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ON QUASI NIL-INJECTIVE MODULES

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Abstract. Module M is called *quasi nil-injective* if for each $m \in Nil(M)$ and each homomorphism $f : mR \to M$, there exists a homomorphism $\bar{f} : M \to M$ such that $\bar{f}(x) = f(x)$ for every $x \in mR$. In this paper, we first obtain some characterizations of the class of quasi nil-injective modules and some known results can be deduced from these characteristics. Next, we apply to ring and obtain some properties of a quasi nil-injective rings. We proved that a ring R is semiprime if only if every right R-module (cyclic) is quasi nil-R-injective.

1. Introduction

Throughout the paper, R is an associative ring with identity $1 \neq 0$ and all modules are unitary R-modules. We write M_R (resp., $_RM$) to indicate that M is a right (resp., left) R-module. Let J (resp., Z_r, S_r) be the Jacobson radical (resp. the right singular ideal, the right socle) of R and $E(M_R)$ the injective hull of M_R. If X is a subset of R, the right (resp. left) annihilator of X in

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R is denoted by $r_R(X)$ (resp., $l_R(X)$) or simply $r(X)$ (resp. $l(X)$). If N is a submodule of M (resp., proper submodule) we write $N \leq M$ (resp., $N < M$). Moreover, we write $N \leq^e M$, $N \ll M$, $N \leq^{\oplus} M$ and $N \leq^{max} M$ to indicate that N is an essential submodule, a small submodule, a direct summand and a maximal submodule of M, respectively. A module M is called uniform if $M \neq 0$ and every non-zero submodule of M is essential in M. A module M is *finite dimensional* (or has *finite rank*) if $E(M)$ is a finite direct sum of indecomposable submodules. A right R-module N is called M-generated if there exists an epimorphism $M^{(I)} \to N$ for some index set I. If the set I is finite, then N is called finitely M-generated. In particular, N is called M-cyclic if it is isomorphic to M/L for some submodule L of M. Hence, any M-cyclic submodule X of M can be considered as the image of an endomorphism of M.

It is well-known that a right R-module Q is called *injective* if for every monomorphism $i : A \to B$, with A, B right R-modules and every R-homomorphism $f : A \to Q$, there exists an R-homomorphism $\bar{f} : B \to Q$ such that $\bar{f}i = f$.

In 1940, Baer has launched an important criterion to test the injectivity of the modules as follows: A right R-module Q is injective if and only if for every homomorphism $i : I \to R_R$ with I a right ideal of R and every homomorphism $f : I \to Q$, there exists a homomorphism $\bar{f} : B \to Q$ such that $\bar{f}i = f$.

From the definition of injectivity and Baer's criterion, two of the extended development of injectivity respectively co-exist. The first is to expand the original definition. A right R-module Q is called C-injective (by [4]) (strong socle-injective (by [2]), resp., if for every monomorphism $i : A \to B$, with every cyclic right R-module A (the socle of B, resp.,); every right R-module B and every R-homomorphism $f : A \to Q$, there exists an R-homomorphism $f : B \to Q$ such that $fi = f$. In this paper, we continue to review the module A in the above chart is just mR with $m \in Nil(M)$, the definition used by the product of the submodules. According to [5], a module M is called *principally quasi-injective* if for every $m \in M$ and every homomorphism $f : mR \to M$, there exists a homomorphism $\bar{f} : M \to M$ such that $\bar{f}(x) = f(x)$ for every $x \in mR$. Some of the results and the relationship between the principally quasi-injective modules and its endomorphism ring has been studied.

According to [5], a module M is called *quasi mininjective* if for every simple submodule N of M and every homomorphism $f : N \to M$, there exists a homomorphism $\bar{f} : M \to M$ such that $\bar{f}(x) = f(x)$ for every $x \in N$. Clearly have

$$\text{principally quasi-injective} \;\Rightarrow\; \text{quasi mininjective.}$$

Besides, the second development of injectivity is interested by many authors. In [6], Nicholson-Yousif has launched a concept called *P-injective* of the modules M if for every $a \in R$ and every homomorphism $f : aR \to M$, there exists a homomorphism $\bar{f} : R_R \to M$ such that $\bar{f}(x) = f(x)$ for every $x \in aR$. They brought out many interesting features of the ring such that R_R is P-injective. In addition, a general case of P-injective modules were also studied and expanded, such as GP-injective modules, AGP-injective modules, ect.

