(n)} L(n) \exp \left\{ \int_n^{a_n} \frac{\eta(t)}{t} dt \right\} \left( 1 + O \left( \frac{1}{\varphi(n)} \right) \right) \\ (n \rightarrow \infty)$$

*and*

$$(2.6) \quad a_n = n L(n)^{1/(1-\delta(n))} \left( 1 + O \left( \frac{1}{\varphi(n)} + \frac{1}{\psi(n)} \right) \right) \quad (n \rightarrow \infty),$$

*where  $\delta$  is some function satisfying  $\eta(a_n) < \delta(n) < \eta(n)$  for all integers  $n \geq x_0$ .*

*Moreover, if there exists a positive constant  $c$  such that*

$$(2.7) \quad \eta(n) \int_{x_0}^n \frac{\eta(t)}{t} dt \rightarrow c \quad (n \rightarrow \infty),$$

*then*

$$(2.8) \quad a_n = (e^c + o(1)) n L(n) \quad (n \rightarrow \infty).$$

### 3. Proof of Theorem 1

To prove Theorem 1, we use an approach already used by Copil and Panaitopol [4] to estimate the size of the  $n$ -th non powerful number.

First, observe that it follows from (2.1) that

$$n = A(a_n) = b_1 a_n^{\lambda_1} + b_2 a_n^{\lambda_2} + R(a_n),$$

so that

$$a_n^{\lambda_1} = \frac{n}{b_1} \left( 1 - b_2 \frac{a_n^{\lambda_2}}{n} - \frac{R(a_n)}{n} \right)$$

thereby implying that

$$(3.1) \quad a_n = \frac{n^{1/\lambda_1}}{b_1^{1/\lambda_1}} \left( 1 - b_2 \frac{a_n^{\lambda_2}}{n} - \frac{R(a_n)}{n} \right)^{1/\lambda_1}.$$

In particular, since both expressions  $b_2 \frac{a_n^{\lambda_2}}{n}$  and  $\frac{R(a_n)}{n}$  goes to 0 as  $n \rightarrow \infty$ , we have  $a_n = \frac{n^{1/\lambda_1}}{b_1^{1/\lambda_1}} (1 + o(1))$  and

$$(3.2) \quad a_n = \frac{n^{1/\lambda_1}}{b_1^{1/\lambda_1}} + O\left(n^{\frac{\lambda_2+1}{\lambda_1}-1}\right).$$

Moreover, for any  $\alpha > 0$ ,

$$(1 - y)^\alpha = 1 - \alpha y + \frac{1}{2} (\alpha^2 - \alpha) y^2 + O(y^3) \quad \text{as } y \rightarrow 0.$$

Thus

$$(3.3) \quad \begin{aligned} & \left( 1 - b_2 \frac{a_n^{\lambda_2}}{n} - \frac{R(a_n)}{n} \right)^{1/\lambda_1} = \\ & = 1 - \frac{b_2}{\lambda_1} \frac{a_n^{\lambda_2}}{n} + \frac{1}{2} \left( \frac{1}{\lambda_1^2} - \frac{1}{\lambda_1} \right) b_2^2 \frac{a_n^{2\lambda_2}}{n^2} + O\left(\frac{R(a_n)}{n}\right). \end{aligned}$$

Substituting this estimate in (3.1) yields

$$(3.4) \quad a_n = \frac{n^{1/\lambda_1}}{b_1^{1/\lambda_1}} - \frac{1}{\lambda_1} \frac{b_2}{b_1^{\frac{\lambda_2+1}{\lambda_1}}} n^{\frac{\lambda_2+1}{\lambda_1}-1} + O\left(n^{\frac{2\lambda_2+1}{\lambda_1}-2}\right).$$

Using this estimate and the fact that  $R(a_n) = o(a_n^{\lambda_3}) = o(n^{\lambda_3/\lambda_1})$ , we can replace the RHS of (3.3) by

$$(3.5) \quad 1 - \frac{1}{\lambda_1} \frac{b_2}{b_1^{\lambda_2/\lambda_1}} n^{\frac{\lambda_2}{\lambda_1}-1} + \frac{1}{2} \left( \frac{2\lambda_2+1}{\lambda_1^2} - \frac{1}{\lambda_1} \right) \frac{b_2^2}{b_1^{2\lambda_2/\lambda_1}} n^{\frac{2\lambda_2}{\lambda_1}-2} + o\left(n^{\frac{\lambda_3}{\lambda_1}-1}\right),$$

which substituted back in (3.4) yields (2.2). ■

#### 4. Proof of Theorem 2

It follows from estimate (2.3) that

$$(4.1) \quad n = A(a_n) = \frac{a_n}{L(a_n)} \left( 1 + O \left( \frac{1}{\varphi(a_n)} \right) \right) \quad (n \rightarrow \infty)$$

and therefore that

$$(4.2) \quad a_n = nL(a_n) \left( 1 + O \left( \frac{1}{\varphi(n)} \right) \right) \quad (n \rightarrow \infty).$$

