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BALL CONVERGENCE COMPARISON BETWEEN TWO SIXTH ORDER NEWTON–JARRATT COMPOSITION METHODS

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Abstract. We compare the convergence balls of two sixth order Newton-Jarratt composition methods used to approximate a locally unique solution of an equation in a Banach space setting. Our convergence conditions involve only the first Fréchet derivative in contrast to earlier studies such as [13, 28] using hypotheses up to the seventh Fréchet-derivative of the operator involved. This way we expand the applicability of these methods. We also provide computable radii of convergence and error bounds based only on Lipschitz constants. We also present examples where earlier results cannot apply but our results apply to solve equations.

1. Introduction

In this study, we are concerned with the problem of approximating a locally unique solution x^* of the nonlinear equation

$$F(x) = 0, \tag{1.1}$$

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where F is a Fréchet-differentiable operator defined on a convex subset D of a Banach space X with values in a Banach space Y. Using mathematical modeling, many problems in computational sciences and other disciplines can be expressed as a nonlinear equation (1.1) [1]–[29]. Closed form solutions of these nonlinear equations exist only for few special cases which may not be of much practical value. Therefore solutions of these nonlinear equations (1.1) are approximated by iterative methods. In particular, the practice of Numerical Functional Analysis for approximating solutions iteratively is essentially connected to Newton-like methods [1]–[29]. The study about convergence matter of iterative procedures is usually based on two types: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls. There exist many studies which deal with the local and semi-local convergence analysis of Newton-like methods such as [1]–[29].

We study the local convergence of the sixth-order three-step methods defined for each $n = 0, 1, 2...$ by

$$
\begin{aligned}
y_n &= x_n - \frac{2}{3}F'(x_n)^{-1}F(x_n)\\
z_n &= x_n - \frac{1}{2}(3F'(y_n) - F'(x_n))^{-1}(3F'(y_n) + F'(x_n))F'(x_n)^{-1}F(x_n)\\
x_{n+1} &= z_n - 2(3F'(y_n) - 3F'(x_n))^{-1}F(z_n),
\end{aligned}
\tag{1.2}
$$

and

$$
\begin{aligned}
y_n &= x_n - \frac{2}{3}F'(x_n)^{-1}F(x_n)\\
z_n &= x_n - \frac{1}{3}(3F'(y_n) - F'(x_n))^{-1}(3F'(y_n) + F'(x_n))F'(x_n)^{-1}F(x_n)\\
x_{n+1} &= z_n - \frac{1}{4}((3F'(y_n) - F'(x_n))^{-1}(3F'(y_n) + F'(x_n)))^2F'(x_n)^{-1}F(z_n),
\end{aligned}
\tag{1.3}
$$

where x_0 is an initial point. The local convergence analysis of method (1.2) was given in [13] in the special case when $X = Y = \mathbb{R}^m$. The method (1.2) and method (1.3) were studied in [13] and [28] using Taylor expansions and hypotheses reaching up to the seventh derivative of function F, when $X = Y = \mathbb{R}^m$. The hypotheses up to the seventh derivative limit the applicability of these methods. As a motivational example, let us define function F on $X = [-\frac{1}{2}, \frac{5}{2}]$ by

$$
F(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0 \end{cases}
$$

Choose $x^* = 1$. We have that

$$\begin{aligned} F'(x) &= 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2, \; F'(1) = 3, \\ F''(x) &= 6x \ln x^2 + 20x^3 - 12x^2 + 10x \\ F'''(x) &= 6 \ln x^2 + 60x^2 - 24x + 22. \end{aligned}$$

Then, obviously function F does not have bounded third derivative in X. Notice that, in-particular there is a plethora of iterative methods for approximating solutions of nonlinear equations [1]–[29]. These results show that if the initial point x_0 is sufficiently close to the solution x^*, then the sequence $\{x_n\}$ converges to x^*. But how close to the solution x^* the initial guess x_0 should be? These local results give no information on the radius of the convergence ball for the corresponding method. We address this question for method (1.2) in Section 2. The same technique can be used to other methods.

