THE FUNCTIONAL EQUATION $f(p+n^4+m^4)=g(p)+h(n^4)+h(m^4)$

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Abstract. The solutions f, g, h of the functional equation

$$f(p+n^4+m^4)=g(p)+h(n^4)+h(m^4)$$

are given under condition that every positive number of the form $32640k$ is the difference of two primes.

1. Introduction

This article is a continuation of [1], [2] and [3].

Let $\mathcal{P}, \mathbb{N}$ and $\mathbb{C}$ be the set of primes, positive integers and complex numbers, respectively. We are interested in solutions of those complex-valued functions f, g, h for which

$$f(p+Q(n)+Q(m))=g(p)+h(Q(n))+h(Q(m))$$

are satisfied for every $p \in \mathcal{P}, n, m \in \mathbb{N}$, where $Q(x) \in \mathbb{Z}[x],\ Q(x)>0$ for every $x \in \mathbb{N}$.

It was proved in [3] that if the functions f, g, h satisfy the above relation for $Q(x)=x^3$, then there exist $A, B, C \in \mathbb{C}$ such that

$$h(n^3)=An^3+B, \quad g(p)=Ap+C$$

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and

$$f(p+n^3+m^3)=A(p+n^3+m^3)+2B+C$$

for every $p\in\mathcal{P}, n,m\in\mathbb{N}$.

In this paper we shall investigate the case $Q(x)=x^4$.

Theorem 1. *Assume that the functions $f,g,h:\mathbb{N}\to\mathbb{C}$ satisfy the relation*

$$(1)\qquad f(p+n^4+m^4)=g(p)+h(n^4)+h(m^4)\quad \textit{for all}\quad p\in\mathcal{P}, n,m\in\mathbb{N}.$$

If every positive number of the form $32640n$ is the difference of two primes, then there are complex numbers A,B,C such that

$$h(n^4)=An^4+B,\quad g(p)=Ap+C$$

and

$$f(p+n^4+m^4)=A(p+n^4+m^4)+2B+C$$

holds for $p\in\mathcal{P}$ and $n,m\in\mathbb{N}$.

2. Lemmas

In the following assume that the arithmetical functions f,g,h satisfy (1). Let

$$S_n:=h(n^4),\quad A:=\frac{S_2-S_1}{15},\quad B:=S_1-A\quad\text{and}\quad C:=g(2)-2A.$$

Let

$$T(g):=\{p\in\mathcal{P}\ \mid\ g(p)=Ap+C\}$$

and

$$T(h):=\{n\in\mathbb{Z}\ \mid\ S_n=An^4+B\}$$

Lemma 1. *We have*

$$(2)\qquad n\in T(h)$$

for every $n\in\mathbb{N}, n\le 20$ and

$$(3)\qquad p\in T(g)$$

for every $p\in\mathcal{P}, p\le 73$.

Proof. We shall prove that (2) holds for every $n \le 20$ and (3) is satisfied for every $p \in \mathcal{B}$, where

$$\mathcal{B} = \{p \in \mathcal{P}, p \le 73\} \cup \{83, 97, 101, 103, 107, 109, 113, 127, 131, 151, 157, 163, 193, 229, 293, 353\}.$$

For numbers $a, b, c, d \in \mathbb{N}$ and $p, q \in \mathcal{P}$, we define I as follows:

$$I := \{(p, a, b, q, c, d) | \ \ p + a^4 + b^4 = q + c^4 + d^4\}.$$

It is obvious from (1) that

$$g(p) + S_a + S_b = g(q) + S_c + S_d \quad \text{if} \quad (p, a, b, q, c, d) \in I. \tag{4}$$

With the help of computer, we computed that the following 54 elements (p, a, b, q, c, d) are in I:

(2, 2, 3, 97, 1, 1), (2, 7, 7, 83, 5, 8), (2, 10, 13, 131, 2, 14), (3, 1, 3, 83, 1, 1), (3, 3, 3, 163, 1, 1), (3, 6, 6, 113, 3, 7), (3, 10, 13, 67, 3, 14), (5, 8, 9, 37, 5, 10), (5, 11, 11, 101, 5, 13), (7, 2, 2, 37, 1, 1), (11, 8, 9, 43, 5, 10), (13, 2, 2, 43, 1, 1), (13, 10, 13, 157, 1, 14), (13, 11, 11, 109, 5, 13), (17, 2, 2, 47, 1, 1), (17, 11, 11, 113, 5, 13), (19, 10, 13, 163, 1, 14), (23, 1, 3, 103, 1, 1), (23, 2, 2, 53, 1, 1), (29, 1, 3, 109, 1, 1), (29, 2, 2, 59, 1, 1), (29, 8, 9, 61, 5, 10), (31, 2, 2, 61, 1, 1), (31, 11, 11, 127, 5, 13), (37, 2, 2, 67, 1, 1), (41, 2, 2, 71, 1, 1), (43, 2, 2, 73, 1, 1), (47, 1, 3, 127, 1, 1), (67, 2, 2, 2, 2, 3), (67, 2, 2, 17, 1, 3), (67, 5, 5, 5, 2, 6), (67, 6, 6, 2, 4, 7), (67, 7, 8, 2, 1, 9), (67, 13, 13, 3, 9, 15), (71, 2, 2, 101, 1, 1), (71, 8, 9, 103, 5, 10), (73, 2, 2, 103, 1, 1), (83, 2, 2, 113, 1, 1), (107, 5, 13, 11, 11, 11), (107, 18, 17, 43, 13, 20), (127, 4, 4, 13, 1, 5), (131, 4, 4, 2, 2, 5), (131, 8, 9, 163, 5, 10), (131, 19, 5, 5, 16, 16), (151, 2, 2, 101, 1, 3), (193, 1, 3, 3, 2, 4), (193, 10, 11, 2, 8, 12), (229, 4, 12, 19, 9, 11), (229, 10, 12, 3, 7, 13), (229, 20, 12, 19, 15, 19) (293, 10, 12, 67, 7, 13), (293, 11, 15, 7, 2, 16), (353, 11, 15, 2, 3, 16), (353, 17, 7, 3, 12, 16).

Thus, from these values of I and from (4), we obtain the system of 54 equations with 57 unknowns, namely $S_n, n \in \mathbb{N}, n \le 20$ and $g(p)$ $(p \in \mathcal{B})$ are unknowns. We solve this linear system and with computer one can check that (2) and (3) are true.

Lemma 1 is proved. ■

Lemma 2. *We have*

$$\{p_1 = 2, \ldots, p_{620} = 4583\} \subseteq T(g),$$

where p_i is the i-th prime number.

Proof. First we note from Lemma 1 that $p_i \in T(g)$ for every $i \leq 21$.

One can check that the following elements belong to I:

(5)
$$\begin{cases} (3,30,6,17,23,27), & 30^4+6^4-23^4-27^4=17-3=14, \\ (3,86,34,19,50,84), & 86^4+34^4-50^4-84^4=19-3=16, \\ (7,2,2,37,1,1), & 2^4+2^4-1^4-1^4=37-7=30, \\ (5,8,9,37,5,10), & 8^4+9^4-5^4-10^4=37-5=32, \\ (3,49,5,37,26,48,), & 49^4+5^4-26^4-48^4=37-3=34, \\ (17,1,3,67,2,2), & 1^4+3^4-2^4-2^4=67-17=50, \\ (5,2,6,67,5,5), & 2^4+6^4-5^4-5^4=67-5=62, \\ (3,10,13,67,3,14), & 10^4+13^4-3^4-14^4=67-3=64, \\ (3,1,3,83,1,1), & 1^4+3^4-1^4-1^4=83-3=80 \\ (5,58,61,101,52,65), & 58^4+61^4-52^4-65^4=101-5=96, \\ (5,106,146,131,155,43), & 106^4+146^4-155^4-43^4=131-5=126, \\ (7,71,47,137,58,66), & 71^4+47^4-58^4-66^4=137-7=130, \\ (5,35,19,149,29,31), & 35^4+19^4-29^4-31^4=149-5=144, \\ (3,114,134,193,99,141), & 114^4+134^4-99^4-141^4=193-3=190. \end{cases}$$

Let $\mathcal{U} := \{14,16,30,32,34,50,62,64,80,96,126,130,144,190\}$. It follows easily from (4), (5) and Lemma 1 that

$$g(p) = g(q) + (p-q)A, \quad \text{if} \quad p,q \in \mathcal{P} \quad \text{and} \quad p-q \in \mathcal{U},$$

consequently

$$p \in T(g), \quad \text{if} \quad q \in T(g) \quad \text{and} \quad p-q \in \mathcal{U}. \tag{6}$$

Assume that $p_j \in T(g)$ for all $j < i$, where $21 < i \leq 620$. If $p_i - p_j \in \mathcal{U}$ for some $j < i$, then we shall write $i \in \mathcal{T}$. It is obvious from (6) that if $i \in \mathcal{T}$, then $p_i \in T(g)$.

With the help of computer, among $p_i, i \leq 620$ only $526 \notin \mathcal{T}$. We prove that $p_{526} = 3779 \in T(g)$. Indeed, since $p_{527} - p_{516} = 3793 - 3697 = 96 \in \mathcal{U}$ and $p_{527} - p_{526} = 3793 - 3779 = 14 \in \mathcal{U}$, we have $p_{526} = 3779 \in T(g)$.