In 2007, Wei and Chen ([7]) have given some general cases of P-injective, named nil-injective. A module M is called *nil-injective* if each nilpotent element $a \in R$ and homomorphism $f : aR \to M$, there exists a homomorphism $\bar{f} : R_R \to M$ such that $\bar{f}(x) = f(x)$ for every $x \in aR$. Naturally, we introduce the concept of *quasi nil-injective module*. In this paper, we study the characterizations of the class of such modules and we also obtain some results on the relationship between quasi nil-injective, nil injective and others. And we extend this concept into right quasi nil-injective rings, then we prove some their properties. One of the characterization of a semiprime ring that is every right R-module (cyclic) is quasi nil-R-injective.

2. Some properties of quasi nil-injective modules

Let M be a right R-module, $S := End_R(M)$ and H, K be submodules of M.

Then, in [3], Lomp defined

$$H \star K := \sum \{f(K)|\ f \in Hom(M, H)\}.$$

Definition 2.1. Let H, K be submodules of M. Then $H \star K$ is called the product of two submodules of H and K and is denoted by HK.

From the definition above we have the following comments:

Remark. (i). If $M = R$, the product of two ideals of R, is the ideal products in the common sense; i.e., if I, K are ideals of R,

$$IK = \{\sum_{i \leq k} a_i b_i|\ a_i \in I, b_i \in K, k \in \mathbb{N}^*\}.$$

(ii) $HK \leq H$ for every $K \leq M$. Moreover, if K is a fully invariant submodule (i.e., carried into itselt by every endomorphism of M), we have $HK \leq K$ for every $H \leq M$.

First we have the following properties:

Lemma 2.2. *Let H, K, L be submodules of M. Then*

(1) $H(KL) \leq (HK)L$.

(2) $L(H+K) = LH + LK$.

(3) $LK + HK \leq (L+H)K$.

(4) *If M is projective in $\sigma[M]$,then (1) and (3) will be the equations.*

Proof. By [3, Proposition 3.1]. ■

Let N be a submodule of M and $n \in \mathbb{N}$. We define a family of submodules of N as follow:

$$N^1 = N, N^2 = NN, N^3 = N^2N, \ldots, N^n = N^{n-1}N.$$

Then we have

$$N^n \leq N^{n-1} \leq \cdots \leq N^2 \leq N^1 = N.$$

The submodule N of M is called *nilpotent* if there exists $n \in \mathbb{N}$ such that $N^n = 0$. We denote

$$Nil(M) = \{m \in M \mid mR \text{ is nilpotent } \}.$$

It is clear to check the following lemma:

Lemma 2.3. *Let A, B be submodules of M. If $A \leq B$, then $A^n \leq B^n$ for all n.*

Definition 2.4. A module M is called *quasi nil-injective* if each $m \in Nil(M)$ and each homomorphism $f : mR \to M$, there exists a homomorphism $\bar{f} : M \to M$ such that $\bar{f}(x) = f(x)$ for every $x \in mR$, i.e., the following diagram is commutative:

$$\begin{tikzcd}
 & M & \\
0 \arrow[r] & mR \arrow[u, "f"] \arrow[r, "i"] & M \arrow[ul, dotted, "\bar{f}"']
\end{tikzcd}$$

where $i : mR \to M$ is the inclusion map.

Theorem 2.5. *The following conditions are equivalent for module M with $S = End(M)$:*

(1) *M is quasi nil-injective.*

(2) $l_M(r(m)) = Sm$ *for every* $m \in Nil(M)$.

(3) *If* $r(m) \leq r(m')$ *for every* $m \in Nil(M)$ *and* $m' \in M$, *then* $Sm' \leq Sm$.

Proof. (1) $\Rightarrow$ (2). Let $m \in Nil(M)$ and $x \in l_M(r(m))$. The map $f : mR \to M$ is defined by $f(mr) = xr$ for every $r \in R$. Then f is an R-homomorphism. By (1), there exists an R-homomorphism $\bar{f} : M \to M$ such that $\bar{f}(y) = f(y)$ for every $y \in mR$. It implies $x = f(m) = \bar{f}(m) \in Sm$ and $l_M(r(m)) = Sm$.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1). For each $m \in Nil(M)$ and each $f : mR \to M$. We have $r(m) \leq r(f(m))$. Hence, there exists $\bar{f} \in S$ such that $f(m) = \bar{f}(m)$. Then M is quasi nil-injective. ■

Next we have a another property of the quasi nil-injective modules:

Proposition 2.6. *If* M *is a quasi nil-injective module, then* $l_S(Ker(\alpha) \cap \cap mR) = S\alpha + l_S(m)$ *for every* $m \in M$ *and* $\alpha \in S$ *with* $\alpha(m) \in Nil(M)$.

