COEFFICIENT INEQUALITY FOR TRANSFORMS OF RECIPROCAL OF BOUNDED TURNING FUNCTIONS

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Abstract. In the present paper we introduce a new subclass of analytic functions. We prove a sharp upper bound to the second Hankel determinant associated with the k^{th} root transform $\left[f(z^k)\right]^{\frac{1}{k}}$ of the normalized analytic function $f(z)$, when it belongs to this class, using Toeplitz determinants.

1. Introduction

Let A denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disc $E = \{z : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions. In 1985, Louis de Branges de Bourcia proved the Bieberbach conjecture, i.e.: for a univalent function its n^{th} coefficient is bounded

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by n (see [3]). The bounds for the coefficients give information about the geometric properties of these functions. In particular, the growth and distortion properties of a normalized univalent function are determined by the bound of its second coefficient. The Hankel determinant of f for $q \geq 1$ and $n \geq 1$ was defined by Pommerenke [14] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}, \quad (a_1 = 1).$$

This determinant has been considered by many authors in the literature. For example, Noor [12] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for the functions in S with bounded boundary. Ehrenborg [5] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman [8]. In the recent years several authors have investigated bounds for the Hankel determinant of functions belonging to various subclasses of univalent and multivalent analytic functions in the literature. In particular for $q = 2$, $n = 1$, $a_1 = 1$ and $q = 2$, $n = 2$, $a_1 = 1$, the Hankel determinant simplifies respectively to

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2,$$

$$\text{and} \quad H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2.$$

We refer to $H_2(2)$ as the second Hankel determinant. A familiar result is that for the univalent function given in (1.1) the sharp inequality $H_2(1) = |a_3 - a_2^2| \leq$ ≤ 1 holds true [4]. For a family $\mathcal{T}$ of functions in S, the more general problem of finding sharp estimates for the functional $|a_3 - \mu a_2^2| (\mu \in \mathbb{R}$ or $\mu \in \mathbb{C})$ in popularly known as the Fekete-Szegő problem for $\mathcal{T}$. Ali [2] found sharp bounds for the first four coefficients and sharp estimate for the Fekete-Szegő functional $|\gamma_3 - t\gamma_2^2|$, where t is real for the inverse function of f defined as $f^{-1}(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n \in \widetilde{ST}(\alpha)$, the class of strongly starlike functions of order α $(0 < \alpha \leq 1)$. Janteng, Halim and Darus [7] have considered the functional $|a_2 a_4 - a_3^2|$ and found sharp upper bound for the function f in the subclass RT of S, consisting of functions whose derivative have a positive real part (also called bounded turning functions) studied by Mac Gregor [10] and have shown that if $f \in RT$ then $|a_2 a_4 - a_3^2| \leq \frac{4}{9}$. R. M. Ali, S. K. Lee, V. Ravichandran and S. Supramaniam [1] obtained sharp bounds for the Fekete-Szegő coefficient functional denoted by $|b_{2k+1} - \mu b_{k+1}^2|$ associated with the k^{th} root transform $\left[f(z^k)\right]^{\frac{1}{k}}$ of the function given in (1.1), belonging to certain subclasses of S. The k^{th} root transform for the function f given in (1.1) is defined as

$$F(z) := \left[f(z^k)\right]^{\frac{1}{k}} = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1} \tag{1.2}$$

Motivated by the results obtained by R. M. Ali, S. K. Lee, V. Ravichandran and S. Supramaniam [1], in the present paper, we introduce a new subclass denoted by $\widehat{RT}$ and obtain sharp upper bound to the functional $|b_{k+1}b_{3k+1} - -b_{2k+1}^2|$ for the k^{th} root transform of the function f when it belongs to this class, defined as follows.

Definition 1.1. A function $f(z) \in A$ is said to be function whose reciprocal derivative has a positive real part (also called reciprocal of bounded turning function), denoted by $f \in \widehat{RT}$, if and only if

$$\operatorname{Re} \frac{1}{f'(z)} > 0, \qquad \forall z \in E.$$

2. Preliminary results

Let $\mathcal{P}$ denote the class of functions consisting of p, such that

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ... = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{2.1}$$

which are regular in the open unit disc E and satisfy Re $p(z) \ > 0$ for any $z \in E$. Here $p(z)$ is called the Carathéodory function [4].