In the present study we extend the applicability of the method (1.2) by using hypotheses up to the first derivative of function F and contractions on a Banach space setting. Moreover we avoid Taylor expansions and use instead Lipschitz parameters. Moreover, we do not have to use higher order derivatives to show the convergence of method (1.2). This way we expand the applicability of method (1.2).

The paper is organized as follows. In Section 2 we present the local convergence analysis. We also provide a radius of convergence, computable error bounds and uniqueness result not given in the earlier studies using Taylor expansions. Special cases and numerical examples are presented in the concluding Section 3.

2. Local convergence analysis

We present the local convergence analysis of the method (1.2) and method (1.3) in this section. Let $L_0 > 0$, $L > 0$ and $M \in [1, 3)$ be given parameters. It is convenient for the local convergence analysis of method (1.2) to define some scalar functions and parameters. Define functions g_1, h_1, p and h_p on the interval $[0, \frac{1}{L_0})$ by

$$g_1(t) = \frac{1}{2(1 - L_0 t)}(Lt + \frac{2M}{3}),$$

$$h_1(t) = g_1(t) - 1,$$

$$p(t) = \frac{1}{2}(3L_0 g_1(t) + L_0)t,$$

$$h_p(t) = p(t) - 1$$

and parameters r_A, r_1 by

$$r_A = \frac{2}{2L_0 + L}, \ r_1 = \frac{2(1 - \frac{M}{3})}{2L_0 + L}.$$

Notice that $0 < r_1 < r_A$ and $h_1(r_1) = 0$. We have that $h_p(0) = -1$ and $h_p(t) \to +\infty$ as $t \to \frac{1}{L_0}^-$. It follows from the intermediate value theorem that function h_p has zeros in the interval $(0, \frac{1}{L_0})$. Denote by r_p the smallest such zero. Moreover, define functions g_2, h_2, g_3 and h_3 on the interval $[0, r_p)$ by

$$g_2(t) = \frac{1}{2(1 - L_0 t)}(L + \frac{3ML_0(1 + g_1(t))}{2(1 - p(t))})t, \ h_2(t) = g_2(t) - 1,$$

$$g_3(t) = (1 + \frac{M}{1 - p(t)})g_2(t)$$

and

$$h_3(t) = g_3(t) - 1.$$

We have that $h_2(0) = -1 < 0$ and $h_2(t) \to +\infty$ as $t \to r_p^-$. Denote by r_2 the smallest zero of function h_2 in the interval $(0, r_p)$. Using the definition of r_2, we get that $h_3(0) = -1 < 0$ and $h_3(r_2) = \frac{M}{1-p(r_2)} > 0$. Denote by r_3 the smallest zero of function h_3 in the interval $(0, r_2)$. Set

$$r = \min\{r_1, r_3\}. \tag{2.1}$$

Then, we have that

$$0 < r < r_A \tag{2.2}$$

and for each $t \in [0, r)$

$$0 \leq g_1(t) < 1 \tag{2.3}$$
$$0 \leq p(t) < 1 \tag{2.4}$$
$$0 \leq g_2(t) < 1 \tag{2.5}$$

and

$$0 \leq g_3(t) < 1. \tag{2.6}$$

Let $U(\gamma, \rho)$, $\bar{U}(\gamma, \rho)$, respectively the open and closed balls in X with center $r \in X$ and of radius $r \in X$ and of $\rho > 0$. Next, we present the local convergence analysis of the method (1.2), using the preceding notation.