Proof. For each $m \in M$, $\alpha \in S$ and $\alpha(m) \in Nil(M)$, we have $S\alpha + l_S(m) \leq \leq l_S(Ker(\alpha) \cap mR)$. In the other hand, for each $s \in l_S(Ker(\alpha) \cap mR)$, $s(Ker(\alpha) \cap mR)) = 0$. Moreover, we have $\alpha(m) \in Nil(M)$ and $r(\alpha(m)) \leq \leq r(s(m))$. By the Theorem 2.5, there exists $s' \in S$ such that $s(m) = s'\alpha(m)$ or $s - s'\alpha \in l_S(m)$ and then $s \in S\alpha + l_S(m)$. Thus $l_S(Ker(\alpha) \cap mR) = S\alpha + l_S(m)$. ■

Proposition 2.7. *Every direct summand of a quasi nil-injective module is a quasi nil-injective module.*

Proof. Assume that M is a quasi nil-injective module and N is a direct summand of M. Let $\iota : N \to M$ be the inclusion map and $p : M \to N$ be the projection. Let $n \in Nil(N)$ and $f : nR \to N$ be homomorphism. There exists $k \in \mathbb{N}$ such that $(nR)^k = 0$. We have $(nR)^k = \sum\{f(nR) |\ f \in \in Hom(N, (nR)^{k-1})\} = 0$. On the other hand, for every $g \in Hom(M, (nR)^{k-1})$, then $g(nR) = g\iota(nR) \leq \sum\{f(nR) |\ f \in Hom(N, (nR)^{k-1})\} = (nR)^k$, which implies

$$\sum\{f(nR) |\ f \in Hom(M, (nR)^{k-1})\} = 0.$$

Therefore $n \in Nil(M)$. Since M is a quasi nil-injective module, there exists a homomorphism $\bar{f} \in End(M)$ such that $\bar{f}(x) = f(x)$ for every $x \in nR$. We have $p\bar{f}\iota \in End(N)$ and $p\bar{f}\iota(x) = f(x)$ for every $x \in nR$. Thus N is a quasi nil-injective module. ■

Lemma 2.8. *Assume that* $\phi : N \to M$ *is an isomorphism and* $A, B \leq N$. *Then*

$$\phi(AB) = \phi(A)\phi(B) \text{ and } \phi(A^k) = \phi(A)^k.$$

Proof. By the definition of the product AB we have $AB = \sum\{f(B)|\ f \in Hom(N, A)\}$ and $\phi(A)\phi(B) = \sum\{g(\phi(B))|\ g \in Hom(M, \phi(A))\}$. Then $\phi(AB) = \sum\{\phi(f(B))|\ f \in Hom(N, A)\}$. Next, let $f \in Hom(N, A)$ and $g = \phi|_A f \phi^{-1}$. Then we have $g \in Hom(M, \phi(A))$ and $g(\phi(B)) = \phi|_A f \phi^{-1}(\phi(B)) = \phi|_A f(B) = \phi(f(B))$. This implies $\phi(AB) \leq \phi(A)\phi(B)$. On the other hand, for each $g \in Hom(M, \phi(A))$, set $f = \phi^{-1}|_{\phi(A)} g\phi$, we have $f \in Hom(N, A)$. So $\phi(f(B)) = \phi\phi^{-1}|_{\phi(A)} g\phi(B) = g\phi(B)$ and hence $g\phi(B) \leq \phi(A)\phi(B)$. It implies that

$$\phi(A)\phi(B) \leq \phi(AB).$$

Thus $\phi(AB) = \phi(A)\phi(B)$.

Moreover, we have

$$\begin{aligned}
\phi(A^2) &= \phi(A)\phi(A) = \phi(A)^2 \\
\phi(A^3) &= \phi(A^2A) = \phi(A^2)\phi(A) = \phi(A)^2\phi(A) = \phi(A)^3 \\
&\vdots \\
\phi(A^k) &= \phi(A)^k.
\end{aligned}$$

■

Use the above lemma we have:

Proposition 2.9. *Every module, which is isomorphic to a quasi nil-injective module, is quasi nil-injective.*

Proof. Let M be a quasi nil-injective module, N be a right R-module and $\phi : N \to M$ be an isomorphism. Assume that $n \in Nil(N)$ and $f : nR \to N$ is a homomorphism. There exists $k \in \mathbb{N}$ such that $(nR)^k = 0$. By the Lemma 2.8 we have $\phi(nR)^k = \phi((nR)^k) = 0$. It implies that $(\phi(n)R)^k = 0$ or $\phi(n) \in Nil(M)$. Since M is quasi nil-injective, there exists a homomorphism $g \in End(M)$ such that g is an extension of the homomorphism $\phi f(\phi^{-1}|_{\phi(n)R})$. Let $\bar{f} = \phi^{-1} g\phi \in End(N)$. For every $x \in R$ we have $\bar{f}(nx) = \phi^{-1} g\phi(nx) = \phi^{-1}(\phi f(\phi^{-1}|_{\phi(n)R}))(\phi(nx)) = f(nx)$. Thus $\bar{f}$ is an extension of f. Hence N is a quasi nil-injective module. ■

It is well-known that a minimal right ideal I of R is either a direct summand of R or $I^2 = 0$. The following theorem gives us the same result as in the ring for the module.

