A CONVERGENCE ANALYSIS FOR A CERTAIN FAMILY OF EXTENDED ITERATIVE METHODS: PART I. THEORY

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Abstract. We present local and semilocal convergence results for some extended methods in order to approximate a locally unique solution of a nonlinear equation in a Banach space setting. In earlier studies the operator involved is assumed to be at least once Fréchet-differentiable. In the present study, we assume that the operator is only continuous. This way we expand the applicability of these methods. In Part II of the study, we present some choices of the operators involved in fractional calculus where the operators satisfy the convergence conditions. Moreover, we present a corrected version of the generalized fractional Taylor's formula given in [14].

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of the nonlinear equation

$$F(x) = 0, \tag{1.1}$$

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where F is a continuous operator defined on a subset D of a Banach space X with values in a Banach space Y.

A lot of problems in Computational Sciences and other disciplines can be brought in a form like (1.1) using Mathematical Modelling [7], [11], [15]. The solutions of such equations can be found in closed form only in special cases. That is why most solution methods for these equations are iterative. Iterative algorithms are usually studied based on semilocal and local convergence. The semilocal convergence matter is, based on the information around the initial point to give hypotheses ensuring the convergence of the method; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls as well as error bounds on the distances involved.

We introduce the method defined for each $n = 0, 1, 2, ...$ by

$$x_{n+1} = x_n - A(x_n)^{-1} F(x_n), \tag{1.2}$$

where $x_0 \in D$ is an initial point and $A(x) \in L(X, Y)$ the space of bounded linear operators from X into Y. There is a plethora on local as well as semilocal convergence theorems for method (1.2) provided that the operator A is an approximation to the Fréchet-derivative F' [1, 2, 5 - 15]. In the present study we do not assume that operator A is not necessarily related to F'. This way we expand the applicability of method (1.2). Notice that many well known methods are special case of method (1.2).

Newton's method: Choose $A(x) = F'(x)$ for each $x \in D$.

Steffensen's method: Choose $A(x) = [x, G(x); F]$, where $G : X \to X$ is a known operator and $[x, y; F]$ denotes a divided difference of order one [7, 11, 15].

The so called Newton-like methods and many other methods are special cases of method (1.2).

The rest of the paper is organized as follows. The semilocal as well as the local convergence analysis of method (1.2) is given in Section 2. Some applications from fractional calculus are given in Part II. In particular, we first correct the generalized fractional Taylor's formula, the integral version extracted from [14]. Then, we use the corrected formula in our applications.

2. Convergence analysis

We present the main semilocal convergence result for method (1.2).

Theorem 2.1. *Let $F : D \subset X \to Y$ be a continuous operator and let $A(x) \in L(X,Y)$. Suppose that there exist $x_0 \in D$, $\eta \geq 0$, $p \geq 1$, a function $g : [0,\eta] \to [0,\infty)$ continuous and nondecreasing such that for each $x, y \in D$*

$$A(x)^{-1} \in L(Y,X), \tag{2.1}$$

$$\left\| A(x_0)^{-1} F(x_0) \right\| \leq \eta, \tag{2.2}$$

$$\left\| A(y)^{-1} \left(F(y) - F(x) - A(x)(y-x) \right) \right\| \leq g(\|x-y\|) \|x-y\|^{p+1}, \tag{2.3}$$

$$q := g(\eta)\eta^p < 1 \tag{2.4}$$

and

$$\overline{U}(x_0, r) \subseteq D, \tag{2.5}$$

where,

$$r = \frac{\eta}{1-q}. \tag{2.6}$$

Then, the sequence $\{x_n\}$ generated by method (1.2) is well defined, remains in $\overline{U}(x_0,r)$ for each $n = 0,1,2,\ldots$ and converges to some $x^* \in \overline{U}(x_0,r)$ such that