Since  $L$  is a slowly oscillating function, we have

$$(4.3) \quad L(a_n) = C(a_n) \exp \left( \int_{x_0}^{a_n} \frac{\eta(t)}{t} dt \right) = \frac{C(a_n)}{C(n)} L(n) \exp \left( \int_n^{a_n} \frac{\eta(t)}{t} dt \right).$$

Combining (4.2) and (4.3) proves (2.5).

Now, let  $\alpha = \alpha(n)$  be the unique positive integer satisfying  $2^{\alpha-1}n < nL(a_n) \leq 2^\alpha n$ , so that  $\alpha = \left\lceil \frac{\log L(a_n)}{\log 2} \right\rceil = \frac{\log L(a_n)}{\log 2} + \epsilon(n)$ , where  $0 \leq \epsilon(n) < 1$ . On the one hand, since  $\eta(t)$  is decreasing and positive, we have

$$\begin{aligned} \int_n^{a_n} \frac{\eta(t)}{t} dt &\leq \int_n^{2n} \frac{\eta(t)}{t} dt + \int_{2n}^{2^{2n}} \frac{\eta(t)}{t} dt + \cdots + \int_{2^{\alpha-1}n}^{2^\alpha n} \frac{\eta(t)}{t} dt < \\ &< \eta(n) \log 2 + \eta(2n) \log 2 + \cdots + \eta(2^{\alpha-1}n) \log 2 \leq \\ (4.4) \quad &\leq \eta(n) \alpha \log 2 = \eta(n) \log 2 \left( \frac{\log L(a_n)}{\log 2} + \epsilon(n) \right). \end{aligned}$$

On the other hand,

$$\begin{aligned} \int_n^{a_n} \frac{\eta(t)}{t} dt &\geq \int_n^{2n} \frac{\eta(t)}{t} dt + \int_{2n}^{2^{2n}} \frac{\eta(t)}{t} dt + \cdots + \int_{2^{\alpha-1}n}^{2^\alpha n} \frac{\eta(t)}{t} dt > \\ &> \eta(2n) \log 2 + \eta(4n) \log 2 + \cdots + \eta(2^{\alpha-1}n) \log 2 > \\ (4.5) \quad &> \eta(2^{\alpha-1}n)(\alpha - 1) \log 2 \geq \eta(a_n) \log 2 \left( \frac{\log L(a_n)}{\log 2} + \epsilon(n) \right). \end{aligned}$$

It follows from (4.3), (4.4) and (4.5) that there exists a function  $\delta$  satisfying  $\eta(a_n) < \delta(n) < \eta(n)$  for all integers  $n \geq x_0$  such that

$$\int_n^{a_n} \frac{\eta(t)}{t} dt = \delta(n) \log L(a_n).$$

Combining this result with (2.4), we get

$$(4.6) \quad L(a_n) = L(n)^{1/(1-\delta(n))} \left( 1 + O\left(\frac{1}{\psi(n)}\right) \right).$$

Substituting (4.6) in (4.2) proves (2.6). Finally, (2.8) follows easily from (2.7). Indeed, by (4.6), we have

$$(4.7) \quad L(a_n) = L(n)^{1+\delta(n)+O(\delta^2(n))}.$$

We have

$$L(n)^{\delta(n)} \leq L(n)^{\eta(n)} = C(n)^{\eta(n)} \exp\left(\eta(n) \int_{x_0}^n \frac{\eta(t)}{t} dt\right) = e^c + o(1)$$

and

$$L(n)^{\delta(n)} \geq L(n)^{\eta(a_n)} = L(a_n)^{\eta(a_n)} \left(\frac{L(n)}{L(a_n)}\right)^{\eta(a_n)} = \left(\frac{L(n)}{L(a_n)}\right)^{\eta(a_n)} (e^c + o(1)).$$

Since

$$\log L(a_n) - \log L(n) = \int_n^{a_n} \frac{\eta(t)}{t} dt (1 + o(1)) = \delta(n) \log L(a_n) (1 + o(1)),$$

it follows that

$$\left(\frac{L(n)}{L(a_n)}\right)^{\eta(a_n)} = 1 + o(1)$$

and thus

$$(4.8) \quad L(n)^{\delta(n)} = e^c + o(1).$$

Moreover, using (4.8),

$$L(n)^{\delta(n)^2} \leq \left(L(n)^{\eta(n)}\right)^{\delta(n)} = (e^c + o(1))^{\delta(n)} = 1 + o(1).$$

Combining this last result with (4.8) and (4.7) gives

$$L(a_n) = L(n) (e^c + o(1)) \quad (n \rightarrow \infty).$$

Substituting this estimate in (4.2) yields

$$a_n = (e^c + o(1)) n L(n).$$

■

## 5. Applications of Theorem 1

We provide two applications.