Lemma 2 is proved. ∎

Lemma 3. *We have*

$$\{1,2,\cdots,256\} \subseteq T(h). \tag{7}$$

Proof. Assume that $a \in \{1,2,\cdots,256\}, a > 21$ and $n \in T(h)$ holds for every $n < a$. We shall prove that $a \in T(h)$.

We consider the following 11 equations

$$(8)\qquad |a^4+x^4-y^4-z^4| = p_i-2,\quad 0\le x,y,z\le a-1, 1\le i\le 620,$$

$$(9)\qquad |a^4+x^4-y^4-z^4| = p_i-3,\quad 0\le x,y,z\le a-1, 1\le i\le 620,$$

$$(10)\qquad |a^4+x^4-y^4-z^4| = p_i-5,\quad 0\le x,y,z\le a-1, 1\le i\le 620,$$

$$(11)\qquad |a^4+x^4-y^4-z^4| = p_i-7,\quad 0\le x,y,z\le a-1, 1\le i\le 620,$$

$$(12)\qquad |a^4+x^4-y^4-z^4| = p_i-11,\quad 0\le x,y,z\le a-1, 1\le i\le 620,$$

$$(13)\qquad |a^4+x^4-y^4-z^4| = p_i-13,\quad 0\le x,y,z\le a-1, 1\le i\le 620,$$

$$(14)\qquad |a^4+x^4-y^4-z^4| = p_i-17,\quad 0\le x,y,z\le a-1, 1\le i\le 620,$$

$$(15)\qquad |a^4+x^4-y^4-z^4| = p_i-19,\quad 0\le x,y,z\le a-1, 1\le i\le 620,$$

$$(16)\qquad |a^4+x^4-y^4-z^4| = p_i-23,\quad 0\le x,y,z\le a-1, 1\le i\le 620,$$

$$(17)\qquad |a^4+x^4-y^4-z^4| = p_i-29,\quad 0\le x,y,z\le a-1, 1\le i\le 620,$$

$$(18)\qquad |a^4+x^4-y^4-z^4| = p_i-31,\quad 0\le x,y,z\le a-1, 1\le i\le 620.$$

It is clear that if one of (8)-(18) is soluble, then from Lemma 1 and Lemma 2 we have $a\in T(h)$. With the help of computer, we obtain that one of (8)–(18) is soluble, except if $a = 62, 186, 205, 232, 238, 254$.

Arguing similarly as in the proof of Lemma 2, we can prove that these values also belong to $T(h)$.

It is clear to see that if $(p,a,b,q,c,d)\in I$, $p,q\in\mathcal{P}$, $p,q\le p_{620}=4583$ and three elements of a,b,c,d belong $T(h)$, then the fourth element also belongs to $T(h)$.

If $a = 62$, then $(3,63,13,563,53,53)\in I$ and $(487,34,63,7,39,62)\in I$ imply that $62\in T(h)$.

If $a = 186$, then $(7,102,184,373,187,75)\in I$ and $(3,72,186,2593,31,187)\in$ $\in I$ imply that $186\in T(h)$.

If $a=205$, then $(2,39,209,1987,136,199)$ and $(5,110,205,3413,46,209)\in$ $\in I$ imply that $205\in T(h)$.

If $a=232$, then $(23,234,63,4423,168,217)$ and $(13,110,232,1277,82,234)\in$ $\in I$ imply that $232\in T(h)$.

If $a=238$, then $(11,242,108,1801,169,229)$ and $(7,242,107,3607,137,238)\in$ $\in I$ imply that $238\in T(h)$.

Finally, if $a=254$, then we infer from

$$(23,255,117,3847,199,231),\quad (2,254,203,4513,201,255)\in I$$

that $254\in T(h)$.

Lemma 3 is proved. ■

Lemma 4. *If*

$$32640\mathbb{Z}\subseteq \mathcal{P}-\mathcal{P}, \tag{19}$$

then

$$\begin{aligned} &S_{16n+m}-S_{16n-m}-S_{8n+8m}+S_{8n-8m}=\\ &=S_{2n+2m}-S_{2n-2m}-S_{n+16m}+S_{n-16m}\end{aligned} \tag{20}$$

holds for all $n,m\in\mathbb{N}$.

Proof. It is easy to check that

$$(16n+m)^4-(16n-m)^4-(8n+8m)^4+(8n-8m)^4=-32640nm^3$$

and

$$(2n+2m)^4-(2n-2m)^4-(n+16m)^4+(n-16m)^4=-32640nm^3$$

which, using (19), there are primes p,q such that $-32640nm^3=p-q$, consequently

$$S_{16n+m}-S_{16n-m}-S_{8n+8m}+S_{8n-8m}=g(p)-g(q)$$

and

$$S_{2n+2m}-S_{2n-2m}-S_{n+16m}+S_{n-16m}=g(p)-g(q).$$

These imply (20).