Theorem 2.1. *Let $F : D \subset X \to Y$ be a Fréchet-differentiable operator. Suppose that there exist $x^* \in D$, $L_0 > 0$, $L > 0$ and $M \in [1, 3)$ such that for each $x, y \in D$*

$$F(x^*) = 0, F'(x^*) \neq 0 \tag{2.7}$$

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq L_0\|x - x^*\|, \tag{2.8}$$

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq L\|x - x^*\|, \tag{2.9}$$

$$\|F'(x^*)^{-1}F'(x)\| \leq M \tag{2.10}$$

and

$$\bar{U}(x^*, r) \subseteq D, \tag{2.11}$$

where the radius r is given by (2.1). Then, the sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r) - \{x^*\}$ by method (1.2) is well defined , remains in $U(x^*, r)$ for each $n = 0, 1, 2.....$ and converges to x^*. Moreover, the following estimates hold

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\| < r, \tag{2.12}$$

$$\|z_n - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\| \tag{2.13}$$

and

$$\|x_{n+1} - x^*\| \leq g_3(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|, \tag{2.14}$$

where the "g" functions are defined previously. Furthermore, if there exist $T \in [r, \frac{2}{L_0})$ and $\bar{U}(x^*, T) \in D$, then the limit point x^* is the only solution of the equation $F(x) = 0$ in $\bar{U}(x^*, T) \cap D$.

Proof. We shall show estimates (2.12)–(2.14) using mathematical induction. By hypothesis $x_0 \in U(x^*, r) - \{x^*\}$, (2.1) and (2.8), we have that

$$\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \leq L_0\|x_0 - x^*\| < L_0 r < 1. \tag{2.15}$$

It follows from (2.15) and Banach Lemma on invertible operators [3, 6, 19, 23, 24, 28] that $F'(x_0) \neq 0$ and

$$\|F'(x_0)^{-1}F'(x^*)\| \leq \frac{1}{1 - L_0\|x_0 - x^*\|}. \tag{2.16}$$

Hence, y_0 is well defined by the first sub-step of method (1.2) for $n = 0$. Then, by (1.2), (2.1), (2.2), (2.7), (2.9), (2.10) and (2.16), we get in turn that

$$\begin{aligned}
\|y_0 - x^*\| &= \|(x_0 - x^* - F'(x_0)^{-1}F(x_0)) + \frac{1}{3}F'(x_0)^{-1}F(x_0)\| \leq \\
&\leq \|x_0 - x^* - F'(x_0)^{-1}F(x_0)\| + \\
&\quad + \frac{1}{3}\|F'(x_0)^{-1}F(x^*)\|\|F'(x^*)^{-1}F(x_0)\| \leq \\
&\leq \|F'(x_0)^{-1}F'(x^*)\|\|\int_0^1 F'(x^*)^{-1}(F'(x^* + \theta(x_0 - x^*)) - \\
&\quad - F'(x_0))(x_0 - x^*)d\theta\| + \\
&\quad + \|F'(x_0)^{-1}F'(x^*)\|\|\int_0^1 F'(x^*)^{-1}(F'(x^* + \theta(x_0 - x^*)) - \\
&\quad - F'(x_0))(x_0 - x^*)d\theta\| \leq \\
&\leq \frac{L\|x_0 - x^*\|^2}{2(1 - L_0\|x_0 - x^*\|)} + \frac{M\|x_0 - x^*\|}{3(1 - L_0\|x_0 - x^*\|)} = \\
&= g_1(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r, \qquad (2.17)
\end{aligned}$$

which shows (2.12) for $n = 0$ and $y_0 \in U(x^*, r)$. Next, we show that $3F'(y_0) - -F'(x_0) \neq 0$. Using (2.1), (2.3), (2.8) and (2.17), we get in turn that

$$\begin{aligned}
&\|(2F'(x^*))^{-1}[3F'(y_0) - 3F'(x^*) + F'(x^*) - F'(x_0)]\| \leq \\
&\leq \frac{1}{2}[3\|F'(x^*)^{-1}(F'(y_0) - F'(x^*))\| + \\
&\quad + \|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\|] \leq \\
&\leq \frac{1}{2}[3L_0\|y_0 - x^*\| + L_0\|x_0 - x^*\|] < \\
&< \frac{L_0}{2}(3g_1(\|x_0 - x^*\|) + 1)\|x_0 - x^*\|) = \\
&= p(\|x_0 - x^*\|) < p(r) < 1. \qquad (2.18)
\end{aligned}$$