First, we shall prove that there exists a positive constant  $C$  such that, as  $n \rightarrow \infty$ ,

$$(5.1) \quad \wp_n = \left( \frac{\zeta(3)}{\zeta(3/2)} \right)^2 n^2 - 2 \frac{\zeta(2/3)}{\zeta(2)} \left( \frac{\zeta(3)}{\zeta(3/2)} \right)^{\frac{8}{3}} n^{\frac{5}{3}} + \frac{7}{3} \left( \frac{\zeta(2/3)}{\zeta(2)} \right)^2 \left( \frac{\zeta(3)}{\zeta(3/2)} \right)^{\frac{10}{3}} n^{\frac{4}{3}} + R_0(n),$$

where

$$(5.2) \quad R_0(n) \ll n^{4/3} \exp \left( -C (\log n)^{3/5} (\log_2 n)^{-1/5} \right).$$

In order to prove (5.1), we first recall the 1958 result of Bateman and Grosswald [1]

$$(5.3) \quad P_2(x) := \#\{n \leq x : n \text{ powerful}\} = \frac{\zeta(3/2)}{\zeta(3)} x^{1/2} + \frac{\zeta(2/3)}{\zeta(2)} x^{1/3} + R(x),$$

where

$$(5.4) \quad R(x) \ll x^{1/6} \exp \left( -C (\log x)^{3/5} (\log_2 x)^{-1/5} \right),$$

which is the best known error term and is due to Suryanarayana and Sitaramachandra Rao [14].

Then setting  $\lambda_1 = 1/2$ ,  $\lambda_2 = 1/3$ ,  $\lambda_3 = 1/6$ ,  $b_1 = \frac{\zeta(3/2)}{\zeta(3)}$  and  $b_2 = \frac{\zeta(2/3)}{\zeta(2)}$  in Theorem 1, keeping track of the explicit error term given by (5.2), estimate (5.1) follows.

As a second application, we consider the general case of  $k$ -full numbers.

Recall that, given an integer  $k \geq 2$ , we say that a positive integer  $n$  is said to be  $k$ -full if  $p \mid n$  implies that  $p^k \mid n$ . We denote by  $P_k(x)$  the number of  $k$ -full integers  $\leq x$  and by  $\wp_{n,k}$  the  $n$ -th  $k$ -full number.

Ivić and Shiu [7] obtained that

$$(5.5) \quad P_k(x) = \gamma_{0,k} x^{1/k} + \gamma_{1,k} x^{1/(k+1)} + \cdots + \gamma_{k-1,k} x^{1/(2k-1)} + \Delta_k(x),$$

where the constants  $\gamma_{i,k}$  are given explicitly and  $\Delta_k(x)$  is a suitable error term.



Using (5.5) in the particular case  $k = 3$  and Theorem 1, we can prove that there exists a positive constant  $C$  and constants  $A_3$ ,  $A_4$  and  $A_5$  such that, as  $n \rightarrow \infty$ ,

$$\begin{aligned} \wp_{n,3} = & (A_3)^{-3} n^3 - 3(A_3)^{-\frac{15}{4}} A_4 n^{\frac{11}{4}} - 3(A_3)^{-\frac{18}{5}} A_5 n^{\frac{13}{5}} + \\ & + \frac{21}{4} (A_3)^{-\frac{9}{2}} (A_4)^2 n^{\frac{5}{2}} + R(n), \end{aligned}$$

where

$$R(n) \ll n^{\frac{19}{8}} \exp\left(-C(\log n)^{3/5} (\log_2 n)^{-1/5}\right).$$

**Remark 1.** Observe that explicit values for the constants  $A_i$  were obtained by Shiu [12]. Moreover, for  $k \geq 4$ , as  $n \rightarrow \infty$ , one can prove that

$$(5.6) \quad \wp_{n,k} = \left(\frac{n}{\gamma_{0,k}}\right)^k - k \frac{\gamma_{1,k}}{(\gamma_{0,k})^{\frac{k(k+2)}{k+1}}} n^{\frac{k^2+k-1}{k+1}} - k \frac{\gamma_{2,k}}{(\gamma_{0,k})^{\frac{k(k+3)}{k+2}}} n^{\frac{k^2+2k-2}{k+2}} + R_k(n),$$

where  $R_k(n) \ll n^{\frac{k^2+k-2}{k+1}}$ .