Theorem 2.10. *Let N be a simple submodule of M. Then N is either a direct summand of M or $N^2 = 0$.*

Proof. Let N be a simple submodule of M. Assume that $N^2 \neq 0$, so $\sum\{f(N)|\ f \in Hom(M, N)\} \neq 0$. Then there exists a homomorphism

$f : M \to N$ such that $f(N) \neq 0$. Since N is simple, $f(N) = N$. We have $N = f(N) = f(M)$ and $M = N + Kerf$. On the other hand, we have $N \cap Kerf = Ker(f|_N)$ and $f|_N : N \to N$ be a isomorphism (because N is simple). Thus $N \cap Kerf = 0$ and so $M = N \oplus Kerf$. ■

Apply the theorem above we have the following result:

Corollary 2.11. *Every quasi nil-injective module is quasi mininjective.*

Proof. Let M be a quasi nil-injective module and $f : mR \to M$ be a homomorphism with mR be a simple submodule of M. By Theorem 2.10, mR is either a direct summand of M or $(mR)^2 = 0$. If mR is a direct summand of M, then $f\pi : M \to M$ (with $\pi : M \to mR$ the canonical projection) is an extension of f. If $(mR)^2 = 0$, we have $m \in Nil(M)$. By hypothesis f can be extended to a homomorphism $M \to M$. ■

Assume that N is a simple submodule of M. The notation

$$Soc_N(M) = \sum\{X \leq M |\ X \simeq N\}$$

is called homogeneous components of $Soc(M)$ contains N.

Proposition 2.12. *Assume that M is a quasi nil-injective module and $S = End(M)$. Then:*

(1) *If N is a simple submodule of M, then $Soc_N(M) = SN$.*

(2) *If mR is a simple submodule of M_R, then Sm is a simple submodule of ${}_SM$.*

(3) *$Soc(M_R) \leq Soc({}_SM)$.*

Proof. (1). We always have $SN \leq Soc_N(M)$. Assume that $f : N \to N_1$ is an isomorphism with $N_1 \leq M$. By Corollary 2.11, M is a quasi mininjective module, there exists a homomorphism $\bar{f} : M \to M$ which is an extension of f. So $N_1 = f(N) = \bar{f}(N) \leq SN$ and we have $Soc_N(M) \leq SN$.

(2). Assume that mR is a simple submodule of M_R and $0 \neq \alpha(m) \in Sm$ for $\alpha \in S$. Then $\alpha : mR \to \alpha(m)R$ is an isomorphism. It follows that $\alpha(m)R$ is a simple submodule of M. Since M is quasi nil-injective, M is quasi mininjective, there exists a homomorphism $\bar{\alpha} : M \to M$ is an extension of $\alpha^{-1} : \alpha(m)R \to mR$. Then $m = \alpha^{-1}(\alpha(m)) = \bar{\alpha}(\alpha(m)) \in S\alpha m$. Thus $Sm = S\alpha m$ or Sm is a simple submodule of ${}_SM$.

(3) is deduced from (2). ■

Corollary 2.13 ([5, Proposition 1.3])**.** *Assume that M is a principally quasi-injective module and $S = End(M)$. Then:*

(1) *If N is a simple submodule of M, then $Soc_N(M) = SN$.*

(2) *If mR is a simple submodule of M_R, then Sm is a simple submodule of ${}_SM$.*

(3) $Soc(M_R) \leq Soc({}_SM)$.

3. Quasi nil-injective rings

Let R be a ring, we denote

$$Nil(R_R) = \{x \in R | (xR)^n = 0\}.$$

In general,

$$Nil(R_R) \neq Nil({}_RR),$$

but if R is a communative ring then denote

$$Nil(R) = Nil(R_R) = Nil({}_RR).$$

We first note that $Nil(R)$ is different from $N(R)$ (the set of all nilpotent elements of R) by the following example:

Example 3.1. Let $R = M_n(F)$, F be a field. Since $J(R) = 0$, $Nil(R) = 0$. However, $N(R) \neq 0$ because it contains the triangular matrix with the zero elements on the main diagonal.

Definition 3.2. A ring R is called *right quasi nil-injective* if R_R is quasi nil-injective, i.e., if each $a \in Nil(R_R)$ and each homomorphism $f : aR \to R$, there exists a homomorphism $\bar{f} : R \to R$ such that $\bar{f}(x) = f(x)$ for every $x \in aR$.