Lemma 2.1. ([13], [15]) *If $p \in \mathcal{P}$, then $|c_k| \le 2$, for each $k \ge 1$ and the inequality is sharp for the function $\frac{1+z}{1-z}$.*

Lemma 2.2. ([6]) *The power series for $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ given in* (2.1) *converges in the open unit disc E to a function in $\mathcal{P}$ if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ c_{-2} & c_{-1} & 2 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, \qquad n = 1, 2, 3, \ldots$$

and $c_{-k} = \bar{c}_k$, are all non-negative. They are strictly positive except for $p(z) = \sum_{k=1}^{m} \rho_k p_0(e^{it_k} z)$, $\rho_k > 0$, t_k real and $t_k \neq t_j$, for $k \neq j$, where $p_0(z) = \frac{1+z}{1-z}$; in this case $D_n > 0$ for $n < (m-1)$ and $D_n \doteq 0$ for $n \geq m$.

This necessary and sufficient condition found in [6] is due to Carathéodory and Toeplitz. We may assume without restriction that $c_1 > 0$. On using Lemma 2.2, for $n = 2$, we have

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \bar{c}_1 & 2 & c_1 \\ \bar{c}_2 & \bar{c}_1 & 2 \end{vmatrix} = [8 + 2\mathrm{Re}\,\{c_1^2 c_2\} - 2 \mid c_2 \mid^2 - 4|c_1|^2] \geq 0,$$

which is equivalent to

$$2c_2 = c_1^2 + x(4 - c_1^2), \text{ for some } x, |x| \leq 1. \tag{2.2}$$

For $n = 3$,

$$D_3 = \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \bar{c}_1 & 2 & c_1 & c_2 \\ \bar{c}_2 & \bar{c}_1 & 2 & c_1 \\ \bar{c}_3 & \bar{c}_2 & \bar{c}_1 & 2 \end{vmatrix} \geq 0$$

and is equivalent to

$$|(4c_3 - 4c_1c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2| \leq 2(4 - c_1^2)^2 - 2|(2c_2 - c_1^2)|^2. \tag{2.3}$$

Simplifying the expressions (2.2) and (2.3), we get

$$\begin{aligned} 4c_3 = \{&c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + \\ &+ 2(4 - c_1^2)(1 - |x|^2)z\}, \quad \text{with } |z| \leq 1. \end{aligned} \tag{2.4}$$

To obtain our result, we refer to the classical method initiated by Libera and Zlotkiewicz [9] and used by several authors in the literature.

3. Main result

Theorem 3.1. *If $f(z) \in \widehat{RT}$, then $|b_{k+1}b_{k+3} - b_{k+2}^2| \leq \frac{4}{9k^2}$ with $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and the inequality is sharp.*

Proof. For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \widehat{RT}$, by virtue of Definition 1.1, there exists an analytic function $p \in \mathcal{P}$ in the open unit disc E with $p(0) = 1$ and $\mathrm{Re}\, p(z) > 0$ such that

$$\frac{1}{f'(z)} = p(z) \quad \Longleftrightarrow \quad 1 = f'(z)p(z). \tag{3.1}$$

Replacing $f'(z)$ and $p(z)$ with their equivalent series expressions in (3.1), we have

$$1=\left\{1+\sum_{n=2}^{\infty} n a_n z^{n-1}\right\}\left\{1+\sum_{n=1}^{\infty} c_n z^n\right\}.$$

Upon simplification, we obtain

$$\begin{aligned}1=1&+(c_1+2a_2)z+(c_2+2a_2c_1+3a_3)z^2+\\&+(c_3+2a_2c_2+3a_3c_1+4a_4)z^3+\cdots.\end{aligned} \tag{3.2}$$

Equating the coefficients of like powers of z, z^2 and z^3 respectively on both sides of (3.2), after simplifying, we get

$$a_2=\frac{-c_1}{2};\quad a_3=\frac{1}{3}(c_1^2-c_2);\quad a_4=-\frac{1}{4}(c_3-2c_1c_2+c_1^3). \tag{3.3}$$

For a function f given by (1.1), a computation shows that

(3.4)

$$\begin{aligned}\left[f(z^k)\right]^{\frac{1}{k}}&=\left[z^k+\sum_{n=2}^{\infty} a_n z^{nk}\right]^{\frac{1}{k}}=\\&=\left[z+\frac{1}{k}a_2z^{k+1}+\left\{\frac{1}{k}a_3+\frac{1-k}{2k^2}a_2^2\right\}z^{2k+1}+\right.\\&\quad\left.+\left\{\frac{1}{k}a_4+\frac{1-k}{k^2}a_2a_3+\frac{(1-k)(1-2k)}{6k^3}a_2^3\right\}z^{3k+1}\;+\;\cdots\right].\end{aligned}$$

The equations (1.2) and (3.4) yield;

$$\begin{aligned}b_{k+1}&=\frac{1}{k}a_2\;;\quad b_{2k+1}=\frac{1}{k}a_3+\frac{1-k}{2k^2}a_2^2\;;\\b_{3k+1}&=\frac{1}{k}a_4+\frac{1-k}{k^2}a_2a_3+\frac{(1-k)(1-2k)}{6k^3}a_2^3.\end{aligned} \tag{3.5}$$