Lemma 4 is proved. ■

3. Proof of the Theorem 1.

Assume that $n \in T(h)$ for every $n \in \mathbb{Z}, |n| < N$. From Lemma 3, we have $N > 256$. Let $N = 16n + m$, where $n > 16$ and $m \in \{1, 2, \cdots, 16\}$. We get from (20) that

$$\begin{aligned} S_N &= S_{16n+m} = \\ &= S_{16n-m} + S_{8n+8m} - S_{8n-8m} + S_{2n+2m} - S_{2n-2m} - S_{n+16m} + S_{n-16m} = \\ &= A\Big((16n-m)^4 + (8n+8m)^4 - (8n-8m)^4 + (2n+2m)^4 - (2n-2m)^4 - \\ &- (n+16m)^4 + (n-16m)^4\Big) + B = A(16n+m)^4 + B = AN^4 + B. \end{aligned}$$

Thus we proved that $n \in T(h)$ for every $n \in \mathbb{N}$.

Now we prove $T(g) = \mathcal{P}$.

We check easily that

$$(2n-1)^4 + (n+8)^4 - (2n+1)^4 - (n-8)^4 = 4080n \tag{21}$$

and

$$(3n-1)^4 + (n+27)^4 - (3n+1)^4 - (n-27)^4 = 157440n. \tag{22}$$

Let $U := 4080 = 2^4 \cdot 3 \cdot 5 \cdot 17$ and $V := 157440 = 2^8 \cdot 3 \cdot 5 \cdot 41$. It is obvious that $(U, V) = 240$ and these relations with the fact $T(h) = \mathbb{N}$ imply that

$$g(p) - g(q) = A(p-q) \quad \text{if} \quad p, q \in \mathcal{P}, p \equiv q \pmod{U} \tag{23}$$

and

$$g(r) - g(\pi) = A(r-\pi) \quad \text{if} \quad r, \pi \in \mathcal{P}, r \equiv \pi \pmod{V}. \tag{24}$$

Indeed, if $p, q \in \mathcal{P}, p > q, p \equiv q \pmod{U}$, then $p - q = Un$ for some $n \in \mathbb{N}$. Then from (21) we have

$$(2n-1)^4 + (n+8)^4 - (2n+1)^4 - (n-8)^4 = p - q,$$

consequently

$$\begin{aligned} g(p) - g(q) &= S_{2n-1} + S_{n+8} - S_{2n+1} - S_{n-8} = \\ &= A\Big((2n-1)^4 + (n+8)^4 - (2n+1)^4 - (n-8)^4\Big) = A(p-q). \end{aligned}$$

Thus, (23) is proved. The proof of (24) is similar as the proof of (23).

Let $\mathfrak{M}$ denote the reduced residue system modulo 240 for which every element of $\mathfrak{M}$ is a prime number and coprime to V, i.e.

$$\mathfrak{M} := \{241, 7, 11, 13, 17, 19, 23, 29, 31, 37, 281, 43, 47, 769, 53, 59, 61, 67, 71, 73, 317, 79, 83, 89, 331, 97, 101, 103, 107, 109, 113, 359, 601, 127, 131, 373, 137, 139, 383, 149, 151, 157, 401, 163, 167, 409, 173, 179, 181, 907, 191, 193, 197, 199, 443, 449, 211, 457, 461, 223, 227, 229, 233, 239\}.$$

It follows from Lemma 2 that $\mathfrak{M} \subseteq T(g)$.

Let $p \in \mathcal{P}$ and $p > p_{620} = 4583$. We shall prove that $p \in T(g)$.

First, we note that there is a prime $r \in \mathfrak{M}$ such that $p \equiv r \pmod{240}$. Since $(U, V) = 240$, therefore there exists a $n_1 \in \mathbb{N}$ such that

$$Un_1 + p \equiv r \pmod{V}. \tag{25}$$

We infer from (25), by using the fact $p > 4583$ and from definitions of $U, V, \mathfrak{M}$ that

$$(UV, Un_1 + p) = (V, Un_1 + p) = (V, r) = 1.$$

From Dirichlet's theorem on arithmetic progressions, there is a prime $P = UVn_2 + Un_1 + p$ for some $n_2 \in \mathbb{N}$. Since

$$P \equiv r \pmod{V} \quad \text{and} \quad P \equiv p \pmod{U},$$

we get from (23) and (24) that

$$g(P) - g(r) = A(P - r) \quad \text{and} \quad g(P) - g(p) = A(P - p).$$

This shows that $g(p) = Ap - Ar + g(r) = Ap + C$, and so the proof of the theorem is finished.

The theorem is proved. ■

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