The definition of a nil-injective ring was introduced by Wei and Chen ([7]) and we have:

$$\text{nil-injective} \;\Rightarrow\; \text{quasi nil-injective.}$$

Note that every ring that has zero Jacobson radical is a quasi nil-injective ring. We have the characterizations of quasi nil-injective ring:

Theorem 3.3. *The following conditions are equivalent for a ring R:*

(1) *R is a right quasi nil-injective ring.*

(2) $l(r(a)) = Ra$ *for every* $a \in Nil(R_R)$.

(3) *If* $r(a) \le r(b)$ *for each* $a \in Nil(R_R)$, $b \in M$, *then* $Rb \le Ra$.

(4) $l(r(a) \cap bR) = Ra + l(b)$ *for every* $a, b \in R$ *with* $ab \in Nil(R_R)$.

(5) $l(r(a) \cap bR) = Ra + l(b)$ *for every* $a, b \in R$ *with* $a \in Nil(R_R)$.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) by Theorem 2.5.

(3) $\Rightarrow$ (4) by Proposition 2.6.

(4) $\Rightarrow$ (5). Let $a, b \in R$ with $a \in Nil(R_R)$. Then $ab \in Nil(R_R)$ by Lemma 2.3.

(5) $\Rightarrow$ (1). Let $a \in Nil(R_R)$ and $f : aR \to R_R$ be a homomorphism. Then $r(a) \le r(f(a))$. By (4) we have $f(a) \in lr(f(a)) \le lr(a) = l(r(a) \cap 1.R) = Ra$. It follows that f can be extended to R_R. ■

Corollary 3.4. *Let* $R = \prod_{i\in I} R_i$ *be a direct product of rings. Then* R *is a right quasi nil-injective ring if and only if* R_i *is right quasi nil-injective for all* $i \in I$.

Proof. By Theorem 3.3, $l(r(a_i)) = R_i a_i$, for all $a_i \in Nil((R_i)_{R_i}), i \in I$. For each $b_i \in Nil(R_R), b_i = (0, 0, ..., a_i, 0, 0, ...0), i \in I$ we have $Rb_i = (\prod_{i\in I} R_i)b_i = = \prod_{i\in I}(R_i b_i) = \prod_{i\in I}(R_i a_i) = \prod_{i\in I} l(r(a_i)) = l(r(\prod_{i\in I} a_i)) = l(r(b_i))$. So we are done. ■

Theorem 3.5. *Let* R *be a right quasi nil-injective ring. If* $ReR = R$ *where* $e^2 = e \in R$ *then* eRe *is a right quasi nil-injective ring.*

Proof. Let $a \in Nil(S)$, where $S = eRe$, then $a \in Nil(R_R)$ and so $l_R r_R(a) = = Ra$ by Theorem 3.3. We will show that $Sa = l_S r_S(a)$. In fact, let $x \in \in l_S r_S(a)$. then $r_R(x) \le r_S(x) \le r_S(a)$. Now let $y \in r_R(x)$ then $xy = 0$. Write $1 = \sum_{i=1}{}^{n} u_i e v_i, u_i, v_i \in R$. Clearly $ayu_i e = aeyu_i e = 0$ for all i, because $r_S(x) \le r_S(a)$. Therefore $ay = ay1 = \sum_{i=1}^{n} ayu_i ev_i = 0$, so $y \in r_R(a)$. This implies that $r_R(x) \le r_R(a)$, so $x \in l_R r_R(x) \le l_R r_R(a) = Ra$. Therefore $x = ex \in eRa = eRea = Sa$, which so that $l_S r_S(a) \le Sa$. Hence $Sa = l_S r_S(a)$ and so $eRe = S$ is a right quasi nil-injective ring. ■

Call a ring R a left minannihilator ring [9] if every minimal left ideal K of R is an annihilator, equivalent if $lr(K) = K$.

Corollary 3.6. *Every right quasi nil-injective ring is left minannihilator.*

Proof. Let R be a right quasi nil-injective ring. Assume that Rk is a minimal left ideal of R. If$(Rk)^2 \ne 0$ then $Rk = Re, e^2 = e \in R$. So $lr(Rk) = lr(k) = = lr(Re) = lr(e) = Re = Rk$. If $(Rk)^2 = 0$ then $(kR)^3 = 0$ and so $k \in Nil(R_R)$. Then $lr(Rk) = lr(k) = Rk$. This implies that R is a left minannihilator ring. ■