Hence, we get by (2.18) that

$$\|(3F'(y_0) - F'(x_0))^{-1}F'(x^*)\| \leq \frac{1}{2(1 - p(\|x_0 - x^*\|))}. \qquad (2.19)$$

Hence, z_0 and x_1 are well defined.

Then, by the second sub-step of method (1.2) for $n = 0$, (2.1), (2.4), (2.9), (2.10), (2.16), (2.17) and (2.19), we obtain in turn

$$\begin{aligned}
\|z_0 - x^*\| &\le \|x_0 - x^* - F'(x_0))^{-1}F'(x_0)\| + \\
&\quad + \|[I - \frac{1}{2}(3F'(y_0) - F'(x_0))^{-1} \times \\
&\quad \times (3F'(y_0) + F'(x_0))F'(x_0)^{-1}F(x_0)\| \le \\
&\le \frac{L\|x_0 - x^*\|^2}{2(1 - L_0\|x_0 - x^*\|)} + \frac{3}{2}\|(3F'(y_0) - F'(x_0))^{-1}F'(x^*)\| \times \\
&\quad \times (\|F'(x^*)^{-1}(F'(y_0) - F'(x^*))\| + \|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\|) \times \\
&\quad \times \|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(x_0)\| = \\
&= \frac{L\|x_0 - x^*\|^2}{2(1 - L_0\|x_0 - x^*\|)} + \frac{3L_0M(\|y_0 - x^*\| + \|x_0 - x^*\|)\|x_0 - x^*\|}{4(1 - p(\|x_0 - x^*\|))(1 - L_0\|x_0 - x^*\|)} \le \\
&\le g_2(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r, \qquad (2.20)
\end{aligned}$$

which shows (2.13) for $n = 0$ and $z_0 \in U(x^*, r)$.

Then, using (2.1), (2.5), (2.13), (2.16) and (2.19), we have that

$$\begin{aligned}
\|x_1 - x^*\| &\le \|z_0 - x^*\| + 2\|(3F'(y_0) - F'(x_0))^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(z_0)\| \le \\
&\le \|z_0 - x^*\| + \frac{M\|z_0 - x^*\|}{1 - p(\|x_0 - x^*\|)} = \\
&= [1 + \frac{M}{1 - p(\|x_0 - x^*\|)}]g_2(\|x_0 - x^*\|)\|x_0 - x^*\| = \\
&= g_3(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r, \qquad (2.21)
\end{aligned}$$

which shows (2.14) for $n = 0$ and $x_1 \in U(x^*, r)$. By simply replacing x_0, y_0, z_0, x_1 by x_k, y_k, z_k, x_{k+1} in the preceding estimates we arrive at estimates (2.12) – (2.14). Then, from the estimate $\|x_{k+1} - x^*\| < \|x_k - x^*\| < r$, we deduce that $\lim_{k\to\infty} x_k = x^*$ and $x_{k+1} \in U(x^*, r)$. To show the uniqueness part, let $Q = \int_0^1 F'(y^* + \theta(x^* - y^*)d\theta$ for some $y^* \in \bar{U}(x^*, T)$ with $F(y^*) = 0$. Using (2.8) we get that

$$\begin{aligned}
|F'(x^*)^{-1}(Q - F'(x^*))| &\le \int_0^1 L_0|y^* + \theta(x^* - y^*) - x^*|d\theta \le \\
&\le \int_0^1 L_0(1 - \theta)|x^* - y^*|d\theta \le \frac{L_0}{2}T < 1. \qquad (2.22)
\end{aligned}$$