Simplifying the equations (3.3) and (3.5), we get

$$\begin{aligned}b_{k+1}&=\frac{-c_1}{2k};\quad b_{2k+1}=\frac{1}{24k^2}[(5k+3)c_1^2-8kc_2];\\b_{3k+1}&=-\frac{1}{48k^3}[12k^2c_3-8k(1+2k)c_1c_2+(1+2k)(1+3k)c_1^3].\end{aligned} \tag{3.6}$$

Substituting the values of b_{k+1}, b_{2k+1} and b_{3k+1} from (3.6) in the second Hankel determinant $|b_{k+1}b_{3k+1}-b_{2k+1}^2|$ for the k^{th} transform of the function $f\in\widehat{RT}$, which simplifies to

$$|b_{k+1}b_{3k+1}-b_{2k+1}^2|=\frac{1}{576k^4}\left|72k^2c_1c_3-16k^2c_1^2c_2-64k^2c_2^2+(11k^2-3)c_1^4\right|. \tag{3.7}$$

Substituting the values of c_2 and c_3 from (2.2) and (2.4) respectively from Lemma 2.2 on the right-hand side of (3.7), we have

$$\begin{aligned}(3.8)\quad &\left|72k^2c_1c_3 - 16k^2c_1^2c_2 - 64k^2c_2^2 + (11k^2-3)c_1^4\right| = \\ &= \Big|72k^2c_1 \times \frac{1}{4}\{c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z\} - \\ &\quad - 16k^2c_1^2 \times \frac{1}{2}\{c_1^2 + x(4-c_1^2)\} - 64k^2 \times \frac{1}{4}\{c_1^2 + x(4-c_1^2)\}^2 + \\ &\quad + (11k^2-3)c_1^4\Big|.\end{aligned}$$

Using the triangle inequality and the fact $|z| < 1$, upon simplification, we get

$$\begin{aligned}&\left|72k^2c_1c_3 - 16k^2c_1^2c_2 - 64k^2c_2^2 + (11k^2-3)c_1^4\right| \leq \\ (3.9)\quad &\leq \left|(5k^2-3)c_1^4 + 36k^2c_1(4-c_1^2) + 4k^2c_1^2(4-c_1^2)|x| + \right. \\ &\left. +2k^2(c_1+2)(c_1+16)(4-c_1^2)|x|^2\right|.\end{aligned}$$

Since $c_1 \in [0,2]$, noting that $(c_1+a)(c_1+b) \geq (c_1-a)(c_1-b)$, where $a, b \geq 0$ on the right hand side of (3.9), we have

$$\begin{aligned}&\left|72k^2c_1c_3 - 16k^2c_1^2c_2 - 64k^2c_2^2 + (11k^2-3)c_1^4\right| \leq \\ (3.10)\quad &\leq \left|(5k^2-3)c_1^4 + 36k^2c_1(4-c_1^2) + 4k^2c_1^2(4-c_1^2)|x| + \right. \\ &\left. +2k^2(c_1-2)(c_1-16)(4-c_1^2)|x|^2\right|.\end{aligned}$$

Choosing $c_1 = c \in [0,2]$, applying triangle inequality and replacing $|x|$ by μ on the right-hand side of the above inequality, we have

$$\begin{aligned}&\left|72k^2c_1c_3 - 16k^2c_1^2c_2 - 64k^2c_2^2 + (11k^2-3)c_1^4\right| \leq \\ (3.11)\quad &\leq \left[(5k^2-3)c^4 + 2k^2\left\{18c + 2c^2\mu + (c-2)(c-16)\mu^2\right\} \times (4-c^2)\right] = \\ &= F(c,\mu), \qquad \text{for } 0 \leq \mu = |x| \leq 1.\end{aligned}$$

We next maximize the function $F(c,\mu)$ on the closed region $[0,2] \times [0,1]$. Differentiating $F(c,\mu)$ partially with respect to μ, we get

$$(3.12)\qquad \frac{\partial F}{\partial \mu} = 4k^2\left[c^2 + (c-2)(c-16)\mu\right] \times (4-c^2).$$