$$\|x_{n+1} - x_n\| \leq g(\|x_n - x_{n-1}\|) \|x_n - x_{n-1}\|^{p+1} \leq q \|x_n - x_{n-1}\| \tag{2.7}$$

and

$$\|x_n - x^*\| \leq \frac{q^n \eta}{1-q}. \tag{2.8}$$

Proof. The iterate x_1 is well defined by method (1.2) for $n = 0$ and (2.1) for $x = x_0$. We also have by (2.2) and (2.6) that $\|x_1 - x_0\| = \left\| A(x_0)^{-1} F(x_0) \right\| \leq \leq \eta < r$, so we get that $x_1 \in \overline{U}(x_0,r)$ and x_2 is well defined (by (2.5)). Using (2.3) for $y = x_1$, $x = x_0$ and (2.4) we get that

$$\|x_2 - x_1\| = \left\| A(x_1)^{-1} \left[F(x_1) - F(x_0) - A(x_0)(x_1 - x_0) \right] \right\| \leq$$

$$\leq g(\|x_1 - x_0\|) \|x_1 - x_0\|^{p+1} \leq q \|x_1 - x_0\|,$$

which shows (2.7) for $n = 1$. Then, we can have that

$$\|x_2 - x_0\| \le \|x_2 - x_1\| + \|x_1 - x_0\| \le q\|x_1 - x_0\| + \|x_1 - x_0\| =$$

$$= (1+q)\|x_1 - x_0\| \le \frac{1-q^2}{1-q}\eta < r,$$

so $x_2 \in \overline{U}(x_0, r)$ and x_3 is well defined.

Assuming $\|x_{k+1} - x_k\| \le q\|x_k - x_{k-1}\|$ and $x_{k+1} \in \overline{U}(x_0, r)$ for each $k = 1, 2, \ldots, n$ we get

$$\|x_{k+2} - x_{k+1}\| = \left\| A(x_{k+1})^{-1}\left[F(x_{k+1}) - F(x_k) - A(x_k)(x_{k+1} - x_k)\right]\right\| \le$$

$$\le g(\|x_{k+1} - x_k\|)\|x_{k+1} - x_k\|^{p+1} \le$$

$$\le g(\|x_1 - x_0\|)\|x_1 - x_0\|^p\|x_{k+1} - x_k\| \le q\|x_{k+1} - x_k\|$$

and

$$\|x_{k+2} - x_0\| \le \|x_{k+2} - x_{k+1}\| + \|x_{k+1} - x_k\| + \cdots + \|x_1 - x_0\| \le$$

$$\le \left(q^{k+1} + q^k + \cdots + 1\right)\|x_1 - x_0\| \le \frac{1-q^{k+2}}{1-q}\|x_1 - x_0\| \le$$

$$< \frac{\eta}{1-q} = r,$$

which completes the induction for (2.7) and $x_{k+2} \in \overline{U}(x_0, r)$. We also have that for $m \ge 0$

$$\|x_{n+m} - x_n\| \le \|x_{n+m} - x_{n+m-1}\| + \cdots + \|x_{n+1} - x_n\| \le$$

$$\le \left(q^{m-1} + q^{m-2} + \cdots + 1\right)\|x_{n+1} - x_n\| \le$$

$$\le \frac{1-q^m}{1-q}q^n\|x_1 - x_0\|.$$

It follows that $\{x_n\}$ is a complete sequence in a Banach space X and as such it converges to some $x^* \in \overline{U}(x_0, r)$ (since $\overline{U}(x_0, r)$ is a closed set). By letting $m \to \infty$, we obtain (2.8). ■

Stronger hypotheses are needed to show that x^* is a solution of equation $F(x) = 0$.