It follows from (2.22) and the Banach Lemma on invertible functions that Q is invertible. Finally, from the identity $0 = F(x^*) - F(y^*) = Q(x^* - y^*)$, we deduce that $x^* = y^*$. ■

Similarly, we present the local convergence analysis of method (1.3). But first, we need to define the corresponding radius of convergence. Define functions $\bar{g}_3$ and $\bar{h}_3$ on the interval $[0, r_p)$ by

$$\bar{g}_3(t) = (1 + \frac{M(3L_0(1+g_1(t))t+4)^2}{16(1-p(t))^2(1-L_0 t)})g_2(t)$$

and

$$\bar{h}_3(t) = \bar{g}_3(t) - 1.$$

We have that $\bar{h}_3(0) = -1 < 0$ and $\bar{h}_3(t) \to +\infty$ as $t \to r_p^-$. Denote by r_2 the smallest zero of function h_2 in the interval $(0, r_p)$. Denote by $\bar{r}_3$ the smallest zero of function $\bar{h}_3$ in the interval $(0, r_p)$. Set

$$\bar{r} = \min\{r_1, \bar{r}_3\}. \tag{2.23}$$

Then, we have that

$$0 < \bar{r} < r_A \tag{2.24}$$

and for each $t \in [0, \bar{r})$

$$0 \leq g_1(t) < 1$$
$$0 \leq p(t) < 1$$
$$0 \leq g_2(t) < 1$$

(2.25)

and

$$0 \leq \bar{g}_3(t) < 1. \tag{2.26}$$

Then, we can present the local convergence analysis of method (1.3).

Theorem 2.2. *Let $F : D \subset X \to Y$ be a Fréchet-differentiable operator. Suppose that there exist $x^* \in D$, $L_0 > 0$, $L > 0$ and $M \in [1, 3)$ such that for each $x, y \in D$, (2.7)– (2.10) hold and*

$$\bar{U}(x^*, \bar{r}) \subseteq D, \tag{2.27}$$

where the radius $\bar{r}$ is given by (2.23). Then, the sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r) - \{x^*\}$ by method (1.3) is well defined , remains in $U(x^*, \bar{r})$ for each $n = 0, 1, 2.....$ and converges to x^*. Moreover, the following estimates hold

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\| < r, \tag{2.28}$$

$$\|z_n - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\| \tag{2.29}$$

and

$$\|x_{n+1} - x^*\| \leq \bar{g}_3(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|, \tag{2.30}$$

where the "g" functions are defined previously. Furthermore, if there exist $T \in [r, \frac{2}{L_0})$ *and* $\bar{U}(x^*, T) \in D$*, then the limit point* x^* *is the only solution of the equation* $F(x) = 0$ *in* $\bar{U}(x^*, T) \cap D$*.*

Proof. According to the proof of Theorem 2.1 we only need to show the last estimate (2.30). Using the alst sub-step of method (1.3), (2.8), (2.10), (2.13), (2.16), (2.19), (2.23) and (2.26) we get in turn that

$$\|x_1 - x^*\| \leq \|z_0 - x^*\| +$$

$$+ \frac{3\|F'(x^*)^{-1}(F'(y_0) - F'(x^*))\| + \|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\|^2 M\|z_0 - x^*\|}{16(1 - p(\|x_0 - x^*\|)^2)(1 - L_0\|x_0 - x^*\|)} =$$

$$= [1 + \frac{M(3L_0(1 + g_1(\|x_0 - x^*\|))\|x_0 - x^*\| + 4)^2}{16(1 - p(\|x_0 - x^*\|)^2)(1 - L_0\|x_0 - x^*\|)}]\|z_0 - x^*\| \leq$$

$$\leq \bar{g}_3(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < \bar{r},$$

which shows (2.30) for $n = 0$ and $x_1 \in U(x^*, r)$. ■

Remarks. 1. In view of (2.8) and the estimate

$$\begin{aligned} \|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\| \leq \\ &\leq 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq 1 + L_0\|x - x^*\| \end{aligned}$$

condition (2.10) can be dropped and be replaced by

$$M(t) = 1 + L_0 t,$$

or

$$M = M(t) = 2,$$

since $t \in [0, \frac{1}{L_0})$.