Corollary 3.7. *Every right quasi nil-injective ring is right mininjective.*

Proof. Let R be a right quasi nil-injective ring. To prove that R is a right mininjective ring, we need to show that every minimal right ideal kR of R, $Rk = lr(k)$. Now, assume that kR is any minimal right ideal of R. If $(kR)^2 = 0$, then $k \in Nil(R_R)$. By hypothesis and Theorem 3.2, $Rk = lr(k)$; we are done. If $(kR)^2 \neq 0$, then $kR = eR$, $e^2 = e \in R$. Write $e = kc, c \in R$. Then $k = ek = kck$. Set $g = ck$. Then $g^2 = g, k = kg$ and $Rk = Rg$. Hence $r(k) = r(g)$ and so $Rk = Rg = lr(g) = lr(k)$; we are also done. Therefore R is a right mininjective ring. ■

Remark 3.8. If R is not a right quasi nil-injective ring then the polynomial ring $R[x]$ is not quasi nil-injective. Indeed, by hypothesis, there exists $0 \neq$ $\neq a \in Nil(R_R)$ such that $l_R r_R(a) \neq Ra$ and $(aR)^n = 0$, so $[a(R[x])]^n = 0$ and $a \in Nil(R[x]_{R[x]})$. Hence $l_{R[x]}r_{R[x]}(a) = (l_R r_R(a))[x] \neq (Ra)[x] = (R[x])a$ so $R[x]$ is not quasi nil-injective. But we have $S_r(R[x]) = 0$, so $R[x]$ is a right mininjective ring. Hence there exists a right mininjective ring which is not right quasi nil-injective.

Example 3.9. Let V be a 2-dimensional vector space over a field K. Denote $R = \{\begin{pmatrix} k & v \\ 0 & k \end{pmatrix} \mid k \in K, v \in V\}$. Then R is a commutative ring. Let $x =$ $= \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}$. Then $(xR)^2 = 0$ and $lr(x) \neq Rx$. It follows that R is not quasi nil-injective. Thus the polynomial ring $R[x]$ is a mininjective ring but not quasi nil-injective.

Corollary 3.10. *Every right quasi nil-injective ring is right minsymmetric.*

Proof. It follows from [9, Propotion 2.4] and Corollary 3.7. ■

A ring R is called right Johns [10] if it is right noetherian and every right ideal is an annihilator.

Corollary 3.11. *Every right John, right quasi nil-injective ring is Quasi-Frobenious.*

Proof. By Corollary 3.7 and [10, Theorem 4.6]. ■

Call R right MC_2 ring [11] if $eRa = 0$ implies $aRe = 0$, where $a, e^2 = e \in$ $\in R, eR$ is a minimal right ideal of R.

Corollary 3.12. *Every right quasi nil-injective ring is right MC2.*

Proof. Assume that R is a right quasi nil-injective ring, $eRa = 0, a \in R$ and eR is a minimal right ideal of R, $e^2 = e \in R$. We should be pointed out that $aRe = 0$. If $aRe \neq 0$ then there exists a $b \in R$ such that $abe \neq 0$. We have $abe \in Nil(R_R)$ so $Rabe = lr(abe)$ (by hypothesis). Since $r(e) = r(abe)$ so $Re = Rabe$. Therefore $Re = ReRe = ReRabe = 0$, which is a contradiction. So we are done. ■

Call a ring R n-regular [7] if $a \in aRa$ for all $a \in N(R)$. A ring R is called reduced [7] if $N(R) = 0$.

Proposition 3.13. *The following conditions are equivalent for a ring R:*

1) $Nil(R_R) = 0$.

2) $\forall a \in Nil(R_R), a \in aRa$.

3) *R is a semiprime ring.*

These statement are equivalent if throughout "right" is replaced by "left".

Proof. 1) $\Longrightarrow$ 2) is clear.
2) $\Longrightarrow$ 3) Let I be a nilpotent ideal of R and $a \in I$. Then $a \in Nil(R_R)$. By hypothesis, $a \in aRa$. There exists an $r \in R$ such that $a = ara$. Then $(ar)^2 = ar$, i.e., ar is an idempotent. In the other hand, $a \in Nil(R_R)$ so $(ar)^n = 0$. It follows that $a = 0$ then $I = 0$. In this case the ring R has no nilpotents ideals then R is a semiprime ring.
3) $\Longrightarrow$ 1) As we know that in a semiprime ring, the only nilpotent right or left ideal is 0. Hence $Nil(R_R) = 0$. ■

It is clear that every n-regular ring is semiprime but the converse does not true in general (see [7]). In case of $a \in Nil(R_R)$, we also have the similar definition: Call a ring R nil-regular if $a \in aRa$ for all $a \in Nil(R_R)$ ($a \in \in Nil(_RR)$). It is easy to see that n-regular $\Rightarrow$ nil-regular and the converse doesn't hold, in general.