Proposition 2.2. *Let $F : D \subset X \to Y$ be a continuous operator and let $A(x) \in L(X, Y)$. Suppose that there exist $x_0 \in D$, $\eta \ge 0$, $p \ge 1$, $\psi > 0$, a* *function* $g_1 : [0, \eta] \to [0, \infty)$ *continuous and nondecreasing such that for each* $x, y \in D$

$$A(x)^{-1} \in L(Y, X), \quad \left\| A(x)^{-1} \right\| \leq \psi, \quad \left\| A(x_0)^{-1} F(x_0) \right\| \leq \eta, \tag{2.9}$$

$$\| F(y) - F(x) - A(x)(y - x) \| \leq \frac{g_1(\|x - y\|)}{\psi} \|x - y\|^{p+1}, \tag{2.10}$$

$$q_1 := g_1(\eta) \eta^p < 1$$

and

$$\overline{U}(x_0, r_1) \subseteq D,$$

where,

$$r_1 = \frac{\eta}{1 - q_1}.$$

Then, the conclusions of Theorem 2.1 *for sequence* $\{x_n\}$ *hold with* g_1*,* q_1*,* r_1*, replacing* g*,* q *and* r*, respectively. Moreover,* x^* *is a solution of the equation* $F(x) = 0$.

Proof. Notice that

$$\left\| A(x_n)^{-1} \left[F(x_n) - F(x_{n-1}) - A(x_{n-1})(x_n - x_{n-1}) \right] \right\| \leq$$

$$\leq \left\| A(x_n)^{-1} \right\| \| F(x_n) - F(x_{n-1}) - A(x_{n-1})(x_n - x_{n-1}) \| \leq$$

$$\leq g_1(\|x_n - x_{n-1}\|) \|x_n - x_{n-1}\|^{p+1} \leq q_1 \|x_n - x_{n-1}\|.$$

Therefore, the proof of Theorem 2.1 can apply. Then, in view of the estimate

$$\|F(x_n)\| = \|F(x_n) - F(x_{n-1}) - A(x_{n-1})(x_n - x_{n-1})\| \leq$$

$$\leq \frac{g_1(\|x_n - x_{n-1}\|)}{\psi} \|x_n - x_{n-1}\|^{p+1} \leq q_1 \|x_n - x_{n-1}\|,$$

we deduce by letting $n \to \infty$ that $F(x^*) = 0$. ■

Concerning the uniqueness of the solution x^* we have the following result:

Proposition 2.3. *Under the hypotheses of Proposition* 2.2*, further suppose that*

$$q_1 r_1^p < 1. \tag{2.11}$$

Then, x^* *is the only solution of equation* $F(x) = 0$ *in* $\overline{U}(x_0, r_1)$.

Proof. The existence of the solution $x^* \in \overline{U}(x_0, r_1)$ has been established in Proposition 2.2. Let $y^* \in \overline{U}(x_0, r_1)$ with $F(y^*) = 0$. Then, we have in turn that

$$\|x_{n+1} - y^*\| = \left\| x_n - y^* - A(x_n)^{-1} F(x_n) \right\| =$$

$$= \left\| A(x_n)^{-1} \left[A(x_n)(x_n - y^*) - F(x_n) + F(y^*) \right] \right\| \leq$$

$$\leq \left\| A(x_n)^{-1} \right\| \|F(y^*) - F(x_n) - A(x_n)(y^* - x_n)\| \leq$$

$$\leq \psi \frac{g_1(\|x_n - y^*\|)}{\psi} \|x_n - y^*\|^{p+1} \leq q_1 r_1^p \|x_n - x^*\| < \|x_n - y^*\|,$$

so we deduce that $\lim_{n\to\infty} x_n = y^*$. But we have that $\lim_{n\to\infty} x_n = x^*$. Hence, we conclude that $x^* = y^*$. ■

Next, we present a local convergence analysis for the method (1.2).