2. The results obtained here can be used for operators F satisfying autonomous differential equations [3, 6, 19] of the form

$$F'(x) = G(F(x))$$

where G is a continuous operator. Then, since $F'(x^*) = G(F(x^*)) = G(0)$, we can apply the results without actually knowing x^*. For example, let $F(x) = e^x - 1$. Then, we can choose: $G(x) = x + 1$.

3. The local results obtained here can be used for projection methods such as the Arnoldi's method, the generalized minimum residual method (GMRES), the generalized conjugate method(GCR) for combined Newton/finite projection methods and in connection to the mesh independence principle can be used to develop the cheapest and most efficient mesh refinement strategies [3, 4, 5, 6, 7].

4. The parameter $r_A = \frac{2}{2L_0+L}$ was shown by us to be the convergence radius of Newton's method [3, 6]

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \text{ for each } n = 0, 1, 2, \cdots \tag{2.31}$$

under the conditions (2.8)–(2.10). It follows from the definitions of radii r that the convergence radius r of these preceding methods cannot be larger than the convergence radius r_A of the second order Newton's method (2.31). As already noted in [3, 6] r_A is at least as large as the convergence ball given by Rheinboldt [26]

$$r_R = \frac{2}{3L}.$$

In particular, for $L_0 < L$ we have that

$$r_R < r_A$$

and

$$\frac{r_R}{r_A} \to \frac{1}{3} \ as \ \frac{L_0}{L} \to 0.$$

That is our convergence ball r_A is at most three times larger than Rheinboldt's. The same value for r_R was given by Traub [28].

5. It is worth noticing that the studied methods are not changing when we use the conditions of the preceding Theorems instead of the stronger conditions used in [13, 28]. Moreover, the preceding Theorems we can compute the computational order of convergence (COC) defined by

$$\xi = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) / \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$

This way we obtain in practice the order of convergence.

3. Numerical examples

The numerical examples are presented in this section.

Example 3.1. Let $X = Y = \mathbb{R}^3, D = \bar{U}(0,1), x^* = (0,0,0)^T$. Define function F on D for $w = (x,y,z)^T$ by

$$F(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

Then, the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that using the (2.8) conditions, we get $L_0 = e - 1, L = e, M = 2$. The parameters are

$$r_A = 0.3249,\ r_1 = 0.1293,\ r_p = 0.1356,\ r_2 = 0.0560,$$

$$r_3 = 0.0086 = r,\ \bar{r}_3 = 0.0392 = \bar{r}.$$

Example 3.2. Let $X = Y = C[0,1]$, the space of continuous functions defined on $[0,1]$ and be equipped with the max norm. Let $D = \overline{U}(0,1)$ and $B(x) = F''(x)$ for each $x \in D$. Define function F on D by

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{3.1}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi \in D.$$

Then, we get that $x^* = 0$, $L_0 = 7.5$, $L = 15$, $M = 2$. The parameters for method are

$$r_A = 0.0667,\ r_1 = 0.0296,\ r_p = 0.0305,\ r_2 = 0.0125,$$

$$r_3 = 0.0019 = r,\ \bar{r}_3 = 0.0088 = \bar{r}.$$

Example 3.3. Returning back to the motivational example at the introduction of this study, we have $L_0 = L = 146.6629073$, $M = 2$. The parameters are

$$r_A = 0.0045,\ r_1 = 0.0015,\ r_p = 0.0016,\ r_2 = 0.0007,$$

$$r_3 = 0.0001 = r,\ \bar{r}_3 = 0.0005 = \bar{r}.$$

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