Recall that a ring R is right PP [7] if every principal right ideal of R is projective as a right R-module. A ring R is right PS [12] if every minimal right ideal is projective as a right R-module. A ring R is right PP if every principal right ideal of R is projective as a right R-module. A ring R is said to be right NPP [7] if Ra_R is projective for all $a \in N(R)$.

Call a ring aR_R right $NilPP$ if aR_R is projective for all $a \in Nil(R_R)$. Hence right PP rings, Von Neumann regular rings, reduced rings and right NPP rings are right $NilPP$.

Proposition 3.14.

(1) *Every right $NilPP$ ring is right non-singular.*

(2) *Every right $NilPP$ ring is right PS.*

(3) *Let R be a ring such that the polynomial ring $R[x]$ is a right $NilPP$ ring. Then R is a right $NilPP$ ring.*

Proof. (1) Let $0 \neq a \in Z_r(R) = \{a \in R \mid r(a) \leq^e R\}$ so all $u \in R, au = 0$ then $a \in Nil(R_R)$. Since R is right $NilPP$, aR is projective. So $r(a)$ is a direct summand of R as a right R-module. But $a \in Z_r(R), r(a)$ must be essential in R_R, which is a contradiction. Hence $Z_r(R) = 0$, so R is a non-singular ring.

(2) By (2) $Z_r(R) = 0$ and $Z_r(R) \cap S_r(R) = 0$. By [13], R is a right PS ring.

(3) Let $a \in Nil(R_R)$, we must verify that $r_R(a) = eR$, $e^2 = e \in R$. Indeed, we have $a \in Nil(R[x]_{R[x]})$, and $r_{R[x]}(a) = gR[x]$, by hypothesis. Let $g = g_0 + g_1x + g_2x^2 + \cdots + g_nx^n$ where $g_i \in R, i = 0, 1, 2, \cdots, n$. Thus $g_0^2 = g_0$ and $r_R(a) = g_0R$, which implies that R is a right $NilPP$ ring. ■

Example 3.15. Let F be a division ring and $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. It is easy to see that $Nil(R_R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. Then R is not right quasi nil-injective. In fact, let $0 \neq x \in F$, then $R\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & Fx \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ and $lr(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}) = = \begin{pmatrix} 0 & 0 \\ F & F \end{pmatrix}$. Clearly, $R\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \neq lr(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix})$. On the other hand, R is right PP so R is right $NilPP$.

A module M_R is called quasi-nil-R-injective if, for any $a \in Nil(R_R)$, then every homomorphism from aR to M can be extended to R_R.

Theorem 3.16. *The following conditions are equivalent for a ring R.*

(1) *R is a semiprime ring.*

(2) *Every right R-module is quasi nil-R-injective.*

(3) *Every cyclic right R-module is quasi nil-R-injective.*

(4) *R is a right quasi nil-injective, right $NilPP$ ring.*

These statement are equivalent if throughout "right" is replaced by "left".

Proof. (1) $\Longrightarrow$ (2). Assume that M is a right R-module and $f : aR \to M$ is any right R-homomorphism for $a \in Nil(R_R)$. By (1), $a = aba, b \in R$. Write $e = ab$ then $e^2 = abab = ab = e$ and $a = ea$. Set $m = f(e)$ then $f(x) = mx$ which implies that M_R is quasi nil-R-injective.

(2) $\Longrightarrow$ (3). It is clear.

(3) $\Longrightarrow$ (4). Clearly R is a right quasi nil-injective ring by (3). Assume that $a \in Nil(R_R)$ then aR is quasi nil-injective by (3). Call $f : aR \to aR$ the identity map. By (3), there exists a homomorphism $g : R_R \to aR$ such that g is an extension of f. It follows that $a = f(a) = g(1)a \in aRa$. Write $g(1) = ab$ for some $b \in R$. Then $a = ca = aba, c^2 = baba = ba = c$ and $aR = cR$ is a projective right R-module.

(4) $\Longrightarrow$ (1) Suppose that $a \in Nil(R_R)$. By (4) and Theorem 3.3, $Ra = lr(a)$. Since R is a right $NilPP$ ring, $r(a) = (1-e)R, e^2 = e \in R$. Therefore $Ra = Re$. Write $e = ca, c \in R$ then $a = ae = aca \in aRa$, which implies that R is a semiprime ring. ■

Remark 3.17. The class of quasi nil-R-injective modules and of nil-injective module are different. Since if they are the same then the class of semiprime rings and the class of n-regular coincides. It can't happen since in [7], there exists a semiprime ring which is not n-regular. For example, the trivial extension $R = T(\mathbb{Z}, \mathbb{Z}_{2^\infty})$ is semiprime which is not n-regular.