Proposition 2.4. *Let $F : D \subset X \to Y$ be a continuous operator and let $A(x) \in L(X, Y)$. Suppose that there exist $x^* \in D$, $p \geq 1$, a function $g_2 : [0, \infty) \to [0, \infty)$ continuous and nondecreasing such that for each $x \in D$*

$$F(x^*) = 0, \quad A(x)^{-1} \in L(Y, X),$$

$$\left\| A(x)^{-1} \left[F(x) - F(x^*) - A(x)(x - x^*) \right] \right\| \leq \tag{2.12}$$
$$\leq g_2(\|x - x^*\|) \|x - x^*\|^{p+1},$$

and

$$\overline{U}(x^*, r_2) \subseteq D,$$

where r_2 is the smallest positive solution of equation

$$h(t) := g_2(t) t^p - 1.$$

Then, sequence $\{x_n\}$ generated by method (1.2) for $x_0 \in U(x^*, r_2) - \{x^*\}$ is well defined, remains in $U(x^*, r_2)$ for each $n = 0, 1, 2, \ldots$ and converges to x^*. Moreover, the following estimates hold

$$\|x_{n+1} - x^*\| \leq g_2(\|x_n - x^*\|) \|x_n - x^*\|^{p+1} < \|x_n - x^*\| < r_2.$$

Proof. We have that $h(0) = -1 < 0$ and $h(t) \to +\infty$ as $t \to +\infty$. Then, it follows from the intermediate value theorem that function h has positive zeros. Denote by r_2 the smallest such zero. By hypothesis $x_0 \in U(x^*, r_2) - \{x^*\}$. Then, we get in turn that

$$\|x_1 - x^*\| = \left\|x_0 - x^* - A(x_0)^{-1} F(x_0)\right\| =$$

$$= \left\|A(x_0)^{-1}\left[F(x^*) - F(x_0) - A(x_0)(x^* - x_0)\right]\right\| \leq$$

$$\leq g_2(\|x_0 - x^*\|)\|x_0 - x^*\|^{p+1} < g_2(r_2) r_2^p \|x_0 - x^*\| =$$

$$= \|x_0 - x^*\| < r_2,$$

which shows that $x_1 \in U(x^*, r_2)$ and x_2 is well defined. By a simple inductive argument as in the preceding estimate we get that

$$\|x_{k+1} - x^*\| = \left\|x_k - x^* - A(x_k)^{-1} F(x_k)\right\| \leq$$

$$\leq \left\|A(x_k)^{-1}\left[F(x^*) - F(x_k) - A(x_k)(x^* - x_k)\right]\right\| \leq$$

$$\leq g_2(\|x_k - x^*\|)\|x_k - x^*\|^{p+1} < g_2(r_2) r_2^p \|x_k - x^*\| = \|x_k - x^*\| < r_2,$$

which shows $\lim_{k\to\infty} x_k = x^*$ and $x_{k+1} \in U(x^*, r_2)$. ■

Remark 2.1. (a) Hypothesis (2.3) specializes to Newton-Mysowski-type, if $A(x) = F'(x)$ [7], [11], [15]. However, if F is not Fréchet-differentiable, then our results extend the applicability of iterative algorithm (1.2).

(b) Theorem 2.1 has practical value although we do not show that x^* is a solution of equation $F(x) = 0$, since this may be shown in another way.

(c) Hypothesis (2.12) can be replaced by the stronger

$$\left\|A(x)^{-1}\left[F(x) - F(y) - A(x)(x - y)\right]\right\| \leq g_2(\|x - y\|)\|x - y\|^{p+1}.$$

The preceding results can be extended to hold for two point methods defined for each $n = 0, 1, 2, \ldots$ by

$$x_{n+1} = x_n - A(x_n, x_{n-1})^{-1} F(x_n), \tag{2.13}$$

where $x_{-1}, x_0 \in D$ are initial points and $A(w, v) \in L(X, Y)$ for each $v, w \in D$. If $A(w, v) = [w, v; F]$, then method (2.13) reduces to the popular secant method, where $[w, v; F]$ denotes a divided difference of order one for the operator F. Many other choices for A are also possible [7], [11], [15].