**Remark 2.** Observe that explicit values for the constants  $\gamma_{i,k}$  are given in Bateman and Grosswald [1] and Erdős and Szekeres [6]. Moreover, for  $k > 4$ , additional terms on the right hand side of (5.6) can be provided.

## 6. Applications of Theorem 2

We provide three applications.

1. Fix a positive integer  $k$  and let

$$A = A_k = \{n \in \mathbb{N} : \omega(n) = k\} = \{a_n : n \in \mathbb{N}\},$$

where  $\omega(n)$  stands for the number of distinct prime factors of  $n$ . It is well known that, as  $x \rightarrow \infty$ ,

$$A(x) = \frac{x}{L(x)} \left(1 + O\left(\frac{1}{\log \log x}\right)\right),$$

where  $L(x) = \frac{(k-1)! \log x}{(\log_2 x)^{k-1}}$  (see for instance Theorem 10.4 in the book of De Koninck and Luca [5]). It follows from Theorem 2 that

$$a_n = n \frac{(k-1)! \log n}{(\log_2 n)^{k-1}} \left(1 + O\left(\frac{1}{\log \log n}\right)\right) \quad (n \rightarrow \infty).$$

2. Consider the set  $A$  of those integers  $n \geq 2$  such that  $\lfloor z(n) \rfloor - \lfloor z(n-1) \rfloor = 1$ , where  $z(n) = n/e^{\sqrt{\log n}}$ . Using a computer, we easily obtain the first elements of  $A$ , so that we may write

$$A = \{3, 9, 16, 24, 33, 42, 51, 61, 71, 82, 93, \dots\} = \{a_n : n \in \mathbb{N}\}.$$

Clearly  $|A(x) - z(x)| \leq 1$  for all  $x \geq 2$ . Then, since condition (2.7) of Theorem 2 is satisfied with  $c = 1/2$ , we get from (2.8) that

$$a_n = (\sqrt{e} + o(1)) ne^{\sqrt{\log n}} \quad (n \rightarrow \infty).$$

3. Let  $W = \{a_1, a_2, \dots\}$  be the set of those positive integers which can be written as the sum of two squares. It has been known since Euler that a positive integer can be represented as a sum of two squares if and only if each of its prime factors of the form  $4k + 3$  occurs with an even power, so that

$$W = \{2, 5, 8, 10, 13, 17, 18, 20, 25, 26, 29, 32, 34, 37, 40, 41, 45, 50, \dots\}.$$

In 1908, Landau [10] showed that

$$W(x) = (B + o(1)) \frac{x}{\sqrt{\log x}} \quad (x \rightarrow \infty),$$

where  $B = \frac{1}{\sqrt{2}} \prod_{p \equiv 3 \pmod{4}} \left( \sqrt{1 - \frac{1}{p^2}} \right)^{-1} = 0.7642236 \dots$ . In 1986, Shiu [13] showed that

$$(6.1) \quad W(x) = \frac{Bx}{\sqrt{\log x}} \left( 1 + O\left( \frac{1}{\log x} \right) \right).$$

Since one can show that

$$(6.2) \quad L(x) := \frac{1}{B} \sqrt{\log x} = \frac{1}{B} \exp \left\{ \int_e^x \frac{1}{2 \log t} \frac{dt}{t} \right\},$$

it follows from (6.1) and (6.2) that the corresponding functions  $\varphi(x)$ ,  $\eta(x)$ ,  $C(x)$  and  $\psi(x)$  from the statement of Theorem 2 are given by

$$\varphi(x) = \log x, \quad \eta(x) = \frac{1}{2 \log x}, \quad C(x) = 1/B, \quad \psi(x) = \infty.$$

Hence, it follows from (2.5) that

$$a_n = \frac{1}{B} n \sqrt{\log a_n} \left( 1 + O\left( \frac{1}{\log n} \right) \right).$$

Thus, using the logarithm on this formula, one can improve it to

$$a_n = \frac{1}{B} n \sqrt{\log n} \left( 1 + \frac{1}{4} \frac{\log \log n}{\log n} + O\left( \frac{1}{\log n} \right) \right) \quad (n \rightarrow \infty).$$

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**J.-M. De Koninck and V. Ouellet**

Département de mathématiques et de statistique

Université Laval

Québec

Québec G1V 0A6

Canada

`jmdk@mat.ulaval.ca`

`vincent.ouellet.7@ulaval.ca`