Call a ring R right $NilC_2$ if aR_R projective implies $aR = eR, e^2 = e \in R$ for all $a \in Nil(R_R)$.

Example 3.18. The trivial extension $R = T(\mathbb{Z}, \mathbb{Z}_{2^\infty})$ is a commutative ring. Since $Nil(R_R) \subseteq J(R), R$ is right $NilC_2$.

Proposition 3.19.

(1) *Every right quasi nil-injective, right $NilPP$ ring is right $NilC_2$.*

(2) *Every right $NilC_2$ ring is right MC_2.*

(3) *If $R[x]$ is a right $NilC_2$ ring, then so is R.*

Proof. (1) Let R be a right quasi nil-injective ring. Suppose that $a \in Nil(R_R)$ and aR_R is projective. Then $r(a) = gR, g^2 = g \in R$. By hypothesis and Theorem 3.3, we have $R(1-g) = l(gR) = lr(a) = Ra$. Write $1-g = ca$ and $e = ac$. Then $a = a(1-g) = aca = ea, e^2 = e$ and $aR = eR$. It implies that R is a right $NilC_2$ ring.

(2) By definition.

(3) Suppose that $a \in Nil(R_R)$ and aR_R is projective. Then $r_R(a) = = eR, e^2 = e \in R$. Since $r_{R[x]}(a) = eR[x]$ and $a \in Nil(R[x]_{R[x]}), aR[x]_{R[x]}$ is projective. Therefore $aR[x] = hR[x], h^2 = h \in R[x]$ by hypothesis. Let $h = h_0 + h_1x + h_2x^2 + \cdots + h_nx^n$ where $h_i \in R, i = 1, 2, \cdots, n$. Clearly $aR = h_0R, h_0^2 = h_0$. ■

Proposition 3.20.

(1) *R is a semiprime ring if and only if R is a right $NilC_2$, right $NilPP$ ring.*

(2) *If R is a semiprime ring, then $Nil(R_R) \cap J(R) = 0$.*

Proof. (1) By Theorem 3.19 we have every semiprime ring is a right $NilC_2$, right $NilPP$ ring. Conversely, let $a \in Nil(R_R)$, since R is right $NilPP$, aR_R is projective. Since R is a right $NilC_2$ ring, $aR = eR$, $e^2 = e \in R$. Thus $a = ea \in aRa$. Hence R is a semiprime ring by Proposition 3.16.

(2) It is obvious. ■

Proposition 3.21. *Let R be a right quasi nil-injective ring and $a \in Nil(R_R)$, $b \in R$.*

(1) *If $\sigma : aR \to bR$ is epic then there exists $\phi : Rb \to Ra$ is monic.*

(2) *If $aR \cong bR$ then $Rb \cong Ra$.*

Proof. (1) Call $u \in R$ with $\sigma(x) = u(x)$ for all $x \in aR$. There exists $v \in R$ such that $ua = \sigma(a) = bv$. $\phi(y) = yv, y \in Rb, v \in R$. Then $\phi(rb) = = rbv = r\sigma(a) = rua \in Ra$ so ϕ is a left R-homomorphism. If $\phi(rb) = 0$ then $rua = rbv = 0$. Since σ is an epimorphism, then $b = \sigma(ac), c \in R, b = uac$ and $rb = ruac = 0$ which implies that ϕ is a monic.

(2) Let ϕ, u, v, σ as (1). By hypothesis, $a \in Nil(R_R)$ then $\sigma(a) \in Nil(R_R)$. Since $r(a) = r(\sigma(a)), R\sigma(a) = lr(\sigma(a)) = lr(a) = Ra$. Thus $Ra = Rua$, which implies that ϕ is epic. So ϕ is isomorphism. ∎

Proposition 3.22. *Let R be a right quasi nil-injective ring.*

(1) *If K is a singular simple right ideal of R, then RK is the homogeneous component of $S_l(R)$ containing K.*

(2) *If R is I-finite, then $R \cong R_1 \times R_2$, where R_1 is semisimple and every simple right ideal of R_2 is nilpotent.*

Proof. (1) Assume $K = kR, k \in R$ and $\sigma : K \to S$ be a right R-homomorphism, where S is a right ideal of R. By hypothesis K is a singular right ideal of R, we have $K^2 = 0$ so $(kR)^2 = 0$, then $k, \sigma(k) \in Nil(R_R)$. By Theorem 3.3, $Rk = lr(k) = lr(\sigma(k)) = R\sigma(k)$. Hence $S = \sigma(k)R \subseteq RkR \subseteq \subseteq RK$, so K-component is in RK. The other inclusion always holds.

(2) By Corollary 3.10, this is an immediate consequence of [9, Theorem 1.12]. ∎

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