For $0 < \mu < 1$, for fixed c with $0 < c < 2$ and for every $k \in \mathbb{N}$, from (3.12), we observe that $\frac{\partial F}{\partial \mu} > 0$. Therefore, $F(c,\mu)$ is an increasing function of μ and hence it cannot have maximum value at any point in the interior of the closed region $[0,2] \times [0,1]$. Moreover, for fixed $c \in [0,2]$, we have

$$(3.13)\qquad \max_{0 \leq \mu \leq 1} F(c,\mu) = F(c,1) = G(c).$$

Therefore, replacing μ by 1 in $F(c,\mu)$, upon simplification, we obtain

$$G(c) = -(k^2+3)c^4 - 40k^2c^2 + 256k^2, \tag{3.14}$$

$$G'(c) = -4(k^2+3)c^3 - 80k^2c. \tag{3.15}$$

From (3.15), we observe that $G'(c) \leq 0$, for every $c \in [0,2]$ and for every k. Therefore, $G(c)$ becomes a decreasing function of c in the interval $[0,2]$, whose maximum value occurs at $c=0$ only. From (3.14), the maximum value of $G(c)$ is given by

$$G_{max} = G(0) = 256k^2. \tag{3.16}$$

From the relations (3.11) and (3.16), we get

$$\left|72k^2c_1c_3 - 16k^2c_1^2c_2 - 64k^2c_2^2 + (11k^2-3)c_1^4\right| \leq 256k^2. \tag{3.17}$$

Simplifying the expressions (3.7) and (3.17), we obtain

$$\left|b_{k+1}b_{3k+1} - b_{2k+1}^2\right| \leq \frac{4}{9k^2}. \tag{3.18}$$

By setting $c_1 = \ c = 0$ and selecting $x = 1$ in the expressions (2.2) and (2.4), we find that $c_2 = 2$ and $c_3 = 0$ respectively. Using these values in (3.17), we observe that equality is attained, which shows that our result is sharp. For these values, we derive the extremal function, given by

$$\frac{1}{f'(z)} = 1 + 2z^2 + 2z^4 + ... = \frac{1+z^2}{1-z^2}. \tag{3.19}$$

This completes the proof of our Theorem. ■

Remark 3.1. Choosing $k = 1$ in (3.18), the result coincides with that of Janteng, Halim and Darus [7]. From this, we conclude that the upper bound to the second Hankel determinant of a function whose derivative has a positive real part and a function whose reciprocal derivative has a positive real part is the same.

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References

[1] **Ali, R. M. Lee, S. K. Ravichandran. V and Supramaniam. S,** The Fekete-Szegő coefficient functional for transforms of analytic functions, *Bull. Iranian Math. Soc.*, **35**(2) (2009), 119–142.

[2] **Ali, R. M.,** Coefficients of the inverse of strongly starlike functions, *Bull. Malays. Math. Sci. Soc. (second series)*, **26** (2003), 63–71.

[3] **de Branges de Bourcia, Louis,** A proof of Bieberbach conjecture, *Acta Mathematica,* **154** (1985), 137–152.

[4] **Duren, P. L.,** *Univalent Functions*, Vol. 259 of Grundlehren der Mathematischen Wissenschaften, Springer, New York, USA, 1983.

[5] **Ehrenborg, R.,** The Hankel determinant of exponential polynomials, *Amer. Math. Monthly*, **107** (2000), 557–560.

[6] **Grenander, U. and Szegő, G.,** *Toeplitz Forms and their Applications*, Second edition. Chelsea Publishing Co., New York, 1984.

[7] **Janteng, A. Halim, S. A. and Darus, M.,** Coefficient inequality for a function whose derivative has a positive real part, *J. Inequal. Pure Appl. Math.*, **7** (2006), 1–5.

[8] **Layman, J. W.,** The Hankel transform and some of its properties, *J. Integer Seq.*, **4** (2001), 1–11.

[9] **Libera, R. J. and Zlotkiewicz, E. J.,** Coefficient bounds for the inverse of a function with derivative in $\mathcal{P}$, *Proc. Amer. Math. Soc.,* **87** (1983), 251–257

[10] **Mac Gregor, T. H.,** Functions whose derivative have a positive real part, *Trans. Amer. Math. Soc.,* **104** (1962), 532–537.

[11] **Noonan, J. W. and Thomas, D. K.,** On the second Hankel determinant of areally mean p - valent functions, *Trans. Amer. Math. Soc.,* **223** (1976), 337–346.

[12] **Noor, K. I.,** Hankel determinant problem for the class of functions with bounded boundary rotation, *Rev. Roumaine Math. Pures Appl.,* **28** (1983), 731–739.

[13] **Pommerenke, Ch.,** *Univalent Functions*, Vandenhoeck and Ruprecht, Gottingen, 1975.

[14] **Pommerenke, Ch.,** On the coefficients and Hankel determinants of univalent functions, *J. Lond. Math. Soc.,* **41** (1966), 111–122.

[15] **Simon. B.,** *Orthogonal Polynomials on the Unit Circle, Part 1. Classical Theory,* Vol. 54, American Mathematical Society Colloquium Publications, Providence (RI), American Mathematical Society, 2005.

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