If we simply replace $A(x)$ by $A(y, x)$ in the proof of Proposition 2.2 we arrive at the following semilocal convergence result for method (2.13).

Theorem 2.5. *Let $F : D \subset X \to Y$ be a continuous operator and let $A(y,x) \in L(X,Y)$ for each $x, y \in D$. Suppose that there exist $x_{-1}, x_0 \in D$, $\eta \geq 0$, $p \geq 1$, $\psi > 0$, a function $g_1 : [0,\eta] \to [0,\infty)$ continuous and nondecreasing such that for each $x, y \in D$:*

$$A(y,x)^{-1} \in L(Y,X), \quad \left\|A(y,x)^{-1}\right\| \leq \psi, \tag{2.14}$$

$$\min\left\{\|x_0 - x_{-1}\|, \left\|A(x_0,x_{-1})^{-1} F(x_0)\right\|\right\} \leq \eta,$$

$$\|F(y) - F(x) - A(y,x)(y-x)\| \leq \frac{g_1(\|x-y\|)}{\psi}\|x-y\|^{p+1}, \tag{2.15}$$

$$q_1 < 1, \quad q_1 r_1^p < 1$$

and

$$\overline{U}(x_0, r_1) \subseteq D,$$

where,

$$r_1 = \frac{\eta}{1-q_1}$$

and q_1 is defined in Proposition 2.2.

Then, sequence $\{x_n\}$ generated by method (2.13) is well defined, remains in $\overline{U}(x_0, r_1)$ for each $n = 0, 1, 2, \ldots$ and converges to the only solution of equation $F(x) = 0$ in $\overline{U}(x_0, r_1)$.

Moreover, the estimates (2.7) and (2.8) hold with g_1, q_1 replacing g and q, respectively.

Concerning, the local convergence of the iterative algorithm (2.13) we obtain the analogous to Proposition 2.4 result.

Proposition 2.6. *Let $F : D \subset X \to Y$ be a continuous operator and let $A(y,x) \in L(X,Y)$. Suppose that there exist $x^* \in D$, $p \geq 1$, a function $g_2 : [0,\infty)^2 \to [0,\infty)$ continuous and nondecreasing such that for each $x, y \in D$*

$$F(x^*) = 0, \quad A(y,x)^{-1} \in L(Y,X),$$

$$\left\|A(y,x)^{-1}[F(y) - F(x^*) - A(y,x)(y-x^*)]\right\| \leq$$

$$\leq g_2(\|y-x^*\|, \|x-x^*\|)\|y-x^*\|^{p+1}$$

and

$$\overline{U}(x^*, r_2) \subseteq D,$$

where r_2 *is the smallest positive solution of equation*

$$h(t) := g_2(t,t)\,t^p - 1.$$

Then, sequence $\{x_n\}$ *generated by method* (2.13) *for* $x_{-1}, x_0 \in U(x^*, r_2) - \{x^*\}$ *is well defined, remains in* $U(x^*, r_2)$ *for each* $n = 0, 1, 2, \ldots$ *and converges to* x^*. *Moreover, the following estimates hold*

$$\|x_{n+1} - x^*\| \le g_2(\|x_n - x^*\|, \|x_{n-1} - x^*\|)\,\|x_n - x^*\|^{p+1} <$$

$$< \|x_n - x^*\| < r_2.$$

Remark 2.2. In Part II, we present some choices and properties of operator $A(y,x)$ from fractional calculus satisfying the crucial estimate (2.15) in the special case when,

$$g_1(t) = c\psi \text{ for some } c > 0 \text{ and each } t \ge 0.$$

(see the end of Part II for a possible definition of the constant c).

Hence, Theorem 2.5 can apply to solve equation $F(x) = 0$. Other choices for operator $A(x)$ or operator $A(y,x)$ can be found in [5]–[7], [9]–